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Three Essays on Economic Interactions under Bounded Rationality

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The undersigned Oktay Surucu, in his quality of doctoral candidate for a Ph.D. degree in Economics granted by the Universitá Ca' Foscari Venezia attests that the research exposed in this dissertation is original and that it has not been and it will not be used to pursue or attain any other academic degree of any level at any other academic institution, be it foreign or Italian.

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Preface

The issues explored in this work concern economic interactions under bounded rationality. Each chapter considers these interactions from different angles.

In the first chapter, we characterize the optimal contract when a profit-maximizing monopolist faces consumers that are diverse in preferences and in their bounded rationality. The bounded rationality is due to the individual's cognitive limitation that prevents him from making context-independent decisions and therefore leads to biased choices. Each consumer is affected differently from this limitation, i.e., consumers are diversely context-biased. Using the standard second-degree price discrimination model, we characterize the optimal menu of contracts and show that the seller can increase his profit by exploiting consumers' bounded rationality. Our main result is that the optimal contract has at most three menus even when the number of types is infinite. The decision between two and three menus contract depends on the distribution of the parameter θ that models the level of context-bias of consumers. The contract with two menus discriminates consumers based only on their preferences, whereas the contract with three menus partitions the consumer set into three sets based on their preferences and level of context-bias.

The second chapter is concerned with the interaction between fully and boundedly rational agents in situations where their interests are perfectly aligned. The cognitive limitations of the boundedly rational agent do not allow him to fully understand the market conditions and lead him to take non-optimal decisions in some situations. Using categorization to model bounded rationality, we show that the fully rational agent can manipulate information to help decreasing the expected loss caused by the boundedly rational agent. Assuming different types for the boundedly rational agent, who differ only in the categories used, we show that the fully rational agent may learn the type of the boundedly rational agent along their interaction. Using this additional information, the outcome can be improved and the amount of manipulated information can be decreased. Furthermore, as the length of the interaction gets longer the probability that the fully rational agent learns the type of the boundedly rational agent increases.

The third chapter studies a model where a team of agents with limited problem-solving ability face a disjunctive task over a large solution space. We provide sufficient conditions for the following four statements. First, two heads are better than one: a team of two agents will solve the problem even if neither agent will not. Second, teaming up does not guarantee success: if the agents are not sufficiently creative, any team (regardless of its size) may fail to solve the problem. Third, "defendit numerus": even if the agent's problem-solving ability is adversely affected by the complexity of the solution space, small teams of more than two people will still solve the problem. Fourth, groupthink impairs the power of diversity: if agents' abilities are positively correlated, larger teams are necessary to solve a problem.

Chapter 1

Discrimination over Price and Quality when Agents are Context-Biased

1.1 Introduction

The assumption of full rationality is widely used in economic literature since it simplifies economic models substantially and makes them analytically tractable. Wide literature initiated by Amos Tversky, Daniel Kahneman, and their collaborators presents experimental evidence that human beings depart systematically from full rationality due to cognitive limitations.

The following experiment, performed by Simonson and Tversky (1992), exhibits an abnormal decision pattern. There were two versions of the experiment. In the first version, participants were asked to choose an item among three alternatives; high quality brand of tissue, high quality brand of towel (a substitute product for tissue) and low quality brand of towel. In the second version, low quality brand of towel was replaced by its substitute, i.e., by low quality brand of tissue. Under the assumption of full rationality, these two sets of alternatives are equivalent. That is because a low quality brand item is always offered together with the high quality brand of the same item in both sets of alternatives. Since low quality items are strictly dominated by high quality ones, the decision of an individual would simplify to choose between a high quality brand of towel and a high quality brand of tissue and therefore the results of the experiment would be same in both versions. In other words, under the assumption of full rationality the preference between alternatives is independent of the context. However, it is shown that the market share of high quality brand towel in the first version is significantly greater than its share in the second version. Even though the low quality items are almost never chosen, their existence affects the choice of agents. This suggests that consumer preferences are influenced by the context, i.e., they are context-biased. Furthermore, we observe that a significant number of people tend to choose the high quality item whose low quality is also present among alternatives. This shows that introducing an inferior option in a category may help to increase the attractiveness of that category.

The above fact may be a possible explanation why shopkeepers prefer to put fancy and highly expensive or very cheap but low quality products on the window of their shop even though the probability to sell these products is very low: to attract attention to their shops. These type of products, that are defined as 'pure attention grabbers' in Eliaz and Spiegler (2010), mostly provide low utility to consumers, either because they are of high quality but too expensive or because they are cheap but of low quality. The following example, inspired by Eliaz and Spiegler (2010), illustrates the usage of these products. A consumer who wants to buy a new cellular phone may notice that a store offers a model that is very small in size. In order to inspect this model, he enters the store and figures out that the other features of this model are not that good (e.g., it does not have a camera). Being already in the store, the consumer may browse other models and find one that suits his needs. Thus, even though the phone that is small in size is not sold, it grabs the attention of consumers and directs them to the store.

The main purpose of the paper is to study the effects of heterogeneity in the level of context-bias of agents. For this purpose, we revisit the standard second degree price model under the assumption that agents differ both in preferences and in the level of context-bias. Using this setting, we examine what set of contracts would an ordinary profit-maximizing monopoly offer when faced with such consumers. We use representation of choice borrowed from Barbos (2010) where the variable $\theta \in [0,1]$ models the individual's level of context-bias. In line with the terminology used in Eliaz and Spiegler (2010), we say that an agent with $\theta = 0$ is *fully sophisticated* and an agent with $\theta = 1$ is *fully naive.* There are two sources that identify the type of a consumer: his valuation for the quality and his level of context-bias. For the first entry, there are two alternatives: high and low. When it comes to the second entry, we consider first the case where it has two possible values, θ_1 and θ_2 ($\geq \theta_1$). We call θ_1-consumers sophisticated and θ_2-consumers naive. In this case, there are four types in the market: naive with high valuation, sophisticated with high valuation, naive with low valuation, sophisticated with low valuation. Assuming that the seller knows θ_1, θ_2 and the probability of each type in the market, we show that the optimal contract has either two menus that discriminates consumers based only on their valuation for the quality or three menus that partitions the consumer set into three sets as: naive with high valuation, sophisticated with high valuation and low valuation consumers. We further show that the decision between contracts with two and three menus depends on θ_1, θ_2 and the probability of naive with high valuation type. Then, we move to a more realistic case where we characterize the optimal menu of contracts with a continuum of context-biased types. In other words, the second entry that identifies the type of a consumer, θ, takes values in a continuum. We assume that the seller does not observe θ but he knows the distribution of it. We show that with continuous context-biased types, the optimal contract has at most three menus. Whenever the fully sophisticated consumer with high valuation is present then the optimal contract has exactly three menus. In the case of a contract with three menus, the set of consumers is partitioned into three subsets. One of the subsets is composed of low valuation consumers whereas the other two belong to high valuation consumers. The seller has to optimally decide on a threshold for θ so that the contract discriminates high valuation types depending on this threshold.

The idea of discriminating between consumer types according to their cognitive features first appears in Rubinstein (1993). In this paper, monopolistic behavior when consumers differ in their ability to process information regarding the economic market is analyzed. It is shown that by complicating the price offers, a monopolist can limit the number of consumers accepting his offer. A crucial assumption that is made in this paper is that the high type (the type with low cost of production) is more sophisticated than the low type, i.e., the high type is more capable of processing information than the low type. This assumption narrows down the real life situations that can be covered by Rubinstein's model. In our model, however, we do not have such a restrictive assumption. Piccione and Rubinstein (2003) model cognitive differences among agents using the concept of DeBruijin sequences. They show that price fluctuations that are independent of economic fundamentals can emerge in equilibrium when consumers have diverse ability to understand market behavior. Furthermore, this can be recognized only by more sophisticated agents. Eliaz and Spiegler (2006) study a model in which agents differ in their ability to forecast changes in their future tastes. They show that more naive types are more heavily exploited and generate a greater profit for the principal, which is one of the conclusions that we too arrive at in this paper. Barbos (2010) studies a model of choice from options grouped into categories that accounts for the context-bias and shows that the proposed model yields a representation that is unique. By using this representation in a second degree price discrimination model, he shows that sellers facing context-biased consumers can increase their profits by exploiting their bounded rationality. The main difference between this paper and ours is that Barbos assumes implicitly that all the consumers have same θ, i.e., they are homogeneous in their context-bias level, whereas we relax this assumption by letting θ vary in $[0, 1]$ and study the optimal contract when agents are diversely context-biased.

The organization of the paper is as follows. In Section 1.2, we describe the model by giving the assumptions and the representation of preferences. In Section 1.3, we examine and characterize the optimal menu of contracts in discrete case, and continue with the continuous case in Section 1.4. Finally Section 2.5 concludes the paper, while the proofs are presented in Appendix.

1.2 The Model

We study a second degree price discrimination model that shows how a monopolist facing consumers differing in preference and in the level of context-bias could exploit the bounded rationality of consumers and increase the profit.

Products on the market are characterized by quality-price pair, $(q, p) \in \mathbb{R}^2_+$, and utilities of consumers depend on these two variables. Consumers differ in their tastes for quality; $\lambda \in (0,1)$ share of consumers have low valuation for the quality denoted by $v_l(.)$ and the rest have a higher valuation for the quality denoted by $v_h(.)$. The utility of a consumer is given by

$$u_i(q, p) = v_i(q) - p, \tag{1.1}$$

where $i \in \{h, l\}$. The single crossing property is satisfied ($v'_h(q) > v'_l(q)$, for all q) and outside option for the consumers is assumed to be zero. The profit of the seller is the amount he receives from customers for the products he sells minus the total cost of production that depends on the quality of the products. The cost function, $c(q)$, is strictly increasing, differentiable and convex.

We use the reference-dependent representation proposed by Barbos (2010) for choices of consumers. This representation is consistent with the experimental evidence by Simonson and Tversky (1992), presented in the introduction. In this setup agents make two sequential decisions. They choose first a category among the exogenously given categories and then an item from that category. Being context-biased affects only the first decision. This is due to the fact that the presence of an inferior option in a category may increase the attractiveness of that category, as suggested by the experiment of Simonson and Tversky (1992). A consumer makes his first decision by comparing his own valuations of the exogenously given categories and chooses the category with the highest valuation. The valuation of a consumer for a finite category A is given by

$$V(A, \theta) = \max_{\{x \in A | u(x) \geq 0\}} u(x) - \theta \min_{\{y \in A | u(y) \geq 0\}} u(y), \tag{1.2}$$

where $\theta \in [0, 1]$ stands for the individual's level of context-bias. Observe that when $\theta = 0$, the second term of equation (1.2) disappears and the individual's valuation for each category is based only on the product that provides the maximum utility in that category. In this case, the choice will be optimal. Thus, $\theta = 0$ represents the situation where the individual is context-unbiased (fully sophisticated). For an individual with a strictly positive θ, the resulting choice may deviate from the optimal one, meaning that the individual is more naive than the fully sophisticated individual. Finally, the case $\theta = 1$ represents the situation where the consumer is fully naive. Furthermore, the above formulation considers only items that provide non-negative utilities, i.e., items that provide higher utility than the outside option. This means that consumers take into account products that they might end up buying when evaluating a category. In the case when all the items in one category have negative value for the consumer, we assume that his valuation for this category is zero.

The second decision that consumers make is to pick an item from the chosen category. All consumers make this decision optimally by choosing the product that maximizes their utility. That is, given that a consumer chooses the category A in the first stage of his decision process, his second decision is determined by

$$c(A) = \arg\max_{x \in A} u(x).$$

The following example illustrates how the decision process works. Consider the first version of the experiment from Simonson and Tversky (1992). Individuals are asked to choose between a high quality brand of tissue, a high quality brand of towel and a low quality brand of towel denoted by y, x and x', respectively. Assume that the utilities of the alternatives for a consumer with $\theta = 0.5$ are $u(y) = 12$, $u(x) = 10$ and $u(x') = 5$. The first phase of the decision process is to select between the category of tissues and the category of towels by comparing their valuations:

$$\begin{aligned} V(\text{Tissues}, \theta = 1/2) &= u(y) - \theta\, u(y) \\ &= 12 - 1/2 * 12 = 6 \\ V(\text{Towels}, \theta = 1/2) &= u(x) - \theta\, u(x') \\ &= 10 - 1/2 * 5 = 7.5 \end{aligned}$$

This individual chooses the category of towels since its valuation is higher. In the second phase, he concentrates on the category of towels and chooses the high quality one (x), since $u(x) > u(x')$. In this example, we see that even though the high quality brand of tissue (y) provides the individual with the highest utility, he ends up choosing the high quality brand of towel (x) because of his context-bias. Observe that consumers with $\theta < \frac{2}{7}$ make the optimal choice since they are sophisticated enough not to be fooled by the attractiveness of the towel category.

We start our analysis by assuming that there are two types of consumers according to their level of context-bias: a sophisticated type with $\theta = \theta_1$ and a naive type with $\theta = \theta_2 > \theta_1$. The seller knows the exact values of θ_1, θ_2 and their distribution. After characterizing the optimal contract in Section 1.3, we replace this assumption by a less restrictive and more realistic one in Section 1.4. We assume that there are infinitely many possible types based on the level of context-bias ($\theta \in [0, 1]$) and work on the optimal contract when the seller does not observe this variable, but knows its distribution.

1.3 Discrete Case

There are two classes of consumers according to their level of context-bias: consumers with θ_1 (sophisticated consumers), and consumers with $\theta_2 (> \theta_1)$ (naive consumers). Overall, there are totally four types on the market with the following probabilities:

- sophisticated with high valuation ($(1 - \lambda)(1 - \alpha_h)$)
- naive with high valuation ($(1 - \lambda)\alpha_h$)
- sophisticated with low valuation ($\lambda(1 - \alpha_l)$)
- naive with low valuation ($\lambda\alpha_l$)

The values given in the parentheses are the probabilities of types, where $\lambda \in (0,1)$ denotes the probability of being low valuation, $\alpha_h \in (0,1)$ and $\alpha_l \in (0,1)$ are the conditional probabilities of being naive given the consumer has high and low valuation, respectively.

The seller knows all the parameters described so far and wants to discriminate between types in order to maximize his profit. Thus, the optimal contract has at least two menus, each of which is designed for some types on the market. In other words, a contract with a single menu is strictly dominated by a contract with two menus since the seller is able to discriminate between low and high valuation types and increase his profit by using a contract with two menus. Furthermore, the optimal contract can have at most four menus since there are four types in the market. In the next subsection, we characterize the optimal contract with two menus and show that there is no need for the seller to discriminate between low valuation types, i.e., designing different menus for sophisticated with low valuation type and naive with low valuation type does not increase the profit. This fact suggests that the optimal contract has either two or three menus. The contract with two menus, characterized in Section 1.3.1, discriminates between high and low valuation consumers, whereas the contract with three menus discriminates between naive with high valuation, sophisticated with high valuation and low valuation consumers. After characterizing the contract with three menus in Section 1.3.2, we show that in some situations the optimal contract has two and in some others it has three menus.

1.3.1 Contract with Two Menus

There are two products in each menu of the contract; one of them is to be sold to consumers who choose that menu in the first phase of the decision process and the other product is there only to attract some types (e.g., the low brand of towel in the example introduced at the beginning of Section 1.2) and we call these products as primary and secondary product, respectively. Since we have only two menus, this contract discriminates between high and low valuation consumers. Figure 1.1 is an illustration for this contract, where the first menu is designed for high valuation consumers and the second menu is for low valuation consumers. The primary and secondary products are denoted by (q_i, p_i) and (q_i', p_i') respectively, where $i = h$ for the first and $i = l$ for the second menu.

The seller's problem is

$$\max_{\{((q_i,p_i),(q_i',p_i'))\in\mathbb{R}_+^2\times\mathbb{R}_+^2\}} (1-\lambda)\,[p_h - c(q_h)] + \lambda\,[p_l - c(q_l)] \qquad \text{s.t. for } i\in\{h,l\} \quad (1.3)$$

$$u_i(q_i', p_i') \geq 0 \quad (1.3.1)$$

$$u_i(q_i, p_i) \geq u_i(q_i', p_i') \quad (1.3.2)$$

$$V_i\big(\{(q_i,p_i),(q_i',p_i')\},\theta_j\big) \geq V_i\big(\{(q_{-i},p_{-i}),(q_{-i}',p_{-i}')\},\theta_j\big) \quad j\in\{1,2\} \quad (1.3.3)$$

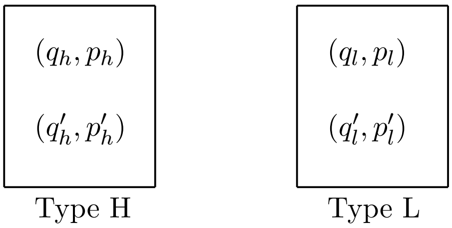

Figure 1.1: Contract with Two Menus

The first incentive restriction (1.3.1) guarantees that the secondary product in the menu designed for consumers of type i provides higher utility than the outside option which is assumed to be zero. The restriction (1.3.2) guarantees that primary products provide higher utility than the secondary products for the respective type. Finally, the restriction (1.3.3) ensures that type i's valuation for the menu designed for him is higher than his valuation for the other menu. In other words, (1.3.3) guarantees that each consumer selects the menu that is designed for him.

Lemma 1 characterizes the solution to this optimization problem.

Lemma 1 *The FOCs to the seller's optimization problem (1.3) are:*

$$c'(q_h) = v'_h(q_h) \tag{1.4}$$
$$p_h = v_h(q_h) - (1-\theta_1)\left[v_h(q_l) - v_l(q_l)\right] \tag{1.5}$$

$$c'(q_l) = v'_l(q_l) - \frac{1-\lambda}{\lambda}(1-\theta_1)\left[v'_h(q_l) - v'_l(q_l)\right] \tag{1.6}$$
$$p_l = v_l(q_l) \tag{1.7}$$

See Appendix A.1. The implications of the FOCs are as follows:

i. The comparison of equation (1.4) to (1.6), together with the fact that single crossing property is satisfied, shows that $c'(q_h^*) > c'(q_l^*)$, and the convexity of $c(.)$ ensures $q_h^* > q_l^*$.

ii. The equation (1.7) implies that the consumers with low valuation pay a price that is equal to their valuation for the quality of the product they are buying. Put differently, they end up with zero utility. Therefore,

  - Consumers with low valuation cannot be exploited more. Thus, it is unnecessary to discriminate these consumers based on their level of context-bias. In other words, the contract with four menus is never optimal.
  - We have $V_l\big(\{(q_l^*, p_l^*), (q_l'^*, p_l'^*)\}, \theta_{lj}\big) = 0 \ \forall j \in \{1, 2\}$, since $u_l(q_l^*, p_l^*) = 0$. Therefore, the incentive restriction (1.3.3) ensures that $u_l(q_h^*, p_h^*) \leq 0$ and this implies $p_h^* > p_l^*$. The economic interpretation of this implication is as follows:

The fact that low valuation consumers end up with zero utility ensures that their valuation for the menu designed for them is zero. But we also know that their valuation for the other menu (the menu designed for high valuation consumers) must be less than the valuation for this menu because of the incentive compatibility constraint. Therefore, the utility of the product that is sold to consumers with high valuation is negative for consumers with low valuation. We already know that the product sold to high valuation consumers has a higher quality than the one sold to low valuation consumers ($q_h^* > q_l^*$). Therefore, in order for its utility to be negative for low valuation type, the price of the high quality product must be greater ($p_h^* > p_l^*$).

iii. Both the quality and the price of the product sold to low valuation consumers and the price of the product sold to high valuation consumers (q_l^*, p_l^*, p_h^*) in the optimal contract depend on θ_1 but not θ_2. The reasoning for this is the following: the fact that the consumers with θ_1 are more sophisticated than the ones with θ_2 implies that whenever the incentive constraint for sophisticated type is satisfied, the one for naive type is already satisfied. Therefore, the contract is designed based on θ_1 rather than θ_2.

Finally, the profit of the seller from the contract with two menus is

$$\begin{aligned}\Pi_2^* &= (1-\lambda)[p_h^* - c(q_h^*)] + \lambda[p_l^* - c(q_l^*)] \\ &= (1-\lambda)\Big[v_h(q_h^*) - (1-\theta_1)\left(v_h(q_l^*) - v_l(q_l^*)\right) - c(q_h^*)\Big] + \lambda[v_l(q_l^*) - c(q_l^*)], \quad (1.8)\end{aligned}$$

where q_h^* and q_l^* are the solutions to equations (1.4) and (1.6), respectively. The profit is increasing in θ_1 since as θ_1 increases, the sophisticated type becomes more naive and therefore can be exploited more heavily.

1.3.2 Contract with Three Menus

In this subsection, we characterize the contract with three menus using the same settings of the previous subsection. Having three menus, this contract partitions consumers into three sets. As we have already observed in the previous subsection, all low valuation consumers are considered together without making any discrimination based on their level of context-bias. Therefore, this contract has one menu designed for sophisticated with high valuation consumers, another menu for naive with high valuation consumers and the last menu for consumers with low valuation. Figure 1.2 shows an illustration of this contract. The primary and secondary products are denoted by (q_k, p_k) and (q'_k, p'_k) respectively, where $k = a$ for the first menu, designed for sophisticated with high valuation consumers, $k = b$ for the second menu, designed for naive with high valuation consumers and $k = c$ for the third menu designed for all low valuation consumers.

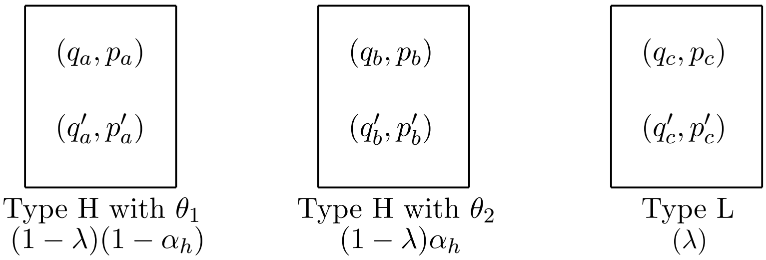

Figure 1.2: Contract with Three Menus

The optimization problem in this case is as follows:

$$\max_{\{((q_k,p_k),(q'_k,p'_k))\in\mathbb{R}^2_+\times\mathbb{R}^2_+\}} (1-\lambda)(1-\alpha_h)\left[p_a - c(q_a)\right] + (1-\lambda)\alpha_h\left[p_b - c(q_b)\right] + \lambda\left[p_c - c(q_c)\right] \quad (1.9)$$

s.t. for $k \in \{a, b, c\}$

$$u_h(q'_a, p'_a) \geq 0, \qquad u_h(q'_b, p'_b) \geq 0, \qquad u_l(q'_c, p'_c) \geq 0, \quad (1.9.1)$$

$$u_h(q_a, p_a) \geq u_h(q'_a, p'_a), \qquad u_h(q_b, p_b) \geq u_h(q'_b, p'_b), \qquad u_l(q_c, p_c) \geq u_l(q'_c, p'_c), \quad (1.9.2)$$

$$V_h\big(\{(q_a,p_a),(q'_a,p'_a)\},\theta_1\big) \geq \max\Big\{V_h\big(\{(q_b,p_b),(q'_b,p'_b)\},\theta_1\big), V_h\big(\{(q_c,p_c),(q'_c,p'_c)\},\theta_1\big)\Big\}, \quad (1.9.3)$$

$$V_h\big(\{(q_b,p_b),(q'_b,p'_b)\},\theta_2\big) \geq \max\Big\{V_h\big(\{(q_a,p_a),(q'_a,p'_a)\},\theta_2\big), V_h\big(\{(q_c,p_c),(q'_c,p'_c)\},\theta_2\big)\Big\}, \quad (1.9.4)$$

$$V_l\big(\{(q_c,p_c),(q'_c,p'_c)\},\theta_j\big) \geq \max\Big\{V_l\big(\{(q_a,p_a),(q'_a,p'_a)\},\theta_j\big), V_l\big(\{(q_b,p_b),(q'_b,p'_b)\},\theta_j\big)\Big\}, \quad (1.9.5)$$

for $j \in \{1, 2\}$.

The first three incentive restrictions (1.9.1) guarantee that each secondary product provides higher utility than the outside option, which is assumed to be zero, to the specific type for whom the menu is designed. The second three restrictions (1.9.2) ensure that the primary products in each menu provide higher utility than that menu's secondary product to the specific type. Finally the last three restrictions ((1.9.3), (1.9.4) and (1.9.5)) guarantee that each consumer selects the menu that is designed for his type.

Lemma 2 characterizes the solution to this maximization problem.

Lemma 2 *The FOCs to the seller's optimization problem (1.9) are:*

$$c'(q_a) = v'_h(q_a) \tag{1.10}$$
$$p_a = v_h(q_a) - \big[v_h(q_c) - v_l(q_c)\big] \tag{1.11}$$

$$c'(q_b) = v'_h(q_b) \tag{1.12}$$
$$p_b = v_h(q_b) - (1-\theta_2)\big[v_h(q_c) - v_l(q_c)\big] \tag{1.13}$$

$$c'(q_c) = v'_l(q_c) - \frac{1-\lambda}{\lambda}\big(1-\alpha_h\,\theta_2\big)\big[v'_h(q_c) - v'_l(q_c)\big] \tag{1.14}$$
$$p_c = v_l(q_c) \tag{1.15}$$

See Appendix A.2.

The implications of the FOCs can be listed as follows:

i. Comparing equation (1.10) with (1.12) we see that $q^*_a = q^*_b$. The products that are sold to sophisticated consumers with high valuation and to naive consumers with high valuation have the same quality.

ii. The fact that $q^*_a = q^*_b$ and the comparison of equation (1.11) with (1.13) yield $p^*_a < p^*_b$. The seller sells the same quality product to consumers with high valuations but charges the naive type more. Therefore, naive high valuation consumers are more heavily exploited and generate a greater profit for the seller.

iii. The implications of the FOCs of the problem of designing optimal contract with two menus regarding the quality and the price of the product sold to low valuation consumers are still valid. That is, $q^*_c < q^*_a$ and $p^*_c < p^*_a$.

iv. The quality of the product sold to low valuation consumers (q^*_c) and therefore the prices of all primary products (p^*_a, p^*_b, p^*_c) depend on θ_2 but not θ_1. This is because the incentive of the sophisticated high valuation consumers to choose the menu designed for them is given through their preferences, not through the fact that they are context-biased. The secondary product designed for them has the same utility as the primary product (see Claim 8 in Appendix A.2). This fact implies that the seller does not have to use the secondary product for that menu. He can achieve his goal by using only the primary product for that menu. However, the utility of the secondary product used for the menu of naive with high valuation consumers is zero, that is, different than the utility of the primary product in the same menu (see Claim 5 in Appendix A.2). Therefore, the incentive of the naive with high valuation consumers to choose the menu designed for them is given through both their preferences and the fact that they are context-biased. Finally, the utility of the secondary product used for the menu of low valuation consumers is equal to the utility of the primary product in the same menu (see Claim 6 in Appendix A.2). This implies that the incentive compatibility constraint of low valuation consumers is satisfied through their preferences, not through their context-bias. Hence among all consumers, only naive with high valuation consumers are exploited depending on their context-bias.

Finally, the profit of the seller from the contract with three menus is

$$\Pi_3^* = (1-\lambda)\,(1-\alpha_h)\,[p_a^* - c(q_a^*)] + (1-\lambda)\,\alpha_h\,[p_b^* - c(q_b^*)] + \lambda[p_c^* - c(q_c^*)]$$

$$\begin{aligned}&= (1-\lambda)\,(1-\alpha_h)\,\big[v_h(q_a^*) - (v_h(q_c^*) - v_l(q_c^*)) - c(q_a^*)\big] \\ &\quad + (1-\lambda)\,\alpha_h\,\big[v_h(q_b^*) - (1-\theta_2)\,(v_h(q_c^*) - v_l(q_c^*)) - c(q_b^*)\big] + \lambda[v_l(q_c^*) - c(q_c^*)]\end{aligned}$$

$$= (1-\lambda)\,\big[v_h(q_a^*) - c(q_a^*) - (1-\alpha_h\,\theta_2)\,(v_h(q_c^*) - v_l(q_c^*))\big] + \lambda[v_l(q_c^*) - c(q_c^*)] \quad (1.16)$$

where q_a^* and q_c^* are the solutions to equations (1.10) and (1.14), respectively. Π_3^* is increasing in θ_2, since this is the level of context bias of the type that is discriminated based on the cognitive limitations. Naturally, Π_3^* is also increasing as the probability of this type (α_h) is increasing.

Comparing the profits of the contract with two menus (Π_2^*) given in (1.8) and three menus (Π_3^*) given in (1.16), we see that

$$\Pi_3^* \geq \Pi_2^* \;\Leftrightarrow\; \alpha_h \geq \frac{\theta_1}{\theta_2} \quad (1.17)$$

The contract with two menus discriminates between high and low valuation consumers. The incentive compatibility constraint for the high valuation consumers must be binding for sophisticated ones so that both sophisticated and naive consumers with high valuation choose the menu designed for them. Therefore the profit in this case depends on the level of context-bias of sophisticated consumers (θ_1) and the above inequality (1.17) implies that the contract with two menus must be used when θ_1 is high enough. However the contract with three menus, on top of discriminating between high and low valuation consumers, further discriminates high valuation consumers based on their level of context-bias. In particular, this contract exploits the naive consumers with high valuation whose level of context bias is θ_2 and the conditional probability of being naive given high valuation type is α_h. The inequality (1.17) implies that this contract needs to be used when the multiplication $\alpha_h\,\theta_2$ is high enough.

1.4 Continuous Case

In this section we characterize the optimal menu of contracts with a continuum of types according to the level of context-bias. We keep all the assumptions related to the consumers' preferences made in previous sections, but now we assume that the seller does not observe θ. However, he knows that it is distributed according to a cdf $F(\theta)$ on $[0,1]$ for consumers with high valuation and $G(\theta)$ for consumers with low valuation again with support $[0,1]$. We further assume that $F(\theta)$ is differentiable and $F'(\theta) > 0$. We have no specific assumption on G since, as it is in the discrete case, we show that also in the continuous case the optimal contract has one and only one menu designed for all low valuation consumers, i.e., low valuation consumers are no further discriminated based on their cognitive limitation.

Our first observation for the continuous case is that since we are dealing with a continuum of types, there are infinitely many possible ways to discriminate consumers. In order to characterize the optimal contract in this case, first we need to figure out the number of menus that the optimal contract has. The following Lemma narrows down the possibilities from infinity to two.

Lemma 3 *The optimal contract has at most three menus.*

See Appendix A.3.

The following example is an illustration of Lemma 3 that shows a contract with four menus is dominated by a contract with three menus.

Example. Assume that the valuations of high and low types are $v_h(q) = 2\ln(q+1)$ and $v_l(q) = \ln(q+1)$, and θ is distributed uniformly on $[0,1]$. Consider the contract with four menus given in Figure 1.3, where each product is characterized by quality-price pair.

Menu-A	Menu-B	Menu-C	Menu-D
$(e^{11}-1, 12)$	$(e^{11}-1, 14)$	$(e^{11}-1, 18)$	$(e^{10}-1, 10)$
$(e^{50}-1, 90)$	$(e^{50}-1, 95)$	$(e^{50}-1, 100)$	$(e^{10}-1, 10)$

Figure 1.3: Example of a Contract with Four Menus

Menu-A is designed for high valuation consumers whose θ belongs to interval $\in [0, 0.4)$, whereas Menu-B is also for high valuation consumers but with $\theta \in [0.4, 0.8)$. Menu-C is for the rest of high valuation consumers and finally Menu-D is designed for all low valuation consumers. Figure 1.4 gives all the utilities provided by the products of the contract for high/low valuation consumers.

Our aim is to show that this contract is strictly dominated by another contract with three menus. Before introducing the other contract, we want to demonstrate that this menu satisfies all the incentive restrictions. From Figure 1.4 we see that all the secondary products in Menu-A, Menu-B and Menu-C offer a positive utility to whom

Menu-A	Menu-B	Menu-C	Menu-D
$10/-1$	$8/-3$	$4/-7$	$10/0$
$10/-40$	$5/-45$	$0/-50$	$10/0$

Figure 1.4: Utilities provided by the Contract with Four Menus

the menus are designed, i.e., to high valuation consumers. Furthermore, the primary products in these menus provide high valuation consumers with a utility that is greater than the one provided by the secondary products. Furthermore both the primary and the secondary product in Menu-D offer zero utility to low valuation consumers. Finally we need to check for incentive compatibility restrictions. Menu-A, Menu-B and Menu-C offer products with negative values for low consumers therefore they all have zero value for low valuation consumers which is the same as his valuation for Menu-D. This shows that the incentive compatibility constraint for low valuation consumers is satisfied. Now consider the values of menus for high valuation consumers whose θ belong to $[0, 0.4)$:

$$\begin{aligned}
V_h(A, \theta \in [0, 0.4)) &= (1 - \theta)10 \\
V_h(B, \theta \in [0, 0.4)) &= 8 - 5\theta \\
V_h(C, \theta \in [0, 0.4)) &= 4 \\
V_h(D, \theta \in [0, 0.4)) &= (1 - \theta)10
\end{aligned}$$

Thus we have

$$V_h(A, \theta \in [0, 0.4)) = \max_{k \in \{B,C,D\}} \{V_h(k, \theta \in [0, 0.4))\} = V_h(D, \theta \in [0, 0.4))$$

This shows that this type of consumers will choose Menu-A among all menus. In the same way we can show that all the other incentive compatibility constraints are satisfied with this contract. The profit of the seller from this contract is

$$\begin{aligned}
\Pi_4 &= (1 - \lambda)\big\{0.4[12 - c(e^{11}-1)] + 0.4[14 - c(e^{11}-1)] + 0.2[18 - c(e^{11}-1)]\big\} \\
&\quad + \lambda[10 - c(e^{10}-1)] \\
&= (1 - \lambda)\big\{14 - c(e^{11}-1)\big\} + \lambda[10 - c(e^{10}-1)] \qquad (1.18)
\end{aligned}$$

Now consider the contract depicted in Figure 1.5. Comparing this contract with the previous one we see that Menu-A and Menu-D still exist in this contract however Menu-B and Menu-C are replaced by Menu-B'. As in the first contract Menu-A and Menu-D are designed for the high valuation consumers whose θs belong to the interval $[0, 0.4)$

and low valuation consumers, respectively. Menu-B' is constructed for all the other consumers, i.e., for high valuation consumers whose θs belong to $[0.4, 1]$. Another way to view this contract is that it is a version of the previous contract where Menu-B and Menu-C are merged.

Menu-A	Menu-B'	Menu-D
$(e^{11}–1, 12)$	$(e^{11}–1, 16)$	$(e^{10}–1, 10)$
$(e^{50}–1, 90)$	$(e^{50}–1, 100)$	$(e^{10}–1, 10)$

Figure 1.5: Contract with Three Menus

In Figure 1.6 we have all the utilities of high/low valuation consumers provided by each product in the contract. We want to show first that all the incentive restrictions are satisfied with this contract and then that the profit of the seller is higher in this case. All the incentive restrictions for the low valuation consumers are satisfied since

Menu-A	Menu-B'	Menu-D
$10/-1$	$6/-5$	$10/0$
$10/-40$	$0/-50$	$10/0$

Figure 1.6: Utilities provided by the Contract with Three Menus

this type receives zero utility from both the primary and secondary product in the menu designed for him and his valuations for all the menus of the contract are equal and zero. All high valuation consumers receive non-negative utility from the secondary products and this is less than equal to the utility received from the primary products. Now consider the values of menus for high valuation consumers whose θ belong to $[0, 0.4)$:

$$V_h(A, \theta \in [0, 0.4)) = (1 - \theta)10$$
$$V_h(B', \theta \in [0, 0.4)) = 6$$
$$V_h(D, \theta \in [0, 0.4)) = (1 - \theta)10$$

Thus we have

$$V_h(A, \theta \in [0, 0.4)) = \max_{k \in \{B', D\}} \{V_h(k, \theta \in [0, 0.4))\} = V_h(D, \theta \in [0, 0.4))$$

This shows that this type of consumers choose Menu-A among all menus. The values of menus for the rest of the high valuation consumers are

$$V_h(A, \theta \in [0.4, 1]) = (1 - \theta)10$$
$$V_h(B', \theta \in [0.4, 1]) = 6$$
$$V_h(D, \theta \in [0.4, 1]) = (1 - \theta)10$$

Thus we have

$$V_h(B', \theta \in [0.4, 1]) = 6 \geq \max_{k \in \{A, D\}} \{V_h(k, \theta \in [0.4, 1])\} = (1 - \theta)10,$$

showing that this type of consumers choose Menu-B. Therefore, this contract satifies all the incentive restrictions. The profit of the seller from this contract is

$$\begin{aligned} \Pi_3 &= (1 - \lambda)\big\{0.4[12 - c(e^{11} - 1)] + 0.6[16 - c(e^{11} - 1)]]\big\} + \lambda[10 - c(e^{10} - 1)] \\ &= (1 - \lambda)\big\{14.4 - c(e^{11} - 1)\big\} + \lambda[10 - c(e^{10} - 1)] \end{aligned} \tag{1.19}$$

Comparing the profits from each menu given in (1.18) and (1.19) we see that $\Pi_3 > \Pi_4$. This completes the example by showing that any contract with four menus is dominated by a contract with three menus. The last observation before finalizing this example is that here we constructed the contract with three menus by merging Menu-B and Menu-C, but in some other situations we may need to merge Menu-A and Menu-B without changing Menu-C. There are two sources that determine which menus need to be merged: the lowest levels of θ for which the menus are designed (in the example they are 0, 0.4 and 0.8 for Menu-A, Menu-B and Menu-C, respectively) and the distribution of θ.

Without loss of generality we can say that any contract with more than three menus is dominated by a contract with three menus. Thus the optimal contract has either two or three menus. A contract with two menus partitions the set of consumers into two subsets based on the differences in preferences. In other words, it discriminates between high and low valuation consumers. This contract is the same as the one characterized in Section (1.3.1) except that θ_1 is replaced with the lowest possible level of θ for high valuations type, which is assumed to be zero here. A contract with three menus partitions the set of consumers into three subsets as low valuation consumers, high valuation consumers whose θ is lower than a threshold, say $\bar{\theta}$, and high valuation consumers whose θ is greater than $\bar{\theta}$. The seller needs to decide optimally on the threshold level, $\bar{\theta} \in [0, 1]$. This contract resembles the one characterized in Section (1.3.2).

Figure 1.7 shows an illustration of the contract. The primary and secondary products are denoted by (q_k, p_k) and (q'_k, p'_k) respectively, where $k = a$ for the first menu designed for high valuation consumers whose θ is less than the threshold, $\bar{\theta}$, $k = b$ for the second menu designed for high valuation consumers whose θ is greater than the threshold and $k = c$ for the third menu designed for all low valuation consumers.

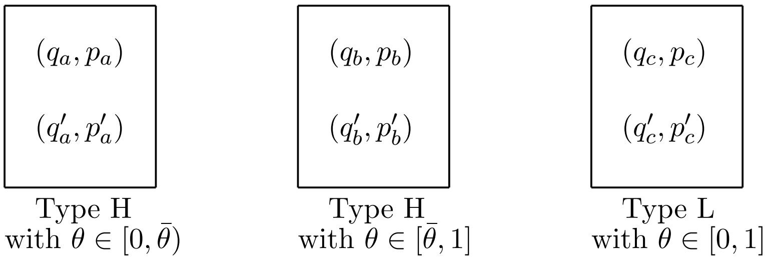

Figure 1.7: Contract with a continuum of types according to the level of context-bias

Observe that the optimal contract has two menus if $\bar{\theta}$ is 0 or 1, otherwise it has three menus.

The seller's optimization problem is

$$\Pi(\bar{\theta}) = \max_{\{((q_k,p_k),(q'_k,p'_k))\in\mathbb{R}^2_+\times\mathbb{R}^2_+\}} (1-\lambda)F(\bar{\theta})\,[p_a - c(q_a)] + (1-\lambda)(1-F(\bar{\theta}))\,[p_b - c(q_b)] + \lambda\,[p_c - c(q_c)]$$

$$\text{s.t. for } k \in \{a, b, c\} \tag{1.20}$$

$$u_h(q'_a, p'_a) \geq 0, \qquad u_h(q'_b, p'_b) \geq 0, \qquad u_l(q'_c, p'_c) \geq 0, \tag{1.20.1}$$

$$u_h(q_a, p_a) \geq u_h(q'_a, p'_a), \qquad u_h(q_b, p_b) \geq u_h(q'_b, p'_b), \qquad u_l(q_c, p_c) \geq u_l(q'_c, p'_c), \tag{1.20.2}$$

$$V_h\big(\{(q_a,p_a),(q'_a,p'_a)\},\theta_1\big) \geq \max\Big(V_h\big(\{(q_b,p_b),(q'_b,p'_b)\},\theta_1\big), V_h\big(\{(q_c,p_c),(q'_c,p'_c)\},\theta_1\big)\Big), \tag{1.20.3}$$

$$V_h\big(\{(q_b,p_b),(q'_b,p'_b)\},\theta_2\big) \geq \max\Big(V_h\big(\{(q_a,p_a),(q'_a,p'_a)\},\theta_2\big), V_h\big(\{(q_c,p_c),(q'_c,p'_c)\},\theta_2\big)\Big), \tag{1.20.4}$$

$$V_l\big(\{(q_c,p_c),(q'_c,p'_c)\},\theta\big) \geq \max\Big(V_l\big(\{(q_a,p_a),(q'_a,p'_a)\},\theta\big), V_l\big(\{(q_b,p_b),(q'_b,p'_b)\},\theta\big)\Big), \tag{1.20.5}$$

$$\text{for } \theta_1 \in [0, \bar{\theta}), \quad \theta_2 \in [\bar{\theta}, 1], \quad \theta \in [0, 1].$$

This optimization problem is very similar to problem (1.9). Here the first menu of the contract is designed for high valuation consumers whose θ lies between $[0, \bar{\theta})$ instead of a specific θ. Likewise, the second menu is designed for high valuation consumers whose θ lies between $[\bar{\theta}, 1]$ instead of a specific one. Furthermore the incentive restrictions (1.20.1) and (1.20.2) are the same as (1.9.1) and (1.9.2) ensuring that each secondary product provides higher utility than the outside option and that the primary products in each menu provide higher utility than that menu's secondary product to the specific type, respectively. The incentive compatibility constraints (((1.20.3), (1.20.4) and (1.20.5))), which guarantee that each consumer selects the menu designed for his type, are slightly different than ((1.9.3), (1.9.4) and (1.9.5)) in that they are constructed for a continuum of types rather than discrete.

Lemma 4 characterizes the solution to this maximization problem.

Lemma 4 *The FOCs to the seller's optimization problem (1.20) are:*

$$c'(q_a) = v_h'(q_a) \tag{1.21}$$
$$p_a = v_h(q_a) - \big[v_h(q_c) - v_l(q_c)\big] \tag{1.22}$$

$$c'(q_b) = v_h'(q_b) \tag{1.23}$$
$$p_b = v_h(q_b) - (1-\bar{\theta})\,\big[v_h(q_c) - v_l(q_c)\big] \tag{1.24}$$

$$c'(q_c) = v_l'(q_c) - \frac{1-\lambda}{\lambda}\,\big(1-\bar{\theta}+\bar{\theta}\,F(\bar{\theta})\big)\,\big[v_h'(q_c) - v_l'(q_c)\big] \tag{1.25}$$
$$p_c = v_l(q_c) \tag{1.26}$$

See Appendix A.4.

The implications of the FOCs, listed below, can be derived using the same reasoning of the contract with three menus in the discrete case given in Section 1.3.2:

i. $q_a^* = q_b^*$ and $p_a^* < p_b^*$. The seller uses the same quality product in the first and the second menu but the price is higher in the second menu that is designed for relatively more naive types.

ii. $q_c^* < q_a^*$ and $p_c^* < p_a^*$. Both the quality and the price of the primary product in the third menu are less than those in the first menu.

iii. Among all consumers, only high valuation consumers with $\theta \in [\bar{\theta}, 1]$ are exploited by context-bias.

The profit of the seller from this contract depends on the threshold ($\bar{\theta}$) in use and is given by

$$\begin{aligned}
\Pi^*(\bar{\theta}) &= (1-\lambda)\,F(\bar{\theta})\,[p_a^* - c(q_a^*)] + (1-\lambda)\,(1-F(\bar{\theta}))\,[p_b^* - c(q_b^*)] + \lambda[p_c^* - c(q_c^*)] \\
&= (1-\lambda)\,F(\bar{\theta})\,\big[v_h(q_a^*) - (v_h(q_c^*) - v_l(q_c^*)) - c(q_a^*)\big] \\
&\quad + (1-\lambda)\,(1-F(\bar{\theta}))\,\big[v_h(q_b^*) - (1-\bar{\theta})\,(v_h(q_c^*) - v_l(q_c^*)) - c(q_b^*)\big] + \lambda[v_l(q_c^*) - c(q_c^*)] \\
&= (1-\lambda)\,\big\{(v_h(q_a^*) - c(q_a^*)) - \big(1-\bar{\theta}+\bar{\theta}\,F(\bar{\theta})\big)[v_h(q_c^*) - v_l(q_c^*)]\big\} + \lambda\,[v_l(q_c^*) - c(q_c^*)]
\end{aligned} \tag{1.27}$$

Now, we need to find the optimal $\bar{\theta}$ that maximizes the profit of the seller. Unfortunately, it is not easy to show that the profit function (1.27) is concave. However, we can make the following observations:

i. $\Pi^*(\bar{\theta})$ is a continuous function on its compact domain $[0, 1]$. According to Weierstrass Theorem, Π^* has a maximum on $[0, 1]$.

ii. $\Pi^*(0) = \Pi^*(1)$. This is because when $\bar{\theta}$ is 0 or 1, all high valuation consumers are considered as whole, i.e., they are not exploited by their context-bias. Therefore, the optimal contract has only two menus discriminating between high and low valuations consumers and ignoring their context-bias. Hence, the optimal contracts when $\bar{\theta} = 0$ and $\bar{\theta} = 1$ are the same.

iii. $\Pi^{*\prime}(0) = (1-\lambda)(v_h(q_l^*) - v_l(q_l^*)) > 0$, therefore $\lim_{\bar{\theta}\to 0^+} \Pi^{*\prime}(\bar{\theta}) > 0$ and $\Pi^{*\prime}(1) = -(1-\lambda)(v_h(q_l^*) - v_l(q_l^*))\, F'(1) < 0$, therefore $\lim_{\bar{\theta}\to 1^-} \Pi^{*\prime}(\bar{\theta}) < 0$. The profit function (1.27) is increasing to the right of 0 and decreasing to the left of 1. Therefore, Π^* does not have its maximum on the boundary. This implies that contract with two menus is never optimal. The reason for this is as follows: the contract with two menus partitions the set of consumers into two as high valuation and low valuation consumers. The incentive compatibility constraint of low valuation consumers is satisfied through the utility arrangements. The fact that they are context-biased is used only for the incentive compatibility constraint of high valuation consumers. Furthermore, since the lower θ is the more sophisticated the consumer becomes, the menu for high valuation consumers is constructed by considering the lowest level of θ, which is assumed to be 0. In other words, this menu is designed based on the fully sophisticated types. Therefore, the resulting contract with two menus ignores the fact that consumers are context-biased. However, the seller can use this fact and design a new menu using the same product that is used for the high valuation type's menu but with a higher price. Therefore, whenever the fully sophisticated with high valuation consumer is among the others, the contract with two menus is not optimal. This fact can be also seen from inequality (1.17) if we set $\theta_1 = 0$. We need to do this since θ_1 stands for the lowest θ in the first menu. After this substitution, we see that the inequality (1.17) is always satisfied and thus the contract with two menus is always dominated by a contract with three menus.

iv. A stationary point, therefore a maximum, of Π^* is given by

$$\bar{\theta} = \frac{1 - F(\bar{\theta})}{F'(\bar{\theta})} \tag{1.28}$$

.

Remark 1 *The observation given in (iii) implies that Lemma 3 can be improved as follows: Whenever the fully sophisticated (fully rational) consumer with high valuation is present, the optimal contract has three menus.*

The following example illustrates a special case where the profit function given in (1.27) is concave that ensures that we have a maximum.

Example. Assume that $F(\theta) = \theta$ on $[0, 1]$ (uniform distribution) and $v_h''(q) - v_l''(q) \leq$

$M < 0$, where $M = \frac{\lambda(v_l''(q) - c''(q))}{(1-\lambda)(1-\theta+\theta^2)}$. In this case Π^* is concave and the profit is maximized when $\bar{\theta} = 1/2$ that is found by substituting $F(\theta) = \theta$ into equation (1.28). The optimal contract partitions the set of consumers into three: high valuation consumers with $\theta < 1/2$, high valuation consumers with $\theta \geq 1/2$ and low valuation consumers.

1.5 Conclusion

This paper studies the standard second degree price discrimination model by taking into account the fact that agents are diversely bounded rational. We consider bounded rationality that is due to individual's context-bias, which is a cognitive limitation that misleads the individual in making his choice depending on the context. Using this fact, sellers can exploit context-biased consumers by making some arrangements in the context of the contract. We study the optimal contract for a profit maximizing monopoly under the assumption that there are two sources that identify the type of a consumer: the valuation for quality, which can be either high or low, and the level of context-bias.

We first study the discrete case, where there are only two levels of context-bias among the consumers: naive and sophisticated consumers. We show that in some situations the optimal contract has two menus separating high and low valuation consumers. This contract sells a low quality but cheap product to low valuation consumers and a high quality but expensive product to high valuation consumers. In other situations consumers are partitioned as sophisticated with high valuation, naive with high valuation and low valuation by the optimal contract with three menus. This contract again provides low valuation consumers with a low quality and cheap product, and high valuation consumers with a high quality and expensive product but charges naive with high valuation consumers more than sophisticated type for the same quality product.

We continue our analysis with the continuous case where we assume that the variable measuring the level of context-bias, θ, is distributed according to a cdf $F(\theta)$ on $[0, 1]$. We characterize the optimal contract and show that it has at most three menus. If the fully sophisticated with high valuation consumer is present, the optimal contract has exactly three menus. For this contract the seller needs to decide optimally on a threshold such that it partitions consumers into three as follows: high valuation consumers whose θs are less than the threshold, high valuation consumers whose θs are greater than the threshold and low valuation consumers. This contract resembles the three-menus contract of discrete case. It provides low valuation consumers with a low quality and cheap product, and high valuation consumers with a high quality and expensive product but charges high valuation consumers with higher θs more than the ones with lower θs for the same quality product.

Our conclusions suggest a seller manufacturing jeans, for example, open three different stores. One store is for low valuation consumers where he sells low quality jeans with low prices. For example an outlet store where he sells defective jeans. Another store is for high valuation consumers with low θs where he sells high quality jeans with reasonable prices. Finally the third store is for high valuation consumers with high θs where he sells high quality jeans with high prices, for example a very fancy store in the most popular shopping center of the city next to other high-society stores.

Appendix A

Proofs

A.1 Proof of Lemma 1

In this subsection we examine the optimization problem given in (1.3). We refer the two menus depicted in Figure 1.1 as Menu-H and Menu-L, respectively. After giving some claims that simplify the problem, we finalize this subsection with the proof of Lemma 1.

Claim 1 *The optimal solution of the problem in (1.3) requires that the utility of the high valuation consumers from the secondary product offered in Menu-H is zero, i.e.,* $u_h(q'_h, p'_h) = 0$.

Assume by contradiction that $u_h(q'_h, p'_h) > 0$. Take an ϵ that satisfies

$$(i) \quad 0 < \epsilon \leq u_h(q'_h, p'_h), \tag{A.1}$$

$$(ii) \quad u_l(q'_h, p'_h) < \epsilon. \tag{A.2}$$

Observe that the choice of ϵ is feasible since $u_h(q,p) > u_l(q,p)\ \forall (q,p) \in \mathbb{R}^2_+$. Consider the following adjustment:

$$p^a_h = p_h + \delta\epsilon, \tag{A.3}$$

$$p'^{\,a}_h = p'_h + \epsilon, \tag{A.4}$$

where the superscript a stands for the adjustment of the variable and $\delta \in (0, \theta_1]$. Now, we want to show that under this adjustment the constraints of the problem are still satisfied. This will complete the proof of the claim since the objective function increases strictly after the adjustment.

Observe first that the adjustment does not affect the incentive restrictions (1.3.1) and (1.3.2) for low valuation type ($(1.3.1)_l$ and $(1.3.2)_l$ henceforth). The incentive restriction $(1.3.1)_h$ still holds by the choice of ϵ: $u_h(q'_h, p'^{\,a}_h) = u_h(q'_h, p'_h) - \epsilon \geq 0$ (by (A.4) and (A.1)). The adjustment affects $(1.3.2)_h$ as follows:

$$u_h(q_h, p^a_h) = u_h(q_h, p_h) - \delta\epsilon \geq u_h(q'_h, p'_h) - \delta\epsilon > u_h(q'_h, p'_h) - \epsilon = u_h(q'_h, p'^{\,a}_h).$$

Therefore $(1.3.2)_h$ is still satisfied. Concerning the incentive compatibility constraints we have

$$\begin{aligned} V_h^a\big(\{(q_h,p_h^a),(q_h',p_h'^a)\},\theta_j\big) &= u_h(q_h,p_h^a) - \theta_j\, u_h(q_h',p_h'^a) \\ &= \big(u_h(q_h,p_h) - \delta\epsilon\big) - \theta_j\,\big(u_h(q_h',p_h') - \epsilon\big) \\ &= V_h\big(\{(q_h,p_h),(q_h',p_h')\},\theta_j\big) + (\theta_j-\delta)\epsilon \\ &\geq V_h\big(\{(q_l,p_l),(q_l',p_l')\},\theta_j\big) + (\theta_j-\delta)\epsilon \\ &> V_h^a\big(\{(q_l,p_l),(q_l',p_l')\},\theta_j\big) \qquad \text{for } j\in\{1,2\}. \end{aligned}$$

For low valuation types,

$$V_l\big(\{(q_h,p_h),(q_h',p_h')\},\theta_j\big) = \begin{cases} u_l(q_h,p_h) - \theta_j\, u_l(q_h',p_h') & \text{if } u_l(q_h,p_h)\geq u_l(q_h',p_h')\geq 0, \\ \big(1-\theta_j\big)u_l(q_h,p_h) & \text{if } u_l(q_h,p_h)\geq 0\geq u_l(q_h',p_h'), \\ u_l(q_h',p_h') - \theta_j\, u_l(q_h,p_h) & \text{if } u_l(q_h',p_h')\geq u_l(q_h,p_h)\geq 0, \\ \big(1-\theta_j\big)u_l(q_h',p_h') & \text{if } u_l(q_h',p_h')\geq 0\geq u_l(q_h,p_h), \\ 0 & \text{otherwise.} \end{cases}$$

and the fact that $u_l(q_h',{p_h'}^a) = u_l(q_h',p_h') - \epsilon < 0$ implies that

$$V_l^a\big(\{(q_h,p_h^a),(q_h',p_h'^a)\},\theta_j\big) = \begin{cases} \big(1-\theta_j\big)u_l(q_h,p_h^a) & \text{if } u_l(q_h,p_h^a)\geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} &V_l\big(\{(q_h,p_h),(q_h',p_h')\},\theta_j\big) - V_l^a\big(\{(q_h,p_h^a),(q_h',p_h'^a)\},\theta_j\big) \\ &= \begin{cases} \theta_j\big(u_l(q_h,p_h) - u_l(q_h',p_h')\big) + \delta\epsilon(1-\theta_j) & \text{if } u_l(q_h,p_h)\geq u_l(q_h',p_h')\geq 0 \text{ and } u_l(q_h,p_h^a)\geq 0, \\ \delta\epsilon(1-\theta_j) & \text{if } u_l(q_h,p_h)\geq 0\geq u_l(q_h',p_h') \text{ and } u_l(q_h,p_h^a)\geq 0, \\ u_l(q_h',p_h') - u_l(q_h,p_h) + \delta\epsilon(1-\theta_j) & \text{if } u_l(q_h',p_h')\geq u_l(q_h,p_h)\geq 0 \text{ and } u_l(q_h,p_h^a)\geq 0, \\ (1-\theta_j)\big(u_l(q_h',p_h') - u_l(q_h,p_h)\big) + \delta\epsilon(1-\theta_j) & \text{if } u_l(q_h',p_h')\geq 0\geq u_l(q_h,p_h) \text{ and } u_l(q_h,p_h^a)\geq 0, \\ u_l(q_h,p_h) - \theta_j\, u_l(q_h',p_h') & \text{if } u_l(q_h,p_h)\geq u_l(q_h',p_h')\geq 0 \text{ and } u_l(q_h,p_h^a)< 0, \\ \big(1-\theta_j\big)u_l(q_h,p_h) & \text{if } u_l(q_h,p_h)\geq 0\geq u_l(q_h',p_h') \text{ and } u_l(q_h,p_h^a)< 0, \\ u_l(q_h',p_h') - \theta_j\, u_l(q_h,p_h) & \text{if } u_l(q_h',p_h')\geq u_l(q_h,p_h)\geq 0 \text{ and } u_l(q_h,p_h^a)< 0, \\ \big(1-\theta_j\big)u_l(q_h',p_h') & \text{if } u_l(q_h',p_h')\geq 0\geq u_l(q_h,p_h) \text{ and } u_l(q_h,p_h^a)< 0, \\ 0 & \text{otherwise.} \end{cases} \\ &\geq 0 \end{aligned} \tag{A.5}$$

Thus,

$$\begin{aligned} V_l\big(\{(q_l,p_l),(q_l',p_l')\},\theta_j\big) &\geq V_l\big(\{(q_h,p_h),(q_h',p_h')\},\theta_j\big) && \text{(by } (1.3.3)_l) \\ &> V_l^a\big(\{(q_h,p_h^a),(q_h',p_h'^a)\},\theta_j\big) && \text{(by } (A.5)) \end{aligned}$$

We have shown that under the proposed adjustment all the incentive restrictions are still satisfied. Due to the fact that the price of the product that is sold to high valuation consumers (p_h) increases, the profit of the seller increases strictly and this completes the proof.

Remark 2 *Claim (1) implies that* $u_l(q'_h, p'_h) < 0$ *and therefore we have*

$$V_l\big(\{(q_h, p_h), (q'_h, p'_h)\}, \theta_j\big) = \max\{(1-\theta_j)u_l(q_h, p_h), 0\}.$$

For the high valuation types:

$$\begin{aligned} V_h\big(\{(q_h, p_h), (q'_h, p'_h)\}, \theta_j\big) &= u_h(q_h, p_h) - \theta_j u_h(q'_h, p'_h) \\ &= u_h(q_h, p_h). \end{aligned}$$

Claim 2 *The optimal solution of the problem in (1.3) requires that the incentive compatibility constraint for high valuation consumers,* $(1.3.3)_h$*, binds only for the sophisticated high valuation type, i.e.,* $V_h\big(\{(q_h, p_h), (q'_h, p'_h)\}, \theta_1\big) = V_h\big(\{(q_l, p_l), (q'_l, p'_l)\}, \theta_1\big)$*. Furthermore, high types receive same utility from the primary and the secondary products of Menu-L, i.e.,* $u_h(q_l, p_l) = u_h(q'_l, p'_l)$*.*

Observe that according to Claim (1), the secondary product provides zero utility for both the naive and sophisticated high valuation types. As noted in Remark 2, this implies that their valuation for this menu is the same since having a secondary product with zero utility eliminates the effect of θ. Furthermore, Menu-L is more valuable to the sophisticated high valuation type than to the naive high valuation type. This is because the naive type multiplies the minimum utility with θ_2 and subtracts from the maximum utility, whereas the sophisticated type uses a smaller multiplier, θ_1. Therefore we have $V_h\big(\{(q_l, p_l), (q'_l, p'_l)\}, \theta_1\big) > V_h\big(\{(q_l, p_l), (q'_l, p'_l)\}, \theta_2\big)$.

Now assume by contradiction $V_h\big(\{(q_h, p_h), (q'_h, p'_h)\}, \theta_1\big) > V_h\big(\{(q_l, p_l), (q'_l, p'_l)\}, \theta_1\big)$ and choose an ϵ such that

$$V_h\big(\{(q_h, p_h), (q'_h, p'_h)\}, \theta_1\big) - V_h\big(\{(q_l, p_l), (q'_l, p'_l)\}, \theta_1\big) > \epsilon > 0$$

Consider an increase in the price of the primary product for high valuation types by ϵ, i.e., $p^a_h = p_h + \epsilon$. This adjustment decreases the valuation of the low types for this menu but that does not contradict with the incentive compatibility constraint for this type. The valuation of high types decreases by ϵ for this menu and thanks to the choice of ϵ, this adjustment does not violate $(1.3.3)_h$. The facts that all constraints of the problem are still satisfied and the value of the objective function is strictly increased complete the first part of the proof.

Assume that $u_h(q_l, p_l) > u_h(q'_l, p'_l)$ and therefore $V_h\big(\{(q_l, p_l), (q'_l, p'_l)\}, \theta_1\big) = u_h(q_l, p_l) - \theta_1 u(q'_l, p'_l)$. Consider the following adjustment: (i.) a slight increase in q'_l coupled with a corresponding increase in p'_l so that the utility of the product for the low valuation types is not changed, (ii.) a corresponding increase in p_h so that the incentive compatibility constraint $(1.3.3)_h$ still binds. Mathematically,

$$\begin{aligned} q'^a_l &= q'_l + \epsilon_1 \\ p'^a_l &= p'_l + \epsilon_2 \\ p^a_h &= p_h + \epsilon_3 \end{aligned}$$

such that

$$u_l(q_l'^a, p_l'^a) = u_l(q_l', p_l') \tag{A.6}$$

$$u_h(q_h, p_h^a) = u_h(q_l, p_l) - \theta_1 u(q_l'^a, p_l'^a), \tag{A.7}$$

where $\epsilon_1 > 0$. The adjustment does not violate any of the restrictions of the low types since their utilities for the secondary product are not changed and the increase in p_h does not do anything rather than relaxing the incentive compatibility constraint. Observe that (A.6) implies that $u_h(q_l'^a, p_l'^a) > u_h(q_l', p_l')$ and therefore, for high valuation consumers the value of Menu-L decreases. However this decrease is compensated by the decrease in the value of Menu-H given by (A.7). Thus the adjustment leads the value of the objective function to increase without violating any restrictions of the problem.

Now, assume that $u_h(q_l, p_l) < u_h(q_l', p_l')$ and therefore $V_h\big(\{(q_l, p_l), (q_l', p_l')\}, \theta_1\big) = u_h(q_l', p_l') - \theta_1 u(q_l, p_l)$. We follow the same procedure as above with an exception: instead of a slight increase, we make a slight decrease in q_l', compensated with a corresponding slight decrease in p_l' so that $u_l(q_l'^a, p_l'^a) = u_l(q_l', p_l')$. Once again, we increase p_h so that the incentive compatibility constraint $(1.3.3)_h$ still binds. This adjustment strictly raises the objective function without violating any restrictions of the problem. Therefore we have, $u_h(q_l, p_l) = u_h(q_l', p_l')$.

Remark 3 *Claim (2) implies that*

$$\begin{aligned} V_h\big(\{(q_h, p_h), (q_h', p_h')\}, \theta_1\big) &= V_h\big(\{(q_l, p_l), (q_l', p_l')\}, \theta_1\big) \\ &= u_h(q_l, p_l) - \theta_1 u_h(q_{l'}, p_{l'}) \\ &= (1 - \theta_1) u_h(q_l, p_l) \end{aligned}$$

Furthermore, we have seen in Remark (2) that $V_h\big(\{(q_h, p_h), (q_h', p_h')\}, \theta_1\big) = u_h(q_h, p_h)$. *Combining the two, we get* $u_h(q_h, p_h) = (1 - \theta_1) u_h(q_l, p_l)$.

Claim 3 *The optimal solution of the problem in (1.3) requires* $q_h \geq q_l$.

Assume $q_h < q_l$ and consider the following adjustment:

$$\begin{aligned} q_h^a &= q_h + A \\ q_l^a &= q_l - B \\ p_h^a &= p_h + C \\ p_l^a &= p_l - D \end{aligned}$$

where $q_h + A = q_l - B = (1 - \lambda) q_h + \lambda q_l$. Furthermore the above adjustment is such that the utility of neither the high valuation nor the low valuation type is changed for the product that is designed for them. Mathematically,

$$\begin{aligned} u_h(q_h^a, p_h^a) &= u_h(q_h, p_h) \\ u_l(q_l^a, p_l^a) &= u_l(q_l, p_l). \end{aligned}$$

Therefore we have

$$\begin{aligned} v_h(q_h + A) - v_h(q_h) &= C \\ v_l(q_l) - v_l(q_l - B) &= D. \end{aligned}$$

Now consider the change in the objective function of the seller after the adjustment:

$$\begin{aligned} \Delta\Pi &= (1-\lambda)\big[C - c(q_h + A) + c(q_h)\big] + \lambda\big[-D - c(q_l - B) + c(q_l)\big] \\ &= \big[(1-\lambda)C - \lambda D\big] + \big[(1-\lambda)c(q_h) + \lambda c(q_l) - c((1-\lambda)q_h + \lambda q_l)\big] \end{aligned}$$

The second component of the summation above is positive, since the cost function $c(.)$ is convex. The first component is also positive since

$$\begin{aligned} (1-\lambda)C - \lambda D &= (1-\lambda)\big[v_h(q_h + A) - v_h(q_h)\big] - \lambda\big[v_l(q_l) - v_l(q_l - B)\big] \\ &\geq (1-\lambda)\big[(1-\lambda)v_h(q_h) + \lambda v_h(q_l) - v_h(q_h)\big] \\ &\quad + \lambda\big[(1-\lambda)v_l(q_h) + \lambda v_l(q_l) - v_l(q_l)\big] \\ &= \lambda(1-\lambda)\big[(v_h(q_l) - v_h(q_h)) - (v_l(q_l) - v_l(q_h))\big] > 0. \end{aligned}$$

In the second line above, we use the fact that both valuation functions $(v_h(.), v_l(.))$ are concave, whereas we use the assumption $q_l > q_h$ and single crossing property in the last line. Therefore we have $\Delta\Pi > 0$.

For high valuation type, the utility of the primary product in Menu-H does not change and his utility from the primary product in Menu-L decreases since

$$\begin{aligned} u_h(q_l^a, p_l^a) - u_h(q_l, p_l) &= v_h(q_l - B) - v_h(q_l) + D \\ &= \big(v_h(q_l - B) - v_l(q_l - B)\big) - \big(v_h(q_l) - v_l(q_l)\big) < 0. \end{aligned}$$

Similarly, the adjustment does not violate any of the incentive restrictions of low type since it does not affect his utility of the primary product in Menu-L, whereas it decreases his utility of the primary product in Menu-H since

$$\begin{aligned} u_l(q_h^a, p_h^a) - u_l(q_h, p_h) &= v_l(q_h + A) - v_l(q_h) - C \\ &= \big(v_h(q_h) - v_l(q_h)\big) - \big(v_h(q_h + A) - v_l(q_h + A)\big) < 0. \end{aligned}$$

Hence, the adjustment strictly increases the objective function while not violating any of the constraints. This shows that $q_h \geq q_l$ and completes the proof.

Claim 4 *The optimal solution of the problem in (1.3) requires* $u_l(q_h, p_h) \leq 0$*.*

Assume by contradiction that $u_l(q_h, p_h) > 0$, which implies that $V_l\big(\{(q_h, p_h), (q_h', p_h')\}, \theta_j\big) = u_l(q_h, p_h)\big(1 - \theta_j\big) > 0$ (for $j \in \{1, 2\}$) and therefore by incentive compatibility we have $u_l(q_l, p_l) > 0$. Choose $\epsilon > 0$, $\epsilon_1 > 0$, $\epsilon_2 > 0$, and $\beta > 0$ satisfying the following conditions:

$$(i) \quad v_l(q_l - \epsilon) - p_l = 0 \tag{A.8}$$

$$(ii) \quad \beta = (1 - \theta_1)\big[v_h(q_l) - v_h(q_l - \epsilon)\big] \tag{A.9}$$

$$(iii) \quad v_h(q_l) - v_h(q_l - \epsilon) = v_h(q_l') - v_h(q_l' - \epsilon_1) - \epsilon_2 \tag{A.10}$$

$$(iv) \quad v_l(q_l' - \epsilon_1) - p_l' + \epsilon_2 = 0 \tag{A.11}$$

Consider now the following adjustment:

$$\begin{aligned} p_h^a &= p_h + \beta \\ q_l^a &= q_l - \epsilon \\ q_l'^a &= q_l' - \epsilon_1 \\ p_l'^a &= p_l' - \epsilon_2 \end{aligned}$$

This adjustment increases the objective function strictly since it increases the price of the product sold to high valuation consumers (p_h) and decreases the quality of the product sold to low valuation consumers (q_l), which decreases the cost. We want to show that the proposed adjustment does not violate any of the constraints of the problem (1.3).

We start by examining the constraints of low valuation consumers. The conditions given in (A.8) and (A.11) yield $u_l(q_l^a, p_l^a) = u_l(q_l'^a, p_l'^a) = 0$, therefore both $(1.3.1)_l$ and $(1.3.2)_l$ are still satisfied. Moreover, the valuation of low types for the Menu-L becomes zero. Therefore their valuation for Menu-H needs to be zero as well in order to satisfy $(1.3.3)_l$. This can be shown as follows:

$$\begin{aligned} u_h(q_h^a, p_h^a) &= u_h(q_h, p_h) - \beta \\ &= (1 - \theta_1)u_h(q_l, p_l) - (1 - \theta_1)\big[v_h(q_l) - v_h(q_l - \epsilon)\big] \\ &= (1 - \theta_1)\big[v_h(q_l - \epsilon) - p_l\big] \\ &= (1 - \theta_1)u_h(q_l^a, p_l^a) \\ &\leq u_h(q_l^a, p_l^a), \end{aligned}$$

and therefore

$$\begin{aligned} u_h(q_l^a, p_l^a) &\geq u_h(q_h^a, p_h^a) \\ v_h(q_l^a) - p_l^a &\geq v_h(q_h^a) - p_h^a, \end{aligned}$$

using the fact that $q_h \geq q_l$ that is proved in Claim (3) we get

$$p_h^a - p_l^a \geq v_h(q_h^a) - v_h(q_l^a) \geq v_l(q_h^a) - v_l(q_l^a),$$

and this implies

$$0 = v_l(q_l^a) - p_l^a \geq v_l(q_h^a) - p_h^a = u_l(q_h^a, p_h^a)$$

Thus $V_l\big(\{(q_h^a, p_h^a), (q_h'^a, p_h'^a)\}, \theta_j\big) = \max\{(1-\theta_j)u_l(q_h^a, p_h^a), 0\} = 0$. Having showed that the low types' constraints are still satisfied, we move to the ones of the high types. The high types' secondary product $((q_h', p_h'))$ is not affected by the adjustment, whereas the value of the primary product decreases. The left hand side of $(1.3.2)_h$ becomes

$$\begin{aligned}
u_h(q_h^a, p_h^a) &= u_h(q_h, p_h) - \beta \\
&= (1-\theta_1)u_h(q_l, p_l) - \beta \\
&= (1-\theta_1)\big[v_h(q_l) - p_l\big] - \beta \\
&= (1-\theta_1)\big[v_h(q_l) - p_l\big] - (1-\theta_1)\big[v_h(q_l) - v_h(q_l - \epsilon)\big] \\
&= (1-\theta_1)\big[v_h(q_l - \epsilon) - p_l\big] \\
&> (1-\theta_1)\big[v_l(q_l - \epsilon) - p_l\big] \\
&> 0 = u_h(q_h'^a, p_h'^a).
\end{aligned}$$

In the second line above we use the result given in Remark (3) whereas in the last line we embed the condition (A.8) and the statement of Claim (1), and show that $(1.3.2)_h$ is still satisfied. Finally we check for the incentive compatibility constraint $((1.3.3)_h)$. For the sophisticated type it is still binding after the adjustment:

$$\begin{aligned}
V_h\big(\{(q_l^a, p_l^a), (q_l'^a, p_l'^a)\}, \theta_1\big) &= u_h(q_l^a, p_l^a) - \theta_1 u_h(q_l'^a, p_l'^a) \\
&= v_h(q_l - \epsilon) - p_l - \theta_1\big[v_h(q_l' - \epsilon_1) - p_l' + \epsilon_2\big] \\
&= u_h(q_l, p_l) - \big(v_h(q_l) - v_h(q_l - \epsilon)\big) \\
&\qquad - \theta_1\big[u_h(q_l', p_l') - \big(v_h(q_l') - v_h(q_l' - \epsilon_1) - \epsilon_2\big)\big] \\
&= u_h(q_l, p_l) - \theta_1 u_h(q_l', p_l') - (1-\theta_1)\big(v_h(q_l) - v_h(q_l - \epsilon)\big) \\
&= V_h\big(\{(q_l, p_l), (q_l', p_l')\}, \theta_1\big) - \beta \\
&= V_h\big(\{(q_h, p_h), (q_h', p_h')\}, \theta_1\big) - \beta \\
&= V_h\big(\{(q_h^a, p_h^a), ({q_h'}^a, {p_h'}^a)\}, \theta_1\big).
\end{aligned}$$

In the above set of equations we used the condition (A.10) in the fourth line and the statement of Claim (2) in the last line. For the naive type we have

$$\begin{aligned}
V_h\big(\{(q_h^a, p_h^a), ({q_h'}^a, {p_h'}^a)\}, \theta_2\big) &= V_h\big(\{(q_h^a, p_h^a), ({q_h'}^a, {p_h'}^a)\}, \theta_1\big) \\
&= V_h\big(\{(q_l^a, p_l^a), (q_l'^a, p_l'^a)\}, \theta_1\big) \\
&= (1-\theta_1)u_h(q_l^a, p_l^a) \\
&\geq (1-\theta_2)u_h(q_l^a, p_l^a) = V_h\big(\{(q_l^a, p_l^a), (q_l'^a, p_l'^a)\}, \theta_2\big).
\end{aligned}$$

This shows that $(1.3.3)_h$ is also still satisfied for the naive type. The first line is the result of $u_h({q_h'}^a, {p_h'}^a) = u_h(q_h', p_h') = 0$ and for the second line we use the above proved fact, that is, the incentive compatibility constraint is binding for sophisticated type. Hence the adjustment does not violate any of the restrictive incentives of the problem.

Remark 4 *Claim (4) implies* $V_l\big(\{(q_h, p_h), (q'_h, p'_h)\}, \theta_j\big) = 0$, *therefore the incentive compatibility constraint,* $(1.3.3)_l$, *becomes*

$$V_l\big(\{(q_l, p_l), (q'_l, p'_l)\}, \theta_j\big) = 0.$$

It is binding since otherwise it is possible to increase the objective function by increasing q_l. *Furthermore, this implies*

$$u_l(q_l, p_l) = u_l(q'_l, p'_l) = 0.$$

Proof of Lemma 1 *Using Claim (1),(2), (4) and the observations given in Remark (2), (3) and(4), the problem (1.3) can be simplified as:*

$$\begin{aligned} \max_{\{((q_h,p_h),(q_l,p_l))\in\mathbb{R}^2_+\times\mathbb{R}^2_+\}} \quad & (1-\lambda)\left[p_h - c(q_h)\right] + \lambda\left[p_l - c(q_l)\right] \\ s.t. \quad & u_l(q_l, p_l) = 0 \\ & u_h(q_h, p_h) = (1-\theta_1)u_h(q_l, p_l) \end{aligned}$$

From the two constraints above we get

$$\begin{aligned} p_l &= v_l(q_l) \\ p_h &= v_h(q_h) - (1-\theta_1)\left[v_h(q_l) - p_l\right] \end{aligned}$$

Substituting these values into the objective function and taking the first order conditions with respect to q_l *and* q_h *we get:*

$$\begin{aligned} & c'(q_h) = v'_h(q_h) \\ & p_h = v_h(q_h) - (1-\theta_1)\left[v_h(q_l) - v_l(q_l)\right] \end{aligned}$$

$$\begin{aligned} & c'(q_l) = v'_l(q_l) - \tfrac{1-\lambda}{\lambda}(1-\theta_1)\left[v'_h(q_l) - v'_l(q_l)\right] \\ & p_l = v_l(q_l) \end{aligned}$$

□

A.2 Proof of Lemma 2

In this subsection we examine the optimization problem given in (1.9). We refer the three menus depicted in Figure 1.2 as Menu-A, Menu-B and Menu-C, respectively. After giving some claims that simplify the problem, we finalize this subsection with the proof of Lemma 2.

Claim 5 *The optimal solution of the problem in (1.9) requires that the utility of the naive consumers with high valuation for the secondary product offered in Menu-B is zero, i.e.,* $u_h(q'_b, p'_b) = 0$.

Assume by contradiction that $u_h(q_b', p_b') > 0$ and follow the procedure in the proof of Claim (1) by choosing an ϵ satisfying

$$\begin{aligned} &(i) \quad 0 < \epsilon \leq u_h(q_b', p_b'), \\ &(ii) \quad u_l(q_b', p_b') < \epsilon, \end{aligned}$$

and replacing the adjustment in (A.3) and (A.4) by

$$\begin{aligned} p_b^a &= p_b + \delta\epsilon, \\ p_b'^a &= p_b' + \epsilon, \end{aligned}$$

where $\delta \in (\theta_1, \theta_2)$. Under this adjustment, which does not violate any of the constraints, the value of the objective function increases strictly.

Remark 5 *Claim (5) implies that* $u_l(q_b', p_b') < 0$ *and therefore we have*

$$V_l\big(\{(q_b, p_b), (q_b', p_b')\}, \theta_j\big) = \max\{(1 - \theta_j)u_l(q_b, p_b), 0\}$$

and for the high valuation types:

$$V_h\big(\{(q_b, p_b), (q_b', p_b')\}, \theta_j\big) = u_h(q_b, p_b),$$

where $j \in \{1, 2\}$.

Claim 6 *The optimal solution of the problem in (1.9) requires* $V_h\big(\{(q_a, p_a), (q_a', p_a')\}, \theta_1\big) = V_h\big(\{(q_c, p_c), (q_c', p_c')\}, \theta_1\big)$. *Furthermore, the high types receive same utility from the primary and the secondary product of Menu-C, i.e.,* $u_h(q_c, p_c) = u_h(q_c', p_c')$.

Our first observation is that all the three incentive compatibility constraints ((1.9.3), (1.9.4) and (1.9.5)) are binding. In order to see this, assume by contradiction that (1.9.3) is unbinding. In this case the objective function can be increased by increasing the price of the primary product of Menu-A (p_a). This does not violate any constraint since increasing p_a decreases the attractiveness of Menu-A for other types. Following the same reasoning and Remark (5) we can write (1.9.3),(1.9.4) and (1.9.5) as

$$V_h\big(\{(q_a, p_a), (q_a', p_a')\}, \theta_1\big) = \max\Big\{u_h(q_b, p_b), V_h\big(\{(q_c, p_c), (q_c', p_c')\}, \theta_1\big)\Big\}, \tag{A.12}$$

$$u_h(q_b, p_b) = \max\Big\{V_h\big(\{(q_a, p_a), (q_a', p_a')\}, \theta_2\big), V_h\big(\{(q_c, p_c), (q_c', p_c')\}, \theta_2\big)\Big\}, \tag{A.13}$$

$$V_l\big(\{(q_c, p_c), (q_c', p_c')\}, \theta_j\big) = \max\Big\{V_l\big(\{(q_a, p_a), (q_a', p_a')\}, \theta_j\big), (1 - \theta_j)u_l(q_b, p_b)\Big\}, \tag{A.14}$$

where $j \in \{1, 2\}$.
We want to show $V_h\big(\{(q_c, p_c), (q_c', p_c')\}, \theta_1\big) \geq u_h(q_b, p_b)$ and therefore

$$V_h\big(\{(q_a, p_a), (q_a', p_a')\}, \theta_1\big) = V_h\big(\{(q_c, p_c), (q_c', p_c')\}, \theta_1\big).$$

Assume by contradiction $u_h(q_b,p_b) > V_h\big(\{(q_c,p_c),(q_c',p_c')\},\theta_1\big)$. Now we have

$$u_h(q_b,p_b) > V_h\big(\{(q_c,p_c),(q_c',p_c')\},\theta_1\big) > V_h\big(\{(q_c,p_c),(q_c',p_c')\},\theta_2\big)$$

In this case A.13 implies $u_h(q_b,p_b) = V_h\big(\{(q_a,p_a),(q_a',p_a')\},\theta_2\big)$, whereas A.12 implies $u_h(q_b,p_b) = V_h\big(\{(q_a,p_a),(q_a',p_a')\},\theta_1\big)$. Therefore,

$$\begin{aligned}V_h\big(\{(q_a,p_a),(q_a',p_a')\},\theta_1\big) &= V_h\big(\{(q_a,p_a),(q_a',p_a')\},\theta_2\big)\\ u_h(q_a,p_a)-\theta_1 u_h(q_a',p_a') &= u_h(q_a,p_a)-\theta_2 u_h(q_a',p_a')\\ u_h(q_a',p_a') &= 0\end{aligned}$$

and

$$\begin{aligned}u_h(q_b,p_b) &= V_h\big(\{(q_a,p_a),(q_a',p_a')\},\theta_1\big)\\ &= u_h(q_a,p_a)-\theta_1 u_h(q_a',p_a')\\ &= u_h(q_a,p_a).\end{aligned}$$

Since we have $u_h(q_a,p_a) = u_h(q_b,p_b)$ and $u_h(q_a',p_a') = u_h(q_b',p_b') = 0$, Menu-A and Menu-B are same in the eye of high valuation type and this contradicts with the fact that this contract has 3 menus. Hence, $V_h\big(\{(q_c,p_c),(q_c',p_c')\},\theta_1\big) \geq u_h(q_b,p_b)$ and therefore $V_h\big(\{(q_a,p_a),(q_a',p_a')\},\theta_1\big) = V_h\big(\{(q_c,p_c),(q_c',p_c')\},\theta_1\big)$. This completes the first part of the proof.

For the second part, assume that $u_h(q_c,p_c) > u_h(q_c',p_c')$ and therefore

$$V_h\big(\{(q_c,p_c),(q_c',p_c')\},\theta_1\big) = u_h(q_c,p_c)-\theta_1 u(q_c',p_c').$$

Consider the following adjustment:

$$\begin{aligned}q_c'^a &= q_c'+\epsilon_1\\ p_c'^a &= p_c'+\epsilon_2\\ p_a^a &= p_a+\epsilon_3\end{aligned}$$

such that

$$u_l(q_c'^a,p_c'^a) = u_l(q_c',p_c') \tag{A.15}$$

$$u_h(q_a,p_a^a)-\theta_1 u(q_a',p_a') = u_h(q_c,p_c)-\theta_1 u(q_c'^a,p_c'^a), \tag{A.16}$$

where $\epsilon_1 > 0$. The adjustment does not violate any of the restrictions of the low types since their utilities for the secondary product are not changed and the increase in p_a does not do anything rather than relaxing the incentive compatibility constraint. Observe that (A.15) implies that $u_h(q_c'^a,p_c'^a) > u_h(q_c',p_c')$ and therefore, for high valuation consumers the value of Menu-C decreases. However this decrease is compensated by the decrease in the value of Menu-A given by (A.16). Thus the adjustment leads the value of the objective function to increase without violating any restrictions of the problem.

Now, assume that $u_h(q_c, p_c) < u_h(q'_c, p'_c)$ and therefore $V_h\big(\{(q_c, p_c), (q'_c, p'_c)\}, \theta_1\big) = u_h(q'_c, p'_c) - \theta_1 u(q_c, p_c)$. We follow the same procedure as above with an exception: instead of a slight increase, we make a slight decrease in q'_c, compensated with a corresponding slight decrease in p'_c so that $u_l(q'^a_c, p'^a_c) = u_l(q'_c, p'_c)$. Once again, we increase p_a so that the incentive compatibility constraint still binds. This adjustment strictly raises the objective function without violating any restrictions of the problem. Therefore we have, $u_h(q_c, p_c) = u_h(q'_c, p'_c)$.

Remark 6 *Claim (6) implies*

$$V_h\big(\{(q_a, p_a), (q'_a, p'_a)\}, \theta_1\big) = (1 - \theta_1) u_h(q_c, p_c).$$

Claim 7 *The optimal solution of the problem in (1.9) requires*

$$u_h(q_b, p_b) = V_h\big(\{(q_a, p_a), (q'_a, p'_a)\}, \theta_2\big).$$

We want to show that $u_h(q_a, p_a) - \theta_2 u_h(q'_a, p'_a) \geq u_h(q_c, p_c)(1 - \theta_2)$ therefore by equation (A.13) $u_h(q_b, p_b) = V_h\big(\{(q_a, p_a), (q'_a, p'_a)\}, \theta_2\big)$. Assume by contradiction

$$u_h(q_a, p_a) - \theta_2 u_h(q'_a, p'_a) < u_h(q_c, p_c)(1 - \theta_2).$$

Therefore we have

$$u_h(q_a, p_a) - u_h(q_c, p_c) < \theta_2\big(u_h(q'_a, p'_a) - u_h(q_c, p_c)\big), \tag{A.17}$$

which implies

$$u_h(q_c, p_c) > u_h(q_a, p_a) \geq u_h(q'_a, p'_a). \tag{A.18}$$

Using the result given in Remark (6) we get

$$\begin{aligned} &u_h(q_a, p_a) - \theta_1 u_h(q'_a, p'_a) = \big(1 - \theta_1\big) u_h(q_c, p_c) \\ &\quad\Rightarrow u_h(q_a, p_a) - u_h(q_c, p_c) = \theta_1\big(u_h(q'_a, p'_a) - u_h(q_c, p_c)\big) \end{aligned} \tag{A.19}$$

Substituting equation (A.19) into (A.17) we get

$$\theta_1\big(u_h(q'_a, p'_a) - u_h(q_c, p_c)\big) < \theta_2\big(u_h(q'_a, p'_a) - u_h(q_c, p_c)\big).$$

This implies

$$u_h(q_a, p_a) \geq u_h(q'_a, p'_a) > u_h(q_c, p_c),$$

which contradicts with (A.18), showing that the assumption is wrong.

Claim 8 *The optimal solution of the problem in (1.9) requires* $u_h(q_a, p_a) = u_h(q'_a, p'_a)$.

Assume by contradiction $u_h(q_a, p_a) > u_h(q'_a, p'_a)$. If $\alpha_h > \theta_1/\theta_2$, choose an $\epsilon > 0$ such that

$$\epsilon\,\big(\frac{\theta_2 - \theta_1}{\theta_1}\big) < u_h(q_b, p_b)$$

and make the following adjustment:

$$\begin{aligned}
p_a^a &= p_a - \epsilon \\
p_a'^{\,a} &= p'_a - \epsilon/\theta_1 \\
p_b^a &= p_b + \epsilon\,\big(\frac{\theta_2 - \theta_1}{\theta_1}\big)
\end{aligned}$$

This adjustment does not violate any of the incentive restrictions. For sophisticated with high valuation type, constraints (1.9.1) and (1.9.2) are satisfied by the assumption and by the choice of ϵ. The adjustment does not affect this type's incentive compatibility constraint since it does not make any change in his valuation of Menu-A and of Menu-C but decreases the value of Menu-B. The no-change in the value of Menu-A can be shown as follows:

$$\begin{aligned}
V_h\big(\{(q_a^a, p_a^a), (q_a'^a, p_a'^a)\}, \theta_1\big) &= u_h(q_a^a, p_a^a) - \theta u_h(q_a'^{\,a}, p_a'^{\,a}) \\
&= u_h(q_a, p_a) + \epsilon - \theta_1\big(u_h(q'_a, p'_a) + \epsilon/\theta_1\big) \\
&= u_h(q_a, p_a) - \theta_1 u_h(q'_a, p'_a) \\
&= V_h\big(\{(q_a, p_a), (q'_a, p'_a)\}, \theta_1\big)
\end{aligned}$$

Obviously, the value of Menu-B decreases since the price of the primary product increases and the value of Menu-C is not changing since we do not change anything in that menu. For naive with high valuation type, constraints (1.9.1) and (1.9.2) are satisfied by the choice of ϵ. According to Claim (7) this type's valuation for Menu-B and for Menu-A must be equal. This can be shown as follows:

$$\begin{aligned}
V_h\big(\{(q_b^a, p_b^a), (q_b'^a, p_b'^a)\}, \theta_2\big) &= u_h(q_b^a, p_b^a) \\
&= u_h(q_b, p_b) - \epsilon\,\big(\frac{\theta_2 - \theta_1}{\theta_1}\big) \\
&= u_h(q_a, p_a) - \theta_2 u_h(q'_a, p'_a) - \epsilon\,\big(\frac{\theta_2 - \theta_1}{\theta_1}\big) \\
&= u_h(q_a^a, p_a^a) - \theta_2 u_h(q_a'^{\,a}, p_a'^{\,a}) \\
&= V_h\big(\{(q_a^a, p_a^a), (q_a'^a, p_a'^a)\}, \theta_2\big)
\end{aligned}$$

Finally, since no change is made in Menu-C, the incentive restrictions (1.9.1) and (1.9.2) hold for low types. Moreover for both naive and sophisticated low valuation types the values of Menu-A and Menu-B are decreasing, therefore their incentive compatibility constraints are not violated.

Now consider the change in the objective function:

$$\begin{aligned}\Delta\Pi &= (1-\alpha_h)(-\epsilon) + \alpha_h\,\epsilon\,\big(\frac{\theta_2-\theta_1}{\theta_1}\big)\\ &= \epsilon\,\alpha_h\,\big(\frac{\theta_2}{\theta_1}\big) > 0\end{aligned}$$

Thus, if $\alpha_h > \theta_1/\theta_2$ the adjustment that brings $u_h(q_a, p_a)$ closer to $u_h(q'_a, p'_a)$ increases the objective function. Therefore we have if $\alpha_h > \theta_1/\theta_2$, then $u_h(q_a, p_a) = u_h(q'_a, p'_a)$.

If $\alpha_h \le \theta_1/\theta_2$, contract with 2-Menu beats contract with 3-Menu. We prove this by showing in this case $u_h(q'_a, p'_a) = u_h(q'_b, p'_b) = 0$ and $u_h(q_a, p_a) = u_h(q_b, p_b)$ therefore Menu-A and Menu-B are the same and thus there are not three, but two menus in the optimal contract.

We start with the following adjustment:

$$\begin{aligned}p_a^a &= p_a + \epsilon\\ {p'_a}^a &= p'_a + \epsilon/\theta_1\\ p_b^a &= p_b - \epsilon\,\big(\frac{\theta_2-\theta_1}{\theta_1}\big)\end{aligned}$$

Following the same procedure above it is trivial to show that this adjustment does not violate any incentive restrictions.

Now consider the change in the objective function:

$$\begin{aligned}\Delta\Pi &= (1-\alpha_h)(\epsilon) + \alpha_h\,\epsilon\,\big(\frac{\theta_1-\theta_2}{\theta_1}\big)\\ &= \epsilon\,\big(1-\alpha_h\frac{\theta_2}{\theta_1}\big) \ge 0\end{aligned}$$

Thus, if $\alpha_h \le \theta_1/\theta_2$ the adjustment that removes $u_h(q'_a, p'_a)$ far from $u_h(q_a, p_a)$ increases the objective function. Therefore we have if $\alpha_h \le \theta_1/\theta_2$, then $u_h(q'_a, p'_a) = 0 = u_h(q'_b, p'_b)$. In this case according to Claim (7) we have $u_h(q_a, p_a) = u_h(q_b, p_b)$. Therefore, Menu-A and Menu-B are same and this implies that the contract has 2 menus. We conclude that in the optimal contract with 3 menus we have $u_h(q_a, p_a) = u_h(q'_a, p'_a)$.

Remark 7 *Claim (6) together with Claim (8) implies*

$$\begin{aligned}V_h\big(\{(q_a, p_a), (q'_a, p'_a)\}, \theta_1\big) &= V_h\big(\{(q_c, p_c), (q'_c, p'_c)\}, \theta_1\big)\\ (1-\theta_1)u_h(q_a, p_a) &= (1-\theta_1)u_h(q_c, p_c)\\ u_h(q_a, p_a) &= u_h(q_c, p_c)\end{aligned}$$

Remark 8 *Claim (7) together with Claim (8) implies*

$$\begin{aligned}u_h(q_b, p_b) &= V_h\big(\{(q_a, p_a), (q'_a, p'_a)\}, \theta_2\big)\\ &= (1-\theta_2)u_h(q_a, p_a)\end{aligned}$$

Claim 9 *The optimal solution of the problem in (1.9) requires $q_c \leq q_b$ and $q_c \leq q_a$.*

This can be proved following the same procedure of the proof of Claim (3).

Claim 10 *The optimal solution of the problem in (1.9) requires $q_a = q_b$.*

First assume $q_a > q_b$ and consider the following adjustment:

$$\begin{aligned} q_a^a &= q_a - A \\ q_b^a &= q_b + B \\ p_a^a &= p_a - C \\ p_b^a &= p_b + D \end{aligned}$$

where $q_a - A = q_b + B = (1-\alpha_h)q_a + \alpha_h q_b$. Furthermore the above adjustment is such that the utility of neither the sophisticated nor the naive high valuation type is changed for the product that is designed for them. Mathematically,

$$\begin{aligned} u_h(q_a^a, p_a^a) &= u_h(q_a, p_a) \\ u_h(q_b^a, p_b^a) &= u_h(q_b, p_b). \end{aligned}$$

Therefore we have

$$\begin{aligned} v_h(q_a) - v_h(q_a - A) &= C \\ v_h(q_b + B) - v_h(q_b) &= D. \end{aligned}$$

Since the utilities are not changed, the adjustment does not violate any constraints. Now consider the change in the objective function of the seller after the adjustment:

$$\begin{aligned} \Delta\Pi &= (1-\alpha_h)\big[-C - c(q_a - A) + c(q_a)\big] + \alpha_h\big[D - c(q_b + B) + c(q_b)\big] \\ &= \big[\alpha_h D - (1-\alpha_h)C\big] + \big[(1-\alpha_h)c(q_a) + \alpha_h c(q_b) - c((1-\alpha_h)q_a + \alpha_h q_b)\big] \end{aligned}$$

The second component of the summation above is positive, since the cost function $c(.)$ is convex. The first component is also positive since

$$\begin{aligned} \alpha_h D - (1-\alpha_h)C &= \alpha_h\big[v_h(q_b + B) - v_h(q_b)\big] - (1-\alpha_h)\big[v_h(q_a) - v_a(q_a - A)\big] \\ &= v_h((1-\alpha_h)q_a + \alpha_h q_b) - \alpha_h v_h(q_b) - (1-\alpha_h)v_h(q_a) > 0. \end{aligned}$$

Therefore we have $\Delta\Pi > 0$.

In the second part of the proof we assume $q_a < q_b$ and making a similar adjustment, where we increase q_a and p_a, and decrease q_b and p_b so that the high valuation consumers' utilities do not change, we show that the objective function increases. This shows $q_a = q_b$ and completes the proof.

Claim 11 *The optimal solution of the problem in (1.9) requires*

$$V_l\big(\{(q_a, p_a), (q'_a, p'_a)\}, \theta_j\big) = V_l\big(\{(q_b, p_b), (q'_b, p'_b)\}, \theta_j\big) = 0,$$

for $j \in \{1, 2\}$.

The secondary product in Menu-A can be adjusted in a such way that $u_l(q'_a, p'_a) < 0$ while the utility it provides the high types does not change. This adjustment does not change the profit since secondary products are never sold. Therefore, we have $V_l\big(\{(q_a, p_a), (q'_a, p'_a)\}, \theta_j\big) = (1 - \theta_j)u_l(q_a, p_a)$. Furthermore, in Claim (5) we show that $u_h(q'_b, p'_b) = 0$ and this implies $u_l(q'_b, p'_b) < 0$. Therefore we have $V_l\big(\{(q_b, p_b), (q'_b, p'_b)\}, \theta_j\big) = (1 - \theta_j)u_l(q_b, p_b)$.

Observe that since $q_a = q_b$ and $u_h(q_a, p_a) > u_h(q_b, p_b)$ we have $u_l(q_a, p_a) > u_l(q_b, p_b)$. This implies that if we show that $u_l(q_a, p_a) \leq 0$, it would automatically imply $u_l(q_b, p_b) > 0$ and therefore this would mean that the valuation of the low types for Menu-A and Menu-B are both zero. Hence showing $u_l(q_a, p_a) \leq 0$ is sufficient to complete the proof. To do that we use the following procedure that is similar to the one used in the proof of Claim (4). We assume by contradiction $u_l(q_a, p_a) > 0$, which implies that $V_l\big(\{(q_a, p_a), (q'_a, p'_a)\}, \theta_j\big) = u_l(q_a, p_a)\big(1 - \theta_j\big) > 0$ (for $j \in \{1, 2\}$) and therefore by incentive compatibility for low types we have $u_l(q_c, p_c) > 0$. Choose $\epsilon > 0$, $\epsilon_1 > 0$, $\epsilon_2 > 0$, $\beta_1 > 0$ and $\beta_2 > 0$ satisfying the following conditions:

$$(i) \quad v_l(q_c - \epsilon) - p_c = 0 \tag{A.20}$$

$$(ii) \quad \beta_1 = v_h(q_c) - v_h(q_c - \epsilon) = v_h(q'_c) - v_h(q'_c - \epsilon_1) - \epsilon_2 \tag{A.21}$$

$$(iii) \quad \beta_2 = \beta_1(1 - \theta_2) \tag{A.22}$$

$$(iv) \quad v_l(q'_c - \epsilon_1) - p'_c + \epsilon_2 = 0 \tag{A.23}$$

Consider now the following adjustment:

$$\begin{aligned} p^a_a &= p_a + \beta_1 \\ p'^a_a &= p'_a + \beta_1 \\ p^a_b &= p_b + \beta_2 \\ q^a_c &= q_c - \epsilon \\ q'^a_c &= q'_c - \epsilon_1 \\ p'^a_c &= p'_c - \epsilon_2 \end{aligned}$$

This adjustment increases the objective function strictly since it increases the prices of the products sold to high valuation consumers (p_a and p_b) and decreases the quality of the product sold to low valuation consumers (q_c), which decreases the cost. We want to show that the proposed adjustment does not violate any of the constraints of the problem (1.9).

We start by examining the constraints of low valuation consumers. The conditions given in (A.20) and (A.23) yield $u_l(q^a_c, p^a_c) = u_l(q'^a_c, p'^a_c) = 0$, therefore both (1.9.1) and

(1.9.2) are still satisfied for low types. Moreover, the valuation of low types for the Menu-C becomes zero. Therefore their valuations for both Menu-A and Menu-B need to be zero as well in order to satisfy (1.9.5). This can be shown as follows:

$$\begin{aligned} u_h(q_a^a, p_a^a) &= u_h(q_a, p_a) - \beta_1 \\ &= u_h(q_c, p_c) - [v_h(q_c) - v_h(q_c - \epsilon)] \\ &= v_h(q_c - \epsilon) - p_c \\ &= u_h(q_c^a, p_c^a) \end{aligned}$$

and therefore

$$p_a^a - p_c^a = v_h(q_a^a) - v_h(q_c^a) \geq v_l(q_a^a) - v_l(q_c^a),$$

and this implies

$$0 = v_l(q_c^a) - p_c^a \geq v_l(q_a^a) - p_a^a = u_l(q_a^a, p_a^a)$$

Thus $V_l\big(\{(q_a^a, p_a^a), (q_a'^a, p_a'^a)\}, \theta_j\big) = \max\{(1-\theta_j)u_l(q_a^a, p_a^a), 0\} = 0$. Having showed that the low types' constraints are still satisfied, we move to the ones of the high types. For the sophisticated with high valuation type, the incentive restrictions (1.9.1) and (1.9.2) are not violated by the assumption and the choice of β_1. Now we consider this type's valuation for each menu:

$$\begin{aligned} V_h\big(\{(q_a^a, p_a^a), (q_a'^a, p_a'^a)\}, \theta_1\big) &= (1-\theta_1)u_h(q_a^a, p_a^a) \\ &= (1-\theta_1)\big(u_h(q_a, p_a) - \beta_1\big) \end{aligned} \tag{A.24}$$

$$\begin{aligned} V_h\big(\{(q_b^a, p_b^a), (q_b'^a, p_b'^a)\}, \theta_1\big) &= u_h(q_b^a, p_b^a) \\ &= u_h(q_b, p_b) - \beta_2 \\ &= (1-\theta_2)u_h(q_a, p_a) - \beta_2 \\ &= (1-\theta_2)\big(u_h(q_a, p_a) - \beta_1\big), \end{aligned} \tag{A.25}$$

where we use Remark (8) in the third line.

$$\begin{aligned} V_h\big(\{(q_c^a, p_c^a), (q_c'^a, p_c'^a)\}, \theta_1\big) &= u_h(q_c^a, p_c^a) - \theta_1 u_h(q_c'^a, p_c'^a) \\ &= u_h(q_c, p_c) - \beta_1 - \theta_1\big(u_h(q_c', p_c') - \beta_1\big) \\ &= (1-\theta_1)\big(u_h(q_c, p_c) - \beta_1\big) \\ &= (1-\theta_1)\big(u_h(q_a, p_a) - \beta_1\big), \end{aligned} \tag{A.26}$$

where we use Claim (8) and Remark (7) in the last two lines. Comparing (A.24), (A.25) and (A.26), we have

$$V_h\big(\{(q_a^a, p_a^a), (q_a'^a, p_a'^a)\}, \theta_1\big) = V_h\big(\{(q_c^a, p_c^a), (q_c'^a, p_c'^a)\}, \theta_1\big) > V_h\big(\{(q_b^a, p_b^a), (q_b'^a, p_b'^a)\}, \theta_1\big),$$

therefore the adjustment does not violate the incentive compatibility constraint of the sophisticated with high valuation type, (1.9.3). Moreover it satisfies the requirement of the optimal contract given in Claim (6). For the naive with high valuation type, the incentive restrictions (1.9.1) and (1.9.2) are still satisfied by the assumption $u_l(q_a, p_a) > 0$ and by the choice of β_2. Now we consider this type's valuation for each menu that are very similar to those of the sophisticated with high valuation type with a single difference: instead of θ_1 we have θ_2.

$$V_h\big(\{(q_a^a, p_a^a), (q_a'^a, p_a'^a)\}, \theta_2\big) = (1-\theta_2)\big(u_h(q_a, p_a) - \beta_1\big)$$
$$V_h\big(\{(q_b^a, p_b^a), (q_b'^a, p_b'^a)\}, \theta_2\big) = (1-\theta_2)\big(u_h(q_a, p_a) - \beta_1\big)$$
$$V_h\big(\{(q_c^a, p_c^a), (q_c'^a, p_c'^a)\}, \theta_2\big) = (1-\theta_2)\big(u_h(q_a, p_a) - \beta_1\big)$$

Therefore, for the naive with high valuation type all the menus become equal. This does not violate the incentive compatibility constraint (1.9.4) and shows that the requirement of the optimal contract given in Claim (7) is satisfied. This suggests that the adjustment increases the objective function without violating any of the constraints and completes the proof.

Remark 9 *Claim (11) implies that the incentive compatibility constraint (1.9.5) becomes*

$$V_l\big(\{(q_c, p_c), (q_c', p_c')\}, \theta_j\big) \geq 0$$

for $j \in \{1, 2\}$. *In fact, in the optimal contract this constraint is binding since otherwise it is always possible to increase* p_c *and increase the objective function. Therefore we have* $u_l(q_c, p_c) = u_l(q_c', p_c') = 0$.

Proof of Lemma 2 *Using Claim (5),(8) and the observations given in Remark (7), (8) and(9), the problem (1.9) can be simplified as:*

$$\max_{\{(q_k,p_k)\in\mathbb{R}_+^2\}_{k\in\{a,b,c\}}} (1-\lambda)(1-\alpha_h)\left[p_a - c(q_a)\right] + (1-\lambda)\alpha_h\left[p_b - c(q_b)\right] + \lambda\left[p_c - c(q_c)\right]$$

$$u_l(q_c, p_c) = 0$$
$$u_h(q_a, p_a) = u_h(q_c, p_c)$$
$$u_h(q_b, p_b) = (1-\theta_2)\, u_h(q_a, p_a)$$

From the constraints above we get

$$p_c = v_l(q_c)$$
$$p_a = v_h(q_a) - \left[v_h(q_c) - p_c\right]$$
$$p_b = v_h(q_b) - (1-\theta_2)\left[v_h(q_a) - p_a\right]$$

Substituting these values into the objective function and taking the first order conditions with respect to q_a, q_b *and* q_c *we get*

$$c'(q_a) = v_h'(q_a)$$
$$p_a = v_h(q_a) - \big[v_h(q_c) - v_l(q_c)\big]$$

$$c'(q_b) = v_h'(q_b)$$
$$p_b = v_h(q_b) - (1-\theta_2)\,\big[v_h(q_c) - v_l(q_c)\big]$$

$$c'(q_c) = v_l'(q_c) - \frac{1-\lambda}{\lambda}\big(1 - \alpha_h\,\theta_2\big)\,\big[v_h'(q_c) - v_l'(q_c)\big]$$
$$p_c = v_l(q_c) \qquad \square$$

A.3 Proof of Lemma 3

In this subsection, following some simplifying claims, we give the proof of Lemma 3. For the sake of ease of exposition, we assume that there are three different levels of θ for high valuation consumers. However all the analysis of this subsection can be easily extended to the continuous case. The three levels are $0 \leq \theta_1 < \theta_2 < \theta_3 \leq 1$ with probabilities α_1, α_2 and α_3 such that $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Furthermore we assume by contradiction that the optimal contract has four menus and we want to show that canceling one of the menus increases the objective function.

The first three menus of this contract are designed for high valuation consumers with θ_1, θ_2 and θ_3, respectively, and the last one is designed for all low valuation types. Remark 9 justifies that there is a single menu for all low valuation consumers. In other words, Remark 9 points out that it is possible to construct a feasible contract that gives zero utility to low valuation consumers without discriminating them depending on the level of context-bias. Figure A.1 gives an illustration of this contract. We refer these four menus as Menu-A, Menu-B, Menu-C and Menu-D, respectively.

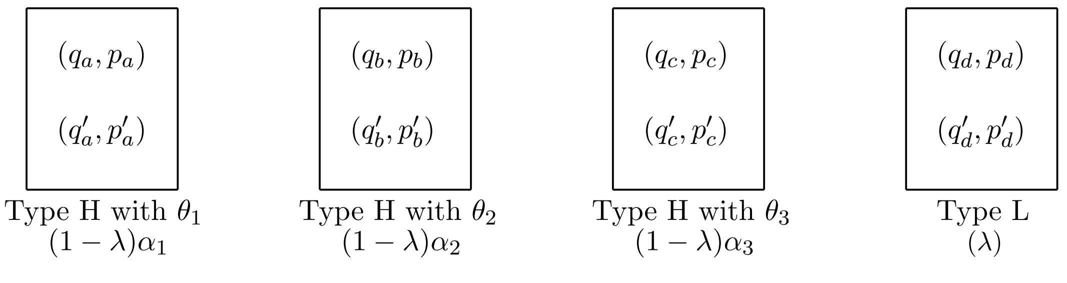

Figure A.1: Contract with Four Menus

We will be characterizing this contract in the following claims in order to show that a contract with three menus beats this contract, i.e., the optimal contract has no more than three menus.

Observe that by assuming that the contract with four menu is optimal, we assume implicitly that this contract satisfies all the incentive restrictions given below:

$$u_h(q'_a, p'_a) \geq 0, \quad u_h(q'_b, p'_b) \geq 0, \quad u_h(q'_c, p'_c) \geq 0 \quad u_l(q'_d, p'_d) \geq 0, \tag{A.27}$$

$$u_h(q_a, p_a) \geq u_h(q'_a, p'_a), \; u_h(q_b, p_b) \geq u_h(q'_b, p'_b), \; u_h(q_c, p_c) \geq u_h(q'_c, p'_c), \; u_l(q_d, p_d) \geq u_l(q'_d, p'_d), \tag{A.28}$$

$$V_h\big(\{(q_a, p_a), (q'_a, p'_a)\}, \theta_1\big) \geq \max_{k\in\{b,c,d\}} V_h\big(\{(q_k, p_k), (q'_k, p'_k)\}, \theta_1\big), \tag{A.29}$$

$$V_h\big(\{(q_b, p_b), (q'_b, p'_b)\}, \theta_2\big) \geq \max_{k\in\{a,c,d\}} V_h\big(\{(q_k, p_k), (q'_k, p'_k)\}, \theta_2\big), \tag{A.30}$$

$$V_h\big(\{(q_c, p_c), (q'_c, p'_c)\}, \theta_3\big) \geq \max_{k\in\{a,b,d\}} V_h\big(\{(q_k, p_k), (q'_k, p'_k)\}, \theta_3\big), \tag{A.31}$$

$$V_l\big(\{(q_d, p_d), (q'_d, p'_d)\}, \theta_j\big) \geq \max_{k\in\{a,b,c\}} V_l\big(\{(q_k, p_k), (q'_k, p'_k)\}, \theta_j\big) \qquad \text{for } \theta_j \in [0, 1]. \tag{A.32}$$

Without loss of generality, the incentive compatibility constraints (A.29)-(A.32) are all binding, since otherwise it is always possible to increase the objective function by increasing the price of the relative good.

Claim 12 *The optimality assumption requires* $u_h(q'_c, p'_c) = 0$.

Follow the procedure in the proof of Claim 1.

Claim 13 *This contract requires* $u_h(q_a, p_a) > u_h(q_b, p_b)$ *and* $u_h(q'_a, p'_a) > u_h(q'_b, p'_b)$.

The incentive compatibility constraints given in (A.29) and (A.30) imply

$$u_h(q_a, p_a) - \theta_1 u_h(q'_a, p'_a) \geq u_h(q_b, p_b) - \theta_1 u_h(q'_b, p'_b) \tag{A.33}$$

$$u_h(q_b, p_b) - \theta_2 u_h(q'_b, p'_b) \geq u_h(q_a, p_a) - \theta_2 u_h(q'_a, p'_a) \tag{A.34}$$

Combining (A.33) and (A.34) we get

$$\theta_2\big(u_h(q'_a, p'_a) - u_h(q'_b, p'_b)\big) \geq u_h(q_a, p_a) - u_h(q_b, p_b) \geq \theta_1\big(u_h(q'_a, p'_a) - u_h(q'_b, p'_b)\big) \tag{A.35}$$

Given the fact that $\theta_2 > \theta_1 \geq 0$, the above inequality shows that $u_h(q'_a, p'_a) - u_h(q'_b, p'_b) \geq 0$. However if $u_h(q'_a, p'_a) - u_h(q'_b, p'_b) = 0$, then by inequality (A.35) we have $u_h(q_a, p_a) = u_h(q_b, p_b)$. In this case Menu-A and Menu-B would be the same, therefore $u_h(q'_a, p'_a) > u_h(q'_b, p'_b)$ and $u_h(q_a, p_a) > u_h(q_b, p_b)$.

Remark 10 *The incentive compatibility constraint given in (A.30) together with the fact that* $u_h(q'_c, p'_c) = 0$ *implies* $u_h(q_b, p_b) - \theta_2 u_h(q'_b, p'_b) \geq u_h(q_c, p_c)$, *therefore we have*

$$u_h(q_a, p_a) > u_h(q_b, p_b) \geq u_h(q_c, p_c). \tag{A.36}$$

Remark 11 *We have*

$$u_h(q_b, p_b) - \theta_1 u_h(q'_b, p'_b) \geq u_h(q_b, p_b) - \theta_2 u_h(q'_b, p'_b) \geq u_h(q_c, p_c)$$

implying that the incentive compatibility constraint given in (A.29) can be replaced with

$$V_h\big(\{(q_a,p_a),(q_a',p_a')\},\theta_1\big)=\max\{V_h\big(\{(q_b,p_b),(q_b',p_b')\},\theta_1\big),V_h\big(\{(q_d,p_d),(q_d',p_d')\},\theta_1\big)\} \tag{A.29'}$$

Remark 12 *One implication of the incentive compatibility constraint given in (A.30) is*

$$u_h(q_b,p_b)-\theta_2 u_h(q_b',p_b')\geq u_h(q_a,p_a)-\theta_2 u_h(q_a',p_a')$$

Therefore,

$$u_h(q_a,p_a)-u_h(q_b,p_b)\leq\theta_2\big(u_h(q_a',p_a')-u_h(q_b',p_b')\big)<\theta_3\big(u_h(q_a',p_a')-u_h(q_b',p_b')\big)$$

implying

$$u_h(q_a,p_a)-\theta_3 u_h(q_a',p_a')<u_h(q_b,p_b)-\theta_3 u_h(q_b',p_b').$$

This observation shows that the incentive compatibility constraint given in (A.31) can be replaced with

$$V_h\big(\{(q_c,p_c),(q_c',p_c')\},\theta_3\big)=\max\{V_h\big(\{(q_b,p_b),(q_b',p_b')\},\theta_3\big),V_h\big(\{(q_d,p_d),(q_d',p_d')\},\theta_3\big)\} \tag{A.31'}$$

Claim 14 $u_h(q_c,p_c)=u_h(q_b,p_b)-\theta_3 u_h(q_b',p_b')$.

Remark 12 implies

$$u_h(q_c,p_c)=\max\{V_h\big(\{(q_b,p_b),(q_b',p_b')\},\theta_3\big),V_h\big(\{(q_d,p_d),(q_d',p_d')\},\theta_3\big)\}.$$

Assume by contradiction that

$$V_h\big(\{(q_d,p_d),(q_d',p_d')\},\theta_3\big)>V_h\big(\{(q_b,p_b),(q_b',p_b')\},\theta_3\big) \tag{A.37}$$

and therefore

$$u_h(q_c,p_c)=V_h\big(\{(q_d,p_d),(q_d',p_d')\},\theta_3\big)=(1-\theta_3)u_h(q_d,p_d). \tag{A.38}$$

The second equality comes from the fact that the product (q_d',p_d') can be arranged in such a way that $u_h(q_d,p_d)=u_h(q_d',p_d')$ while the value of $u_l(q_d',p_d')$ is not changing. For more details see the second part of the proof of Claim 6.

From equation (A.38) we get

$$u_h(q_c,p_c)=(1-\theta_3)u_h(q_d,p_d)<(1-\theta_2)u_h(q_d,p_d)$$

The incentive compatibility constraint (A.30) together with the above inequality implies

$$u_h(q_b,p_b)-\theta_2 u_h(q_b',p_b')=\max\big\{u_h(q_a,p_a)-\theta_2 u_h(q_a',p_a'),\,(1-\theta_2)u_h(q_d,p_d)\big\}$$

Case 1: $u_h(q_b, p_b) - \theta_2 u_h(q'_b, p'_b) = (1 - \theta_2) u_h(q_d, p_d)$
In this case, incentive restrictions (A.27) and (A.28) imply

$$u_h(q_b, p_b) \in \big[(1 - \theta_2) u_h(q_d, p_d), u_h(q_d, p_d)\big] \tag{A.39}$$

Furthermore, the assumption given in (A.37) yields

$$\begin{aligned} u_h(q_b, p_b) - \theta_3 u_h(q'_b, p'_b) &< (1 - \theta_3) u_h(q_d, p_d) \\ (1 - \theta_2) u_h(q_d, p_d) - (\theta_3 - \theta_2) u_h(q'_b, p'_b) &< (1 - \theta_3) u_h(q_d, p_d) \\ (\theta_3 - \theta_2) u_h(q_d, p_d) &< (\theta_3 - \theta_2) u_h(q'_b, p'_b) \\ u_h(q_d, p_d) &< u_h(q'_b, p'_b), \end{aligned}$$

which contradicts with the observation given in (A.39). Therefore Case 1 is not possible.

Case 2: $u_h(q_b, p_b) - \theta_2 u_h(q'_b, p'_b) = u_h(q_a, p_a) - \theta_2 u_h(q'_a, p'_a)$
In this case, we have

$$\begin{aligned} V_h\big(\{(q_a, p_a), (q'_a, p'_a)\}, \theta_1\big) &= u_h(q_a, p_a) - \theta_1 u_h(q'_a, p'_a) \\ &= u_h(q_b, p_b) - \theta_2 u_h(q'_b, p'_b) + (\theta_2 - \theta_1) u_h(q'_a, p'_a) - \theta_1 u_h(q'_b, p'_b) \\ &\quad + \theta_1 u_h(q'_b, p'_b) \\ &= u_h(q_b, p_b) - \theta_1 u_h(q'_b, p'_b) + (\theta_2 - \theta_1)\big(u_h(q'_a, p'_a) - u_h(q'_b, p'_b)\big) \\ &> u_h(q_b, p_b) - \theta_1 u_h(q'_b, p'_b) \end{aligned}$$

Thus (A.29') implies

$$u_h(q_a, p_a) - \theta_1 u_h(q'_a, p'_a) = (1 - \theta_1) u_h(q_d, p_d) \tag{A.40}$$

and therefore

$$u_h(q_a, p_a) \in \big[(1 - \theta_1) u_h(q_d, p_d),\ u_h(q_d, p_d)\big] \tag{A.41}$$

Combining the incentive compatibility constraint (A.31) with the assumption (A.37) and the observation (A.40) we get

$$\begin{aligned} u_h(q_a, p_a) - \theta_3 u_h(q'_a, p'_a) &\leq (1 - \theta_3) u_h(q_d, p_d) \\ u_h(q_a, p_a) - \theta_1 u_h(q'_a, p'_a) - (\theta_3 - \theta_1) u_h(q'_a, p'_a) &\leq (1 - \theta_3) u_h(q_d, p_d) \\ (1 - \theta_1) u_h(q_d, p_d) - (\theta_3 - \theta_1) u_h(q'_a, p'_a) &\leq (1 - \theta_3) u_h(q_d, p_d) \\ (\theta_3 - \theta_1) u_h(q_d, p_d) &\leq (\theta_3 - \theta_1) u_h(q'_a, p'_a) \\ u_h(q_d, p_d) &\leq u_h(q'_a, p'_a) \end{aligned}$$

This shows that $u_h(q_a, p_a) = u_h(q'_a, p'_a) = u_h(q_d, p_d)$, which yields

$$u_h(q_b, p_b) - \theta_2 u_h(q'_b, p'_b) = u_h(q_a, p_a) - \theta_2 u_h(q'_a, p'_a) = (1 - \theta_2) u_h(q_d, p_d)$$

and this takes us back to Case 1. We have already seen that Case 1 is not possible, therefore Case 2 is not possible, too.

This completes the proof.

Claim 15 $u_h(q_b, p_b) - \theta_2 u_h(q'_b, p'_b) = u_h(q_a, p_a) - \theta_2 u_h(q'_a, p'_a)$.

Using Claim 14 the incentive compatibility constraint given in (A.30) can be rewritten as follows:

$$u_h(q_b, p_b) - \theta_2 u_h(q'_b, p'_b) = \max\big\{u_h(q_a, p_a) - \theta_2 u_h(q'_a, p'_a),\ (1-\theta_2)u_h(q_d, p_d)\big\}$$

Assume by contradiction

$$u_h(q_b, p_b) - \theta_2 u_h(q'_b, p'_b) = (1-\theta_2)u_h(q_d, p_d) > u_h(q_a, p_a) - \theta_2 u_h(q'_a, p'_a) \tag{A.42}$$

This assumption implies

$$u_h(q_b, p_b) \in \big[(1-\theta_2)u_h(q_d, p_d),\ u_h(q_d, p_d)\big] \tag{A.43}$$

Furthermore,

$$\begin{aligned}(1-\theta_1)u_h(q_d, p_d) &= (1-\theta_2)u_h(q_d, p_d) + (\theta_2-\theta_1)u_h(q_d, p_d)\\ &= u_h(q_b, p_b) - \theta_2 u_h(q'_b, p'_b) + (\theta_2-\theta_1)u_h(q_d, p_d)\\ &= u_h(q_b, p_b) - \theta_1 u_h(q'_b, p'_b) + (\theta_2-\theta_1)\big(u_h(q_d, p_d) - u_h(q'_b, p'_b)\big)\\ &\geq u_h(q_b, p_b) - \theta_1 u_h(q'_b, p'_b)\end{aligned}$$

After this observation the incentive compatibility constraint (A.29) becomes

$$u_h(q_a, p_a) - \theta_1 u_h(q'_a, p'_a) = (1-\theta_1)u_h(q_d, p_d).$$

Using this we can rewrite the assumption (A.42) as

$$\begin{aligned}(1-\theta_1)(1-\theta_2)u_h(q_d, p_d) &> (1-\theta_1)\big(u_h(q_a, p_a) - \theta_2 u_h(q'_a, p'_a)\big)\\ (1-\theta_2)\big(u_h(q_a, p_a) - \theta_1 u_h(q'_a, p'_a)\big) &> (1-\theta_1)\big(u_h(q_a, p_a) - \theta_2 u_h(q'_a, p'_a)\big)\\ (\theta_2-\theta_1)u_h(q'_a, p'_a) &> (\theta_2-\theta_1)u_h(q_a, p_a)\\ u_h(q'_a, p'_a) &> u_h(q_a, p_a)\end{aligned}$$

which contradicts with the incentive restriction (A.28).

Claim 16 $u_h(q_a, p_a) - \theta_1 u_h(q'_a, p'_a) = (1-\theta_1)u_h(q_d, p_d)$.

Remark 11 implies

$$u_h(q_a, p_a) - \theta_1 u_h(q'_a, p'_a) = \max\{V_h\big(\{(q_b, p_b), (q'_b, p'_b)\}, \theta_1\big), V_h\big(\{(q_d, p_d), (q'_d, p'_d)\}, \theta_1\big)\}.$$

Assume by contradiction that

$$V_h\big(\{(q_b, p_b), (q'_b, p'_b)\}, \theta_1\big) > V_h\big(\{(q_d, p_d), (q'_d, p'_d)\}, \theta_1\big) \tag{A.44}$$

and therefore

$$u_h(q_a,p_a)-\theta_1 u_h(q'_a,p'_a)=V_h\big(\{(q_b,p_b),(q'_b,p'_b)\},\theta_1\big)=u_h(q_b,p_b)-\theta_1 u_h(q'_b,p'_b). \quad \text{(A.45)}$$

Using Claim 15 and equation (A.45) we get

$$u_h(q_a,p_a)-u_h(q_b,p_b)=\theta_2\big(u_h(q'_a,p'_a)-u_h(q'_b,p'_b)\big)=\theta_1\big(u_h(q'_a,p'_a)-u_h(q'_b,p'_b)\big).$$

This implies $u_h(q'_a,p'_a)=u_h(q'_b,p'_b)$, contradicting with Claim 13.

Remark 13 *The incentive compatibility constraints for high valuation consumers (A.29)-(A.31) can now be written as follows:*

$$\begin{aligned} u_h(q_a,p_a)-\theta_1 u_h(q'_a,p'_a)&=(1-\theta_1)u_h(q_d,p_d), && \text{(A.46)}\\ u_h(q_b,p_b)-\theta_2 u_h(q'_b,p'_b)&=u_h(q_a,p_a)-\theta_2 u_h(q'_a,p'_a) && \text{(A.47)}\\ u_h(q_c,p_c)&=u_h(q_b,p_b)-\theta_3 u_h(q'_b,p'_b) && \text{(A.48)}\end{aligned}$$

Proof of Lemma 3 *Take ϵ such that*

$$0<\epsilon\leq\begin{cases}\frac{\theta_2}{1-\theta_2}(u_h(q_b,p_b)-u_h(q'_b,p'_b)), & \textit{if } \alpha_3\theta_3>(\alpha_2+\alpha_3)\theta_2;\\ \theta_2 u_h(q'_b,p'_b), & \textit{otherwise.}\end{cases}$$

Consider the following adjustment:

$$p_b^a=\begin{cases}p_b-\epsilon, & \textit{if } \alpha_3\theta_3>(\alpha_2+\alpha_3)\theta_2;\\ p_b+\epsilon, & \textit{otherwise.}\end{cases}$$

$${p'_b}^a=\begin{cases}p'_b-\epsilon/\theta_2, & \textit{if } \alpha_3\theta_3>(\alpha_2+\alpha_3)\theta_2;\\ p'_b+\epsilon/\theta_2, & \textit{otherwise.}\end{cases}$$

$$p_c^a=\begin{cases}p_c+\epsilon\,\big(\dfrac{\theta_3-\theta_2}{\theta_2}\big), & \textit{if } \alpha_3\theta_3>(\alpha_2+\alpha_3)\theta_2;\\ p_c-\epsilon\,\big(\dfrac{\theta_3-\theta_2}{\theta_2}\big), & \textit{otherwise.}\end{cases}$$

To complete the proof we need to show that the adjustment increases the value of the objective function while not violating any of the incentive restriction. We start with the case when $\alpha_3\theta_3>(\alpha_2+\alpha_3)\theta_2$. First consider the change in the objective function:

$$\begin{aligned}\Delta\Pi&=\alpha_2(-\epsilon)+\alpha_3\,\epsilon\,\big(\frac{\theta_3-\theta_2}{\theta_2}\big)\\ &=\epsilon\,\big(-(\alpha_2+\alpha_3)+\alpha_3\frac{\theta_3}{\theta_2}\big)>0\end{aligned}$$

Now we need to see if this adjustment violates any of the incentive restrictions. It is obvious that the constraints (A.27) and (A.28) for high valuation consumers are satisfied by the choice ϵ. *We want to show that all the incentive compatibility constraints for high valuation consumers (A.46)-(A.48) are still satisfied. The adjustment does not require any change for the products in Menu-A and Menu-D, therefore (A.46) is not affected. The incentive compatibility constraint for high valuation consumer with* θ_2 *(A.47) becomes*

$$\begin{aligned} u_h(q_b^a, p_b^a) - \theta_2 u_h(q_b'^a, p_b'^a) &= u_h(q_b, p_b) + \epsilon - \theta_2\big(u_h(q_b', p_b') + \frac{\epsilon}{\theta_2}\big) \\ &= u_h(q_b, p_b) - \theta_2 u_h(q_b', p_b') + \epsilon - \theta_2 \frac{\epsilon}{\theta_2} \\ &= u_h(q_b, p_b) - \theta_2 u_h(q_b', p_b') \\ &= u_h(q_a, p_a) - \theta_2 u_h(q_a', p_a'). \end{aligned}$$

Furthermore, (A.48) becomes

$$\begin{aligned} u_h(q_b^a, p_b^a) - \theta_3 u_h(q_b'^a, p_b'^a) &= u_h(q_b, p_b) + \epsilon - \theta_3\big(u_h(q_b', p_b') + \frac{\epsilon}{\theta_2}\big) \\ &= u_h(q_b, p_b) - \theta_3 u_h(q_b', p_b') + \epsilon - \theta_3 \frac{\epsilon}{\theta_2} \\ &= u_h(q_c, p_c) - \epsilon\,\big(\frac{\theta_3 - \theta_2}{\theta_2}\big) \\ &= u_h(q_c^a, p_c^a). \end{aligned}$$

This shows that all the incentive restrictions for high valuation consumers are satisfied. Observe that this adjustment decreases the value of Menu-C for all low valuation consumers and this does not affect any of the constraints of this type. However it increases the value of Menu-B only for those whose $\theta < \theta_2$. *This means that in order to complete the proof, we need to show that the increase in the valuation of Menu-B of low valuation consumers whose context-bias is less than* θ_2 *does not violate their incentive compatibility constraint. This can be done in two steps. First we can show that* $q_b \geq q_d$ *using the technique of the proof of Claim 3 and then we can use this fact to show* $V_l\big(\{(q_b, p_b), (q_b', p_b')\}, \theta_j\big) < 0$ *by using the technique of the proof of Claim 4. This implies that it is always possible to find a positive* ϵ' *such that* $V_l\big(\{(q_b^a, p_b^a), (q_b'^a, p_b'^a)\}, \theta_j\big) - \epsilon' \leq 0$. *By updating* ϵ *by* $\epsilon^{new} = \min\{\epsilon, \epsilon'\}$ *we can show that none of the constraints are violated.*

Observe that this adjustment implies $u_h(q_b, p_b) = u_h(q_b', p_b')$. *From the incentive compatibility constraint given in (A.47) we get*

$$u_h(q_a, p_a) - u_h(q_b, p_b) = \theta_2\big(u_h(q_a', p_a') - u_h(q_b', p_b')\big) \leq u_h(q_a', p_a') - u_h(q_b', p_b'),$$

which implies $u_h(q_a, p_a) \leq u_h(q_a', p_a')$ *and therefore we have*

$$u_h(q_a', p_a') = u_h(q_a, p_a) = u_h(q_b, p_b) = u_h(q_b', p_b')$$

The fact that Menu-A and Menu-B are the same implies that this contract has three menus, not four. Obviously this contradicts with the optimality assumption of four-menu contract.

For the case when $\alpha_3\theta_3 \leq (\alpha_2 + \alpha_3)\theta_2$, we proceed likewise. First we consider the change in the objective function:

$$\begin{aligned}\Delta\Pi &= \alpha_2(\epsilon) - \alpha_3\,\epsilon\,\big(\frac{\theta_3 - \theta_2}{\theta_2}\big) \\ &= \epsilon\,\big(\alpha_2 + \alpha_3 - \alpha_3\frac{\theta_3}{\theta_2}\big) \geq 0\end{aligned}$$

Following similar arguments as before, it is possible to prove that the adjustment does not violate any constraints. In this case the adjustment implies $u_h(q'_b, p'_b) = 0$. From the incentive compatibility constraint given in (A.48) we get $u_h(q_c, p_c) = u_h(q_b, p_b)$, therefore Menu-B and Menu-C are the same. This is a contradiction with the assumption that the contract with four menus is optimal. □

A.4 Proof of Lemma 4

In order to prove this lemma, it is enough to realize that each menu in this contract has to be constructed based on the most sophisticated one in the set of the consumers for which that menu is designed. In particular, the first menu that is for the set of high valuation consumers whose θs belong to the interval $[0, \bar{\theta})$ must be constructed in such a way that the incentive constraints of the most sophisticated type in this set (the fully sophisticated consumer) need to be satisfied. All the other consumers in this set (the high valuation consumers for which $\theta \in (0, \bar{\theta})$) follow the fully sophisticated one, since once the incentive constraint of this type is satisfied the incentive constraints of all the other types are already satisfied. In the same way, the second menu must be constructed based on the context-bias level of the most sophisticated type for which this menu is designed, i.e., high valuation consumer with $\bar{\theta}$. Finally, the last menu needs to be concerned with the most sophisticated type among low valuation consumers. This observation concludes that the contract is constructed based on three types on the market; the most sophisticated types from each subset of consumers that the contract partitions. Therefore it is equivalent to the contract examined in Section 1.3.2 with $\theta_1 = 0$, $\theta_2 = \bar{\theta}$ and $\alpha_h = 1 - F(\bar{\theta})$. Using Lemma 2, we get

$$
\begin{aligned}
&c'(q_a) = v_h'(q_a) \\
&p_a = v_h(q_a) - \big[v_h(q_c) - v_l(q_c)\big]
\end{aligned}
$$

$$
\begin{aligned}
&c'(q_b) = v_h'(q_b) \\
&p_b = v_h(q_b) - (1-\bar{\theta})\,\big[v_h(q_c) - v_l(q_c)\big]
\end{aligned}
$$

$$
\begin{aligned}
&c'(q_c) = v_l'(q_c) - \frac{1-\lambda}{\lambda}\,\big(1-\bar{\theta}+\bar{\theta}\,F(\bar{\theta})\big)\,\big[v_h'(q_c) - v_l'(q_c)\big] \\
&p_c = v_l(q_c)
\end{aligned}
$$

□

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Chapter 2

Lying for the Greater Good: Bounded Rationality in a Team

2.1 Introduction

In economic literature, one of the most commonly used assumptions about decision makers is full rationality. When faced with an economic decision problem, a fully rational decision maker has the ability to see and understand what is feasible and what is preferable. Furthermore, he is also able to calculate the optimal course of action given these two constraints. This widely used assumption that simplifies economic models has received many criticisms for overlooking real life situations by ignoring cognitive limitations. Wide literature initiated by Amos Tversky, Daniel Kahneman, and their collaborators provides us with experimental evidence that human beings depart systematically from full rationality due to cognitive limitations. These limitations affect their ability to recognize the available information on markets and their ability to compute. Herbert Simon, the originator of the phrase, defines bounded rationality as "rational choice that takes into account the cognitive limitations of the decision-maker-limitations of both knowledge and computational capacity" (Simon 1987).

Boundedly rational agents try to simplify and structure the economic decision process. One of the possible ways to do this is to use categories. The usage of categories is also supported by psychological evidence that people in environments with abundance of information show the tendency to group events, objects or numbers into categories depending on their perceived similarities (Rosch and Mervis 1975). The social psychologist Gordon Allport states that "the human mind must think with the aid of categories. We cannot possibly avoid this process. Orderly living depends upon it"(Allport 1954, pg 20). Both in economic and social psychological literature, there are many studies aiming to explain human behavior using categorization (e.g. see Macrae and Bodenhausen 2000 or Fryer and Jackson 2008).

The following example illustrates one possible way how the categorization process works. Consider a consumer who wants to buy a new television. There are an overwhelming number of available alternatives on the market. In order to make a decision, the consumer has to compare a long list of attributes among all products. These attributes include a wide variety of technical features (e.g. screen size, aspect ratio, resolution, contrast ratio, sound system, dimension, weight, etc.), price arrangements (price of the product, payment schedule, service fees), brand, warranty, product support, delivery service, etc. Unless the consumer is an expert on televisions, he may have difficulties in making decision because of this long list of items to consider for each product on the market. What happens most of the time is that after eliminating the obviously undesirable products (e.g. too expensive products), the consumer categorizes all the remaining products on the market so that in each category there are products with some similar attributes. One possible categorization process works as follows. At each step of the process, the consumer chooses an attribute, attaches some criteria to the attribute and partitions the set of products based on the criteria. Say, for example, he considers the screen size attribute and the criteria he attaches is if it is less than 45 inches or between 45 and 55 inches or larger than 55 inches. In this way, he partitions the products into three sets as "products with screen sizes less than 45 inches", "products with screen sizes between 45 and 55 inches" and "products with screen sizes higher than 55 inches". He continues the categorization process by choosing another attribute-criterion tuple, say resolution and a threshold for resolution. He further refines each set in his partition based on this new attribute-criterion tuple and obtains a new partition. In particular, he divides each of the three sets into two as high-resolution and low-resolution, and ends up with 6 sets (categories) in his new partition (low resolution-small size, high resolution-small size, low resolution-medium size, high resolution-medium size, low resolution-big size, high resolution-big size). Repeating this process for a number of steps, he ends up with a final partition of products.[1] Each category in this partition includes a subset of products on the market having similar features. He chooses one product from each category as a representative and compares all the representatives. Then he considers only the category whose representative gives the maximum utility. The final decision is made among the products in that category. This process may lead to a non-optimal decision since the consumer considers only a small subset of products (the category whose representative gives him the highest utility) rather than the whole set. Furthermore, another feature of categorization is that even if their preferences are perfectly aligned, the decisions made by different individuals may not be the same. The reason is that the final partition for a consumer is most likely to be different than the final partition of another consumer, since it depends on the number of steps and the criteria the individuals use.

The main purpose of the paper is to analyze the interaction between fully and boundedly rational people. More specifically, we focus on situations in which both agents work together in a team and the boundedly rational agent has to make a deci-

[1] The number of steps depends on the degree of the individual's bounded rationality. In the limit case (when the individual is fully rational, say, an expert on televisions), the number of steps is sufficiently large that each category contains only one product (finest partition).

sion after receiving a message from the fully rational agent. In such setups, although being fully rational, an agent might suffer from possible non-optimal decisions made by boundedly rational agent. We investigate how a fully rational agent can decrease the expected loss due to bounded rationality. We show that this is possible by manipulating information sent to the boundedly rational agent. Furthermore, we focus on what the fully rational agent can infer about the categories used by boundedly rational agent among their interaction and we show that it is possible to decrease the amount of manipulated information.

The following setting about a fully rational boss and his boundedly rational namesake can be considered as a motivating example for our model. The boss, who can be regarded as the principal, is willing to buy arms for hunting animals. Having a criminal record, he does not meet the conditions for registration of arms with the police forces. Therefore he asks his namesake, who does not have any records of criminal commitment, to buy a weapon for him. The namesake, who can be regarded as the agent, has also some connections in the weaponry black market. Therefore he can buy the weapon from either the legal or illegal market. At this point, it is important to note that the problem we are dealing with is not a principal-agent problem, but an instance of team theory initiated by Roy Radner. In principal-agent problems there is a conflict of interest giving rise to agency cost. In our setting, however, this is not the case since the preferences of the boss and his namesake are perfectly aligned.

Our paper takes as a departure point Dow (1991), where an economic decision problem for a boundedly rational agent visiting two stores and searching for the lowest price is modeled. The bounded rationality of the agent comes from his limitations in memory. More specifically, when the agent is in the second store, he cannot remember the exact price in the first store, but only remembers to which category it belongs. The agent makes a decision by comparing the price in the second store with the representative of the category to which the price in the first store belongs. Dow (1991) characterizes the optimal categorization. We depart from Dow's setting by introducing a fully rational agent and examining the interaction between the two agents.

Considering a similar setting to Dow's (1991), Chen, Iyer and Pazgal (2005) and Luppi (2006) examine the price competitions in the market and show that fully rational firms can take advantage of boundedly rational consumers. Chen, Iyer and Pazgal (2005) depart from Dow's setting by introducing two different types of consumers: totally uninformed consumers, who only consider buying from a specific store as long as the price is below their reservation value, and informed consumers with perfect memory, i.e., fully rational consumers. They characterize the Nash equilibrium of the game in which firms choose pricing strategies and consumers with limited memory choose their categories. It is shown that having bounded rational agents in the market softens price competition. A similar setting is used by Luppi (2006), where there are rational firms on one side and boundedly rational consumers on the other side of the market. Consumers categorize the price space and make their decision based on their categories. She demonstrates that in the presence of boundedly rational consumers two firms competing a la Bertrand depart from the standard equilibrium and make positive profits. The difference between these two papers and ours comes basically from the difference in the settings. In our case fully rational and boundedly rational agents are working as a team and their common aim is to improve the outcome. In other words, the fully rational agent is not trying to take advantage of the boundedly rational agent like in Chen, Iyer and Pazgal (2005) and Luppi (2006), but he is trying to learn how to deal with the latter one in order to achieve the common goal.

Another literature strand to which this paper refers is the field of Information Transmission. Crawford and Sobel (1982) analyze costless strategic communication between a better-informed, fully rational sender and a fully rational receiver. The sender categorizes the support of messages and sends the category to which the realized message belongs instead of sending the real value. This situation arises because the players' preferences are not perfectly aligned. The receiver, after reading the signal, takes an action that affects both his and the sender's payoffs. They show that as the preferences become more aligned, the number of categories the sender uses increases, i.e., the signal becomes more informative. The main difference from our model relies on differences in assumptions: full rationality of both agents and differences in preferences.

Although there have been many studies in economic literature on bounded rationality, studies on interaction between fully and boundedly rational agents are limited in number. To our knowledge all these studies concern with how fully rational agents take advantage of boundedly rational agents (see Rubinstein 1993, Piccione and Rubinstein 2003, Eliaz and Spiegler 2006). The main novelty of our paper lies in our team approach. Both type of agents work together to decrease the inefficiency caused by bounded rationality since their preferences are perfectly aligned.

Another interpretation of our model could be done by using the concept of interpreted signals rather than bounded rationality. This concept, introduced by Hong and Page (2009), is based on the assumption that people filter reality into a set of categories. Hong and Page call the predictions that agents make about the value of the variable of interest by using their own categories as interpreted signals. They state that "... two agents' signals differ if the agents rely on different predictive models. This can only occur if agents differ in how they categorize or classify objects, events or data, if agents possess different data, or if agents make different inferences." In our model, we can think that the interpreted signal of the boss and his namesake may differ due to their different ways to categorize the real world. In this case, the action taken by the namesake may cause a loss for the boss because the good bought by his namesake might be less valuable for the boss than the alternative. In order to decrease this expected loss, the boss manipulates the information he sends to his namesake. Moreover, it might be possible to decrease the amount of manipulated information, since the boss might infer the categorization of his namesake among their interaction.

The organization of the paper is as follows. Section 2.2 describes our two-period toy model, gives the details of learning mechanism and presents results obtained using myopic approach. Section 2.4 recaptures the results using a farsighted approach and

Section 2.5 concludes.

2.2 A Toy Model

We consider a two-period decision problem, in which a fully rational boss wants to buy a product in each period. There are two markets having a huge number of alternatives for the product. The first market is more complex than the second one. A possible explanation for this could be that the first market is a legal market with many regulations and the second market is an illegal one with less complexity. The boss can only observe the products in the first market but cannot perform any transaction since he does not have access to neither of the markets. Therefore he asks his boundedly rational namesake, who has access to both markets, to compare products in the two markets and buy from one. However, cognitive limitations of the namesake do not allow him to fully understand the complex (first) market. Being aware of his limitations, he categorizes the price space for the first market to simplify the decision process and uses the representatives of his categories in order to compare the prices in two markets. The objective of the boss is to minimize the expected loss due to the cognitive limitations of his namesake.

It is common knowledge that the boss is fully and the namesake is boundedly rational. It is also known by both parties that the bounded rationality of the namesake is due to his limited ability in understanding the first market. It should be noted that for simplicity we consider only a single number (price) for a product, but in fact this is a combination of many elements, like the type, quality, brand, and age of the product, length of the warranty, payment arrangements and service fees. It is the multiplicity of such items that makes the namesake unable to fully understand the first market. However, the number of elements that are embedded in prices of the second market is less than those of the first market. In case of an illegal market, for example, there are no warranties, no payment arrangements, no service fees, etc. This is what makes the first market more complicated than the second market. In other words, this is the reason why the namesake is unable to fully understand the first market whereas he understands the second market. Being aware of his limitation, the namesake fully trusts his boss. This is because he knows that their preferences are perfectly aligned and that the boss is fully rational, i.e., that the boss does not have any limitations in understanding the market. Furthermore, the namesake is aware of the fact that the boss may lie to him. However he knows that the reason for that is not that the boss wants to take advantage of him but to improve the outcome. Finally, the boss knows that his namesake fully trusts him.

In the first period, the boss observes the price on the first market, p_1^1, and then reports a price to his namesake, p^1 (not necessarily the true observed value). Receiving the report, the namesake understands to which category the reported price belongs. Then he compares the representative of that category with the price on the second market, p_2^1, and decides from which market to buy. Note that he may take a non-optimal action since he uses the representative instead of the realized price for the product in the first market. Finally, he informs his boss about the price on the second market. Therefore, the boss is able to understand whether the decision was optimal or not.

At the beginning of the second period, the boss updates his beliefs about the namesake's categories by looking at the realized prices on both markets and the action of the namesake. Then the first period is repeated. The notations used for the second period are as follows: p_1^2 stands for the realized price on the first market, whereas p_2^2 is the price on the second market, and p^2 is the reported price.

We assume that prices on both markets are independent and distributed uniformly between 0 and 1. There are three possible types for the namesake. All types use two categories, namely, they all partition the price space in two. In order to do that they choose a cutoff price level. Prices lower than the cutoff level belong to the first category (low) and prices higher than the cutoff belong to the second category (high). The representative of each category which is used to compare with the price on the second market is the median of that category. Types differ in their choices of cutoff price level. Type-1 uses 1/4 as the cutoff level and the representative price of his low category is 1/8, whereas it is 5/8 for his high category. Type-2 uses 1/2 as the cutoff level, thus 1/4 and 3/4 are the representatives for his low and high categories, respectively. Finally, type-3 who uses 3/4 as the cutoff level, has 3/8 and 7/8 as the representatives for his low and high categories, respectively. The prior belief of the boss is that all types are equally likely.

Given the number of categories and price distribution, type-2 uses optimal categories. The cutoff that is used by type-1 is lower than what it should be. It can be thought that type-1 believes that the mean of prices is low. On the contrary, type-3 uses a cutoff higher than the optimal. With the same logic, he can be thought as a person who believes that the mean is high. The distances of cutoffs of type-1 and type-3 from the optimal level of cutoff (1/2) are the same but in reverse directions. This is to say that type-1 and type-3 behave symmetrically.

The objective of the boss is to minimize his expected loss caused by bounded rationality. His action is the price that he reports to the namesake. There are 4 different types of available action that are given in Table 2.1. For example, if the boss chooses to report a price in $[0, 1/4]$, then all the types consider their low categories, and use 1/8, 1/4, 3/8 as representative, respectively.

	type-1	type-2	type-3	used prices
$p^1 \in [0, \frac{1}{4}]$	L	L	L	$\{ \frac{1}{8}, \frac{1}{4}, \frac{3}{8} \}$
$p^1 \in [\frac{1}{4}, \frac{1}{2}]$	H	L	L	$\{ \frac{5}{8}, \frac{1}{4}, \frac{3}{8} \}$
$p^1 \in [\frac{1}{2}, \frac{3}{4}]$	H	H	L	$\{ \frac{5}{8}, \frac{3}{4}, \frac{3}{8} \}$
$p^1 \in [\frac{3}{4}, 1]$	H	H	H	$\{ \frac{5}{8}, \frac{3}{4}, \frac{7}{8} \}$

Table 2.1: Action Space

2.3 Myopic Approach

In this section we consider a myopic approach. That is, we assume that the boss is only concerned with the expected loss of the current period, not with the aggregate expected loss. A farsighted approach is considered in the following section. Table 2.2 shows the expected loss for each possible combination of price realizations on the first market (p_1^1) and actions taken by the boss. Each number in bold gives the minimum expected loss for the relevant price realization. Given the myopic approach, the action that corresponds to each bold number is the optimal choice of action for the boss for the relevant price realization. For example, if the boss observes a price on the first market that belongs to interval $[0, 1/8]$, he will report a price that belongs to interval $[0, 1/4]$. At this point we make another assumption about the boss. We assume that he prefers to tell the truth whenever it is among the optimal actions. This assumption together with the fact that $[0, 1/8] \subset [0, 1/4]$ (truth-telling is among optimal actions) imply that the boss reports the true value in this case. However, if $p_1^1 \in [1/4, 3/8]$ it is optimal to report $p^1 \in [0, 1/4]$. In this case, the reported price is less than its true value (the boss under-states the price). The other case in which the boss lies is when $p_1^1 \in [5/8, 3/4]$. The reported price in this case is $p^1 \in [3/4, 1]$, i.e., it is higher than its true value (the boss over-states the price).

Observed Price\ Report	$p^1 \in [0, \frac{1}{4}]$	$p^1 \in [\frac{1}{4}, \frac{1}{2}]$	$p^1 \in [\frac{1}{2}, \frac{3}{4}]$	$p^1 \in [\frac{3}{4}, 1]$
$p_1^1 \in [0, \frac{1}{8}]$	**9**	29	57	93
$p_1^1 \in [\frac{1}{8}, \frac{1}{4}]$	**3**	15	35	63
$p_1^1 \in [\frac{1}{4}, \frac{3}{8}]$	**3**	7	19	39
$p_1^1 \in [\frac{3}{8}, \frac{1}{2}]$	9	**5**	9	21
$p_1^1 \in [\frac{1}{2}, \frac{5}{8}]$	21	9	**5**	9
$p_1^1 \in [\frac{5}{8}, \frac{3}{4}]$	39	19	7	**3**
$p_1^1 \in [\frac{3}{4}, \frac{7}{8}]$	63	35	15	**3**
$p_1^1 \in [\frac{7}{8}, 1]$	93	57	29	**9**

Table 2.2: Expected Loss (common multiplier: $\frac{1}{6\times 8^3}$)

Table 2.2 results in the following reaction function:

$$R(p_1^1) = \begin{cases} \text{report } p^1 \in [0, \frac{1}{4}] & \text{if } p_1^1 \in [\frac{1}{4}, \frac{3}{8}], \\ \text{report } p^1 \in [\frac{3}{4}, 1] & \text{if } p_1^1 \in [\frac{5}{8}, \frac{3}{4}], \\ \text{report the true price} & \text{otherwise.} \end{cases} \tag{2.1}$$

Under-statement of the price occurs only if $p_1^1 \in [1/4, 3/8]$ and receiving this report all types use their low (L) categories (see Table 2.1). However, if $p_1^1 \in [1/4, 3/8]$ and the boss reports the true value of the price rather than under-stating, type-1 uses his high

(H) category whereas type-2 and 3 stick to their low (L) categories. So, it is only type-1 who is affected by under-statement. Since the boss prefers to tell the truth whenever it is among the optimal actions and under-statement does not affect other types, the boss uses this strategy only if type-1 is among possible types when the observed price belongs to interval $[1/4, 3/8]$.

Over-statement of the price occurs only if $p_1^1 \in [5/8, 3/4]$. By the same reasoning above, over-statement affects only type-3, not others. Therefore, the boss uses this strategy only if type-3 is among possible types when $p_1^1 \in [5/8, 3/4]$. Otherwise, he prefers to report the truth.

Figure (2.1) represents the reaction function of the boss. Here, we can observe that the behavior of the boss is symmetric around 1/2. The arrow on the left represents under-statement and in case of under-statement only type-1 switches category, whereas the arrow on the right represents over-statement and only type-3 switches category in this case. As noted earlier, these types behave symmetrically which results in symmetric behavior of the boss.

Figure 2.1: Reaction Function

At the end of the first period, the boss updates his beliefs by looking at the prices realized in both markets and the action taken by the namesake. To see how this works let us consider the following example. Say, $p_1^1 \in [0, 1/8]$, $p_2^1 \in [1/8, 1/4]$. Given the price on the first market, the boss reports the true value (see (2.1)). In this case, the representative price is 1/8 for type-1, 1/4 for type-2 and 3/8 for type-3. The namesake, comparing the representative price with the price on the second market, buys the good from the first market if he is of type-1 and buys from the second market if he is of type-2 or type-3. If the product is bought from the first market, the boss understands that his namesake is of type-1 and updates his belief such that with probability 1 the namesake is of type-1. If instead, it is bought from the second market, the boss updates his belief such that with probability 1/2 the namesake is of type-2 and with probability 1/2 the namesake is of type-3.

Figure 2.2 summarizes the learning process at the end of period-1. Numbers in bold stand for the numbers of possible types of the namesake. The boss starts with three possible and equally likely types. The probability that he learns the exact type, i.e., that the number for possible types is 1, at the end of the first period is $\frac{3}{32} = 0.09375$. The probability that the number of possible types decreases to 2 (elimination of one type) is $\frac{3}{16} = 0.1875$, and finally the probability that the boss learns nothing is $\frac{23}{32} = 0.71875$.

Figure 2.2: Learning Process, 1st Period

The boss starts the second period with updated beliefs. The objective is again to minimize the expected loss caused by bounded rationality. When type-1 is among possible types and the observed price on the first market in the second period (p_1^2) belongs to the interval $[1/4, 3/8]$, he uses the under-statement strategy described above. Furthermore, when type-3 is among possible types and $p_1^2 \in [5/8, 3/4]$, he uses the over-statement strategy. In all the other cases he reports the true observed value. The reaction function for the second period coincides with the one for the first period (2.1) if both type-1 and type-3 are among possible types.

Figure 2.3: Learning Process, 2nd Period

Figure 2.3 summarizes the learning process for the whole game. If the boss figures out the exact type of the namesake (arrives to node 1) at the end of the first period, there is nothing left to learn and he continues the second period with the relevant strategy. If he arrives to node 2 at the end of the first period, the learning process continues and he might either figure out the type and arrive to node 1 or not learn anything and stay in node 2. If he does not learn anything about the type at the end of the first period (stays at node 3), there are three possibilities for the second period. He might figure out the exact type and arrive to node 1, or he might eliminate only one possible type and arrive to node 2, or he might not learn anything and stay at node 3. The overall probability that the boss figures out the exact type of the namesake by the end of the game is 0.19238, that he eliminates only one possible type is 0.29102 and that he does not learn anything is 0.51660.

The transition matrix of the learning process is given in Table 2.3. It is a finite Markov Chain and has three ergodic states. According to the Theorem by Kemeny and Snell (1976), the probability after n steps that the process is in an ergodic state tends to 1, as n tend to infinity. This means that if the game is repeated for n periods the probability that the boss learns the exact type of the namesake tends to 1 as n gets larger.

possible types	{1,2,3}	{1,2}	{1,3}	{2,3}	{1}	{2}	{3}
{1,2,3}	0.71875	0.08333	0.02083	0.08333	0.04167	0.01042	0.04167
{1,2}	0	0.84375	0	0	0.07813	0.07813	0
{1,3}	0	0	0.75000	0	0.12500	0	0.12500
{2,3}	0	0	0	0.84375	0	0.07813	0.07813
{1}	0	0	0	0	1	0	0
{2}	0	0	0	0	0	1	0
{3}	0	0	0	0	0	0	1

Table 2.3: Transition Matrix

The relationship between the number of periods and the probability of learning the exact type is given in Table 2.4. The probability increases in the number of periods, and it becomes almost 1 after 30 periods.

n	p
5	0.46344
7	0.60419
10	0.75543
15	0.89388
20	0.95465
30	0.99178

Table 2.4: Number of periods/probability

A crucial point to be noted is that in this section we use a myopic approach to solve the optimization problem. This means that we assume the boss is concerned only with the expected loss of the period he is in. Whereas with a farsighted approach, he considers the overall expected loss that is the sum of discounted expected losses. However, both approaches yield the same results with the given available types. In this setting, a manipulated message affects only one type, while other types stick to their category that they would consider without the manipulated message. In other words, a strategy that needs to be used in order to decrease the expected loss caused by one type does not conflict with the strategies that need to be used for other types. For example, the under-statement strategy is used whenever type-1 is among possible types. The fact that type-2 and/or type-3 are among possible types does not change this strategy, because it induces only type-1 to change his category, not the other types.

Therefore, the boss can continue to use the reaction function given in (2.1) even if he knows the exact type of the namesake. It should be noted that if he does so, he might report a manipulated price although reporting the true value is also among optimal actions. Even tough this violates our assumption that the boss prefers reporting the truth whenever it is possible, it yields the same expected loss for the boss. This fact ensures that he can use the same reaction function for each period no matter if he is farsighted or myopic. In the following section we show that myopic and farsighted optimizations do not always coincide.

2.4 Farsighted Approach

In this section, we consider a farsighted approach. That is, we assume that the objective of the boss is to minimize the sum of discounted expected losses. We modify the model by changing the possible types. Here, we assume that the namesake has two possible types. The first type uses two categories (low and high) and his cutoff price level is 1/3. Therefore he uses 1/6 as the representative for low category (L) and 2/3 for high category (H). The second type uses three categories (low, medium and high) and his cutoff price levels are 1/3 and 2/3. Thus 1/6, 1/2 and 5/6 are the representative prices for his low (L), medium (M), and high (H) categories, respectively. The prior belief of the boss is that both types are equally likely.

In this setting, the boss can choose his strategy among three different types of action, that are represented in Table 2.5. If he reports a price belonging to $[0, 1/3]$, both types use low categories and 1/6 as representative price. If he reports $p^i \in [1/3, 2/3]$, then type-1 uses his high category and 2/3 as his representative for the first market price, and type-2 uses his medium category and 1/2 as the representative ($i \in \{1, 2\}$ represents the period). Finally, if the boss reports $p^i \in [2/3, 1]$, both types will use high categories and type-1 uses 2/3 whereas type-2 uses 5/6 as representative price.

	type-1	type-2	used prices
$p^i \in [0, \frac{1}{3}]$	L	L	$\{ \frac{1}{6}, \frac{1}{6} \}$
$p^i \in [\frac{1}{3}, \frac{2}{3}]$	H	M	$\{ \frac{2}{3}, \frac{1}{2} \}$
$p^i \in [\frac{2}{3}, 1]$	H	H	$\{ \frac{2}{3}, \frac{5}{6} \}$

Table 2.5: Action Space

We solve the optimization problem by backward induction since we are considering a farsighted approach. If the boss does not learn anything about the type of his namesake during the first period, he starts second period with the belief that both types are equally likely. Following the same reasoning of the previous section, we get the following reaction function:

$$R(p_1^2 |\, type1 \,\&\, type2) = \begin{cases} \text{report } p^2 \in [0, \frac{1}{3}] & \text{if } p_1^2 \in [\frac{1}{3}, \frac{23}{60}], \\ \text{report the true price} & \text{otherwise,} \end{cases} \tag{2.2}$$

where $R(p_1^2 |\, type1 \,\&\, type2)$ stands for the reaction function for the second period given that both type-1 and type-2 are among possible types. And the expected loss in this case is

$$E_2(L |\, type1 \,\&\, type2) = \frac{151}{17280}, \tag{2.3}$$

where $E_2(L)$ denoted the expected loss in the second period. If the boss learns that his namesake is of type-1 during the first period, his reaction function for the second period is

$$R(p_1^2 |\, type1) = \begin{cases} \text{report } p^2 \in [0, \frac{1}{3}] & \text{if } p_1^2 \in [\frac{1}{3}, \frac{25}{60}], \\ \text{report the true price} & \text{otherwise.} \end{cases} \tag{2.4}$$

The expected loss in this case is

$$E_2(L|\, type1) = \frac{7}{576} \tag{2.5}$$

If the boss learns that his namesake is of type-2 during the first period, his reaction function for the second period is to always report the true value, since given the number of categories and the distribution of the price this type uses optimal categorization. In this case, the expected loss is

$$E_2(L|\, type2) = \frac{1}{216} \tag{2.6}$$

Now, we move to the first period. If the boss, after observing the price on the first market, reports $p^1 \in [0, 1/3]$ then both types will use low category and 1/6 as representative price for the first market (see Table 2.5). Since both types will be using the same representative, they will behave in the same way. Therefore, it will be impossible for the boss to distinguish between the two, i.e., the boss will not learn anything about the type of his namesake and will continue with his initial belief. In this case, the overall expected loss will be the sum of the expected loss from the first period and the expected loss of the second period multiplied by the discount factor of the boss, $\delta \in [0, 1]$.

$$\int_{p_1^1}^{1/6} (p_2^1 - p_1^1)\, dp_2^1 + \delta\, E_2(L|\, type1\,\&\, type2). \tag{2.7}$$

If the boss reports $p^1 \in [1/3, 2/3]$ then type-1 will use his high category and 2/3 as representative price for the first market, whereas type-2 will use his medium category and 1/2 as representative (see Table 2.5). If the price realization on the second market is below 1/2 then both types will act in the same way and will buy from second market. If it is greater than 2/3 both types will act again in the same way and will buy from the first market. However, if $p_2^1 \in [1/2, 2/3]$, type-1 will buy from the second whereas type-2 will buy from the first market. Thus, the boss will learn the exact type of his namesake only if $p_2^1 \in [1/2, 2/3]$ and this occurs with probability 1/6. Hence, with this strategy the probability that he boss figures out that his namesake is of type-1 is $\frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$, which is the same for type-2. In this case, the expected loss is

$$\frac{1}{2}\int_{p_1^1}^{2/3} (p_2^1 - p_1^1)\, dp_2^1 + \frac{1}{2}\int_{p_1^1}^{1/2} (p_2^1 - p_1^1)\, dp_2^1 + \delta\Big[\frac{5}{6}E_2(L|type1\&type2) + \frac{1}{12}E_2(L|type1) + \frac{1}{12}E_2(L|type2)\Big] \tag{2.8}$$

If the boss reports $p^1 \in [2/3, 1]$ then type-1 will use his high category and 2/3 as representative price for the first market, whereas type-2 will also use his high category and 5/6 as representative (see Table 2.5). In this case, both types will act in the same way unless the price realization on the second market belongs to $[2/3, 5/6]$. Thus, the boss will learn the exact type of his namesake only if $p_2^1 \in [2/3, 5/6]$ and this occurs with probability 1/6. Hence, the expected loss in this case is

$$\frac{1}{2}\int_{p_1^1}^{2/3}(p_2^1-p_1^1)\,dp_2^1+\frac{1}{2}\int_{p_1^1}^{5/6}(p_2^1-p_1^1)\,dp_2^1+\delta\,[\frac{5}{6}E_2(L|type1\&type2)+\frac{1}{12}E_2(L|type1)+\frac{1}{12}E_2(L|type2)] \tag{2.9}$$

Inserting (2.3), (2.5) and (2.6) into (2.7), (2.8) and (2.9) we derive the reaction function of the boss as follows:

$$R(p_1^1) = \begin{cases} \text{report } p^1 \in [0, \frac{1}{3}] & \text{if } p_1^1 \in [0, a], \\ \text{report } p^1 \in [\frac{1}{3}, \frac{2}{3}] & \text{if } p_1^1 \in [a, \frac{2}{3}], \\ \text{report } p^1 \in [\frac{2}{3}, 1] & \text{if } p_1^1 \in [\frac{2}{3}, 1], \end{cases} \tag{2.10}$$

where $a = \frac{2760-\delta}{7200}$.

Taking into account the assumption that the boss prefers to tell the truth whenever it is among the optimal actions, the above reaction function becomes

$$R(p_1^1) = \begin{cases} \text{report } p^1 \in [0, \frac{1}{3}] & \text{if } p_1^1 \in [\frac{1}{3}, a], \\ \text{report the true price} & \text{otherwise.} \end{cases} \tag{2.11}$$

The reaction function (2.11) shows that the optimal strategy depends on the discount factor of the boss. If the boss concentrates only on the expected loss of the current period and does not take into account future expected losses, i.e., if $\delta = 0$, then myopic and farsighted optimization results coincide. Whenever the boss considers current losses together with future losses (i.e., whenever $\delta \neq 0$), there is a difference, albeit small, between the reaction functions resulted from myopic and farsighted approach.

Here we consider a game with only two periods. Before the last, there is only one period in which the boss can learn something about the type of his namesake. Furthermore, he has only one period, namely the second period, where he can use this information. This is the reason why the difference between reaction functions resulting from myopic and farsighted optimizations is so small. This difference is increasing in the number of periods of the game as well as in δ. A boss with high δ is more concerned about future loss compared to a boss with lower δ. Therefore he is more willing to invest in learning the type of his namesake in order to decrease his future loss.

2.5 Conclusion

We have constructed a model in order to study the interaction between fully and boundedly rational agents when they are parts of the same team and have perfectly aligned preferences. In an environment with abundance of information (type, quality, brand, age of the good, length of the warranty, payment arrangements and service fees), boundedly rational agents are having difficulties in making decision due to their cognitive limitations. In order to simplify the situation, they try to group events, objects or numbers into categories. In our model we consider a boundedly rational agent who partitions the price space into connected sets. The decision made by this agent might be non-optimal in some cases, since he is using categories instead of realized prices and regards prices belonging to the same category as equal.

Assuming different types for the boundedly rational agent and that types differ only in categories they use, we show that during his interaction, the fully rational agent may learn about the type of the boundedly rational agent, and using this additional information, he can improve the outcome. The probability that he learns the type of the boundedly rational agent increases in the length of this interaction, whereas it decreases in the number of available types.

Finally, we show that myopic and farsighted approaches yield different results in some cases, depending on the available types. This difference is caused by the tradeoff between experimenting for the future and starting to cope with the problem right away.

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Chapter 3

The Power of Diversity over Large Disjunctive Tasks

3.1 Introduction

"Two heads are better than one" or "A camel is a horse designed by committee"? This question has long been studied in the literature and raised equivocal research findings. The former adage has been supported by a number of studies indicating that the functional diversity that comes along with the formation of a group results in improved group performance.

Simons et al. (1999) study how the interaction between top management team diversity variables (like perceived environmental uncertainty diversity, education level diversity, functional background diversity, age diversity) and debate influences the financial performance. By analyzing the data from the top management teams of 57 manufacturing companies in electronics industry, they conclude that the interaction of debate with more job-related types of diversity positively affects sales and profitability.

More evidence supporting the first adage is provided by Blinder and Morgan (2005). Their departing point is the observation that many important decisions, including those on monetary policy, are made by groups rather than individuals. In order to study the performance of monetary policies made by committees against the ones made by individuals, they run two laboratory experiments. Even though the set-ups of the experiments are very different from each other, they almost yield the same results: groups are not slower than individual in reaching decisions and groups perform better than individuals.

Kocher and Sutter (2005) study individual versus group decision makers in an experimental beauty-contest game. The results suggest that there is no significant difference between individual and group decision makers in the first round; however, when the game is repeated, groups exhibit more rational behavior; that is, their decisions are closer to the equilibrium of the game. This implies that groups are able to learn and adapt much faster to the environment. Therefore, when competing against individuals, groups outperform them in terms of payoff.

More support for the first adage is offered by Cooper and Kagel (2005), who study differences between individuals and teams in learning and adjusting process. Based on the results from a signaling game experiment, they find that teams, playing more strategically, outperform individuals. Moreover, as it becomes more difficult to learn to play strategically, the advantage of being part of a team is even greater.

The final first-adage-supporting study that we introduce here is from the ecological literature. Liker and Bókony (2009) compare small versus large groups of house sparrows in terms of problem solving performance. Large groups of birds are both faster and better in foraging performance. This superiority of large groups is not only due to high number of attempts but also due to high effectiveness in problem solving and this fact is attributed to the diversity of large groups in skills and experiences by the authors.

However, there have been a number of other studies that indicate that forming a group creates a negative or non-significant impact on performance. This finding is mainly attributed to poor relations impairing collective effort. For example, in brainstorming groups, members are asked to generate as many ideas as possible and encouraged to build on the ideas of others. It is observed that groups generate much fewer ideas then when agents work in isolation. Possible reasons for this unexpected loss in the number of ideas are fear of negative evaluation, social loafing, free-riding and production blocking (Paulus and Yang, 2000).

Similarly, the psychological literature produces two main approaches to explain why individuals might be better than groups in decision making: groupthink and risky shift. Groupthink is mainly self-censorship and group conformity that may lead to oppressing those who disagree, closed-mindedness, stereotyping, and incomplete survey of alternatives/objectives. Risky shift is group polarization that can be explained as the tendency of people to make riskier decisions when they are in a group.

In our model, we eliminate all but one problems listed above by the use of a sequential search procedure and the assumption that the aim of the team is to maximize a given one-to-one objective function. A tentative solution found by an agent is handed over the next agent and he is asked to improve it. This process continues, cycling over all agents, until no one can find further improvements. The sequential nature of this interaction dispels social loafing and free-riding problems. Furthermore there is never a disagreement, since improvements are always objective. This fact cuts out the fear of negative evaluation. Finally there is no group polarization since there is no uncertainty involved in the model. The only problem that could impair the collective effort is groupthink that we embed in our model.

The aim of this paper is to study the effect of diversity in problem-solving ability on the team performance when a disjunctive task over a large solution space is faced. A disjunctive task refers to a task in which all members of a team work together in order to produce a single solution. We model the limitations in agents' abilities by using partitions. We show that under some conditions a team of two agents will succeed in solving a difficult problem with a large solution space. Even though neither can do it alone, when they pool their abilities the power of diversity helps them to achieve the goal. In other words, it takes two to solve a problem. We also provide sufficient conditions under which teaming up does not always guarantee success. This happens when the agents are not sufficiently creative. In this case, forming a team improves the outcome but does not necessarily ensure that the team finds the optimal solution to the problem.

Afterwards, we consider situations in which the ability of agents is adversely affected by the complexity of the solution space. As the solution space gets larger, the problem becomes more complex since the number of elements among which the search is done increases. We use graph theory in order to model the ability of an agent that is diversely affected by the complexity. More specifically, we use the *Erdős-Rényi Model* of random graphs. We show that small sized teams will still solve the problem. The size of a successful team is determined by the cardinality of the solution space, i.e., by the complexity of the problem.

Finally, we deal with the effect of groupthink on the performance of a team. When the members of a team begin to think and act alike, its performance decreases. In other words, groupthink harms the performance and impairs the power of diversity. We study the case in which the abilities of agents are both positively correlated and adversely affected by the complexity of the problem. We show that the size of a team and the magnitude of correlation are the key components in these situations. Unless there is a perfect correlation, a team can still solve the problem but it requires a larger team to achieve this goal.

The organization of the paper is as follows. Section 3.2 describes the model and presents some technicalities used in our analysis. Section 3.3 gives sufficient conditions for a team of two agent to be always successful. We show that our model is equivalent to the model used in Hong and Page (2004), and compare them by pointing out the novelties of our perspective in Section 3.4. Section 3.5 gives sufficient condition under which teaming up does not guarantee success. We continue by assuming that abilities are adversely affected by the complexity of the problem in Section 3.6 and present our result that small teams of more than two people will still solve the problem. Section 3.7 is concerned with groupthink and shows that larger teams are necessary to solve a problem. Finally, Section 3.8 concludes.

3.2 The model

There is a team T of m problem-solving agents of limited ability who attempt to maximize an objective function V that maps a finite set X of n solutions into real numbers. The function $V : X \to \mathbb{R}$ is one-to-one; in particular, it has a unique maximizer at x^*. The task of the team is locating x^* and it can be carried out *disjunctively*: if only one of the agents finds x^*, the task is accomplished. However, due to their limited ability, the agents may fail to do so.

We represent the limited problem-solving ability of each agent i by a partition Π_i of X. Suppressing momentarily subscripts, consider one agent. The mutually disjoint and exhaustive classes constituting the partition Π are called *blocks.* The agent can find the best solution within the block is working on, but he is impervious to the other blocks unless his attention is redirected there by someone else. The *availability map* $\Pi : X \to 2^X \setminus \emptyset$ describes the search space of the agent: for each $x \in X$, $\Pi(x)$ is the set of solutions that he can explore when he is aware of a candidate solution x. The availability map is *consistent*: x is in $\Pi(x)$ for all x in X; that is, the search space of an agent always contains the candidate solution.

We assume that each agent correctly identifies and compares the values of V for each solution he examines. Therefore, given a candidate solution x, he explores the search space $\Pi(x) \subseteq X$ and finds $x_0 = \arg\max_{y\in\Pi(x)} V(y)$: since $V(x_0) \geq V(x)$, the search is always (weakly) improving. More generally, when the agent has access to an initial subset $S \subseteq X$ of candidate solutions, he explores $\Pi(S) = \cup_{x\in S}\Pi(x)$ and finds the solution $x_0 = \arg\max_{y\in\Pi(S)} V(y)$.

When two or more agents work together, they can pool their abilities and expand their search spaces. Following the general framework developed in Hong and Page (2001), it is not necessary to specify the minute details of their interaction but, for the sake of clarity, we provide an illustrative example inspired by sequential search. Number agents from 1 to m. Agent 1 works on the task from an initial set S of candidate solutions and finds a tentative solution x_1^* that he hands over to Agent 2. She uses x_1^* as a starting point and explores the search space $\Pi_2(x_1^*)$ from her partition Π_2, locating a (possibly new) tentative solution $x_2^* = \arg\max\{V(x) : x \in \Pi_2(x_1^*)\}$. Clearly, $V(x_2^*) \geq V(x_1^*)$ so the search is (weakly) improving. The process continues, cycling over all agents, until no one can find further improvements; then the search stops and the current tentative solution becomes final. If, along the process, no agent ever gets to explore that block in his partition that contains x^*, the optimal solution is not found and the team fails (although it may succeed in discovering a very good local optimum).

We formalize the problem-solving ability of a team $T = \{1, 2, \dots, m\}$ as follows. Each agent i in T is associated with a partition Π_i that represents his ability to explore the search space. Starting with an initial set S of candidate solutions, each agent explores $\Pi(S)$; analogously, any team of two agents $i = 1, 2$ explore $\Pi_i(\Pi_{3-i}(S))$; and, more generally, the team *jointly explores* the union of all subsets representable as $\Pi_{i_k}(\Pi_{i_{k-1}} \dots (\Pi_{i_1}(s))) \subseteq S$ for any k and any sequence $i_1, i_2, \dots, i_k$.

When X is finite, there is a simple characterization of the set of solutions that are jointly explored by a team. The finest common coarsening of the partitions $(\Pi_i, i \in T)$ is another partition M called their *meet.* Then, from an initial subset S of candidate solutions, the team jointly explores $M(S)$. In other words, the meet describes the problem solving abilities commonly attained by the team. The next short subsection conveniently collects a few technicalities to be used later in the paper.

3.2.1 Technicalities

We assume that X is a finite set with n elements; when useful, we write it as X_n to make the number of elements evident. We label the elements of X_n with the integers $\{1, \ldots, n\}$. Following custom, we list the blocks of a partition of X in increasing order of their least elements and the elements of each block in increasing order. For instance, the blocks of the partition $\{3, 4, 5\}, \{6, 1\}, \{2\}$ of a set with six elements are listed $\{1, 6\}, \{2\}, \{3, 4, 5\}$. For brevity, we often simplify notation and write the partition as $16|2|345$.

We write $\Pi \preceq \Pi'$ to denote that Π is coarser than Π'. Clearly, an agent endowed with partition Π has a higher problem-solving ability than another agent endowed with Π'. The trivial partition Π_0 that has X as its unique block satisfies the property $\Pi_0 \preceq \Pi$ for any partition Π; therefore, an agent endowed with the trivial partition has the highest problem-solving ability of all and, indeed, will find the global optimum from any starting point. The set of all the partitions of X partially ordered by the refinement relation $\preceq$, is a lattice. In particular, the notation $\Pi_1 \wedge \Pi_2$ denotes the finest coarsening of Π_1 and Π_2; analogously, $\Pi_1 \vee \Pi_2$ stands for their coarsest refinement. The meet of $(\Pi_i, i \in T)$ is $M = \bigwedge_{i \in T} \Pi_i$.

The number of partitions for a (finite) set X_n of n elements is given by the *Bell number* B_n. As an example, consider $X_3 = \{1, 2, 3\}$. The set of all its partitions, denoted by (X_3), is $(X_3) = \{123, 1|23, 2|13, 3|12, 1|2|3\}$ and thus $B_3 = |(X_3)| = 5$. The first few Bell numbers are $B_0 = 1$, $B_1 = 1$, $B_2 = 2$, $B_3 = 5$, $B_4 = 15$, $B_5 = 52$, and $B_6 = 203$. The Bell numbers satisfy the recursive formula

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$$

as well as the Dobinsky's formula

$$B_n = \sum_{k=0}^{+\infty} k^n \cdot \left(\frac{e^{-1}}{k!} \right)$$

according to which B_n is the n-th moment of a Poisson distribution with expected value 1.

We are to study large spaces of possible solutions, when n is large. An asymptotic formula for the Bell numbers as $n \uparrow +\infty$ is

$$B_n \sim \frac{1}{\sqrt{n}} r^{n+1/2} e^{r-n-1}$$

where r is defined as the root of $re^r = n$; see Pitman (2006). It is also known that for every $\varepsilon > 0$ there exists n_ε such that

$$\left(\frac{n}{e \ln n} \right)^n < B_n < \left(\frac{n}{e^{1-\varepsilon} \ln n} \right)^n \tag{3.1}$$

for all $n > n_\varepsilon$; see De Bruijn (1958). Recently, Berend and Tassa (2010) have proved a convenient (albeit less tight) upper bound

$$B_n < \left(\frac{0.792n}{\ln(n+1)}\right)^n$$

that holds for all n.

3.3 Two heads are better than one

This section studies the performance of a randomly chosen team of problem-solvers. We fix both the number n of solution in X_n and the size m of the team T_m. A team is *always successful* if it can find the optimal solution in X_n from any starting point; that is, if the meet of the partitions of its agents is the trivial partition. We assume that the agents in the team are randomly chosen so that the composition of the team is stochastic and we ask what is the probability that the team is always successful.

Since a problem-solver is represented by a partition, we need to construct a model of how agents are randomly drawn from the set of partitions of X_n . To stack the deck, we borrow from Hong and Page (2004) the assumption that no single agent can find the optimum alone.

Assumption 1 (Difficulty). No agent in the team is endowed with the trivial partition.

The simplest way to build a random model of team formation is to assume that agents are independently chosen according to the uniform distributions over all partitions. Formally, let (X_n) denote the set of all partitions of X_n. The number of elements of (X_n) is the Bell number B_n. The uniform distribution on (X_n) assigns probability B_n^{-1} to each partition. We call this the *uniform model.* In order to satisfy the difficulty assumption, it suffices to attach zero probability to the trivial partition and update the uniform model by giving probability $(B_n - 1)^{-1}$ to every nontrivial partition. We call this the *uniform model with difficulty.* This is the model studies in this section. Section 3.5 and 3.6 consider two alternative models.

3.3.1 Small teams over small solution spaces

Consider two agents and three elements, so that $m = 2$ and $n = 3$ with $X = \{1, 2, 3\}$. There are $B_3 = 5$ possible partitions, namely: 123; 1|23; 2|31; 3|12; 1|2|3. Each of these five partitions has identical probability 1/5 and each pair of partitions (one for each agent) has identical probability 1/25 of occurring. Denoting by $\times$ the event that the meet of two partitions is the trivial coarse partition, we obtain the left-hand side of Table 3.1 where it is easy to check that there are 15 favorable events out of 25 possible ones. Since the joint probability distribution is uniform, the probability that a team of two agents is always successful under the uniform model is $15/25 = 0.6$.

	123	1|23	2|31	3|12	1|2|3
123	×	×	×	×	×
1|23	×		×	×	
2|31	×	×		×	
3|12	×	×	×		
1|2|3	×				

	1|23	2|31	3|12	1|2|3
1|23		×	×	
2|31	×		×	
3|12	×	×		
1|2|3				

Table 3.1: When the meet of two partitions is the trivial partition.

Under the difficulty assumption, the probability that an agent is endowed with the trivial partition 123 is zero: so we delete the first row and the first column and get the right-hand side of Table 3.1. Now, there are 6 favorable events out of 16. Thus, under the uniform model with difficulty, the probability that a team of two agents is always successful is $6/16 = 0.375$.

Using similar reasoning, we have computed the probability that a team of $m = 2$ agents is always successful under the uniform model with difficulty for $n = 2, 4, 5$ as well. The results are summarized in Table 3.2. It is obvious that the probability that the team is always successful is higher when no difficulty is assumed.

$n = 2$	$n = 3$	$n = 4$	$n = 5$
0	$6/16 = 0.375$	$90/196 = 0.459$	$1240/2601 \approx 0.477$

Table 3.2: Probability that a team of 2 agents is always successful.

In the case when $n = 2$, there are only two possible partitions: 12 and 1|2. Under the difficulty assumption, the probability that an agent is endowed with the trivial partition 12 is zero. Therefore, there is only one possible partition, namely: 1|2. This means that all the agents are endowed with the same ability and hence there is no diversity. The fact that the meet of a partition with itself is again itself implies that the probability of being always successful for a team with no diversity is zero. This holds no matter how large the team is.

Table 3.2 shows that the probability that a team of 2 agents is always successful is increasing in n under the uniform model. The basic idea behind this fact is the following: In (X_n), the partitions with about an average number of blocks have the highest occurrence. The more the number of blocks goes away from this average, the less the occurrence. For example, the set (X_5) has 51 elements, out of which 25 have three blocks, 15 have two blocks, 10 have four blocks and, finally, 1 has five blocks. Since the uniform model assigns the same probability to each partition in (X_n), the overall probability assigned to partitions with about an average number of blocks is higher. Furthermore, as n increases this overall probability for partitions with about an average number of blocks increases. This leads to the fact that as n increases it is getting more likely for the members of a team to have an average number of blocks in their partitions. Hence, the overall effect of an increase in n results in an increase in the probability that a team of 2 agents is always successful. In fact; Theorem 3.3.2, given in the next section, proves that this probability goes to one when $n \uparrow \infty$.

3.3.2 Small teams over large solution spaces

This section studies what is the probability that a team of m people is always successful under the uniform model when the space X_n of solutions grows large. We find that (with or without difficulty), the probability approaches one for any $m \geq 2$ when $n \uparrow \infty$. On the other hand, the probability that an agent is always successful is zero by the difficulty assumption. (And even without difficulty it goes down to zero as $1/B_n$.) This is a sharp difference: when the space of solutions is sufficiently large, we are almost sure that no single agent is able to solve it but a team of even only two people will. Tossing many solutions around unclench the power of diversity.

Let P_{mn} the probability that the meet of $m \geq 2$ partitions of X_n chosen under the uniform model with difficulty is the trivial partition. Then, $P_{mn} \to 1$ as $n \uparrow \infty$. Denote by P'_{mn} the probability that the meet is the trivial partition under the uniform model without difficulty. Theorem 5 in Pittel (2000) shows that

$$P'_{mn} = 1 - O\left(\frac{\log^{m+1} n}{n^{m-1}}\right)$$

from which it follows immediately that $\lim_{n\uparrow\infty} P'_{mn} = 1$. We establish the claim by proving that $P'_{mn}/P_{mn} \to 1$ as $n \uparrow \infty$.

The difficulty assumption removes the trivial partition from the set of possible partitions of each agent. This decreases by one the number of equally likely partitions for each agent, from B_n to $B_n - 1$. On the other hand, this removal does not affect the total number of m-tuples of partitions whose meet is *not* the trivial partition. Denote this number by W_n. Then

$$\lim_{n\uparrow\infty} \frac{P'_{mn}}{P_{mn}} = \frac{W_n/(B_n)^m}{W_n/(B_n-1)^m} = \left(\frac{B_n - 1}{B_n}\right)^m = 1.$$

Note that the proof shows also that $P'_{mn} \to 1$, so Theorem 3.3.2 holds for the uniform model regardless of whether difficulty is assumed or not.

3.4 A comparison with Hong and Page

The seminal reference for the power of diversity is the work of Hong and Page (2001, 2004), recently popularized in Page (2007). This section shows that the model used in Hong and Page (2004) for the limited problem-solving ability of an agent is equivalent to ours. Moreover, we abide by the same three main assumptions (Difficulty, Diversity, and Uniqueness) made there. However, even though we have made a conscious effort to stay close to the paradigm, our perspective is novel in two important respects.

First, from a methodological point of view, we introduce partitions as an effective tool to represent the limitations in problem-solving ability. Second, and perhaps more importantly, from a substantial point of view, the main result in Hong and Page (2004) assumes that the number m of agents is sufficiently large while the solution space X_n has a fixed finite size n. We take the opposite point of view and let n grow large while keeping fixed the cardinality of the team of problem solvers who are in charge of tackling it. They are interested in large teams; we study large solution spaces. So we view the two papers as complementary with regard to exploring the power of diversity.

We also argue that proving the power of diversity over a large solution space X_n is more challenging than establishing it for a large team of m agents. If we assemble a larger team to work over a fixed solution space, it stands to reason that it should have a better chance to succeed. On the other hand, consider a team of fixed size working on an ever larger solution space. It is far less obvious that the team should always be successful. A simple example may help to ground this intuition.

Table 3.3 exhibits the probability that a team of $m = 2, 4, 8, 16$ agents is always successful for $n = 3, 4$ under the uniform model with difficulty. All values are rounded

Uniform	$m = 2$	$m = 4$	$m = 8$	$m = 16$
$n = 3$	0.37500	0.82031	0.98831	0.99995
$n = 4$	0.45918	0.91977	0.99881	1^-

Table 3.3: Probability that a team of m agents is always successful over X_n.

to the closest fifth decimal digit. We write 1^- when the probability is within 10^{-5} from 1. Assuming ever larger teams over a given solution space X_n corresponds to reading a row in the table from left to right: as expected, for small values of n, the probability that a team of m is always successful is rapidly approaching 1. Our paper, on the other hand, is concerned with what happens when we descend a column in a table.

The general framework in Hong and Page (2001) describes the problem-solving ability of an agent as the pairing of his perspective with a set of heuristics. The *perspective* is the agent's internal representation of a problem; the *heuristics* are the algorithms he applies to locate solutions. To facilitate the mathematical study of disjunctive tasks, Hong and Page (2004) suppress their own distinction between perspectives and heuristics and characterize each agent i by a mapping $\varphi_i : X \to X$ and a probability distribution ν on X. The initial distribution ν has full support and it is the same for all agents: it is used to randomly generate the starting point of the search process; without loss of generality, let ν be the uniform distribution on X.

For each x, $\varphi_i(x)$ denotes the local solution found by Agent i when he starts his search at x. Assumption 0 in Hong and Page (2004) states two properties valid for any mapping φ_i and encapsulating the assumption that each Agent i is intelligent:

(0.a) $V(\varphi_i(x)) \geq V(x)$ for all x in X;

(0.b) $\varphi_i(\varphi_i(x)) = \varphi(x)$ for all x in X.

Assumption (0.a) is the obvious requirement that the agent never finds a local solution worse than his starting point. Assumption (0.b) states that the local solution found starting at x cannot be further improved upon by the agent and that the final point of his search is unique. This is natural if we take the point of view that Agent i tries his best given his set of heuristics.

Under Assumption 0, it takes a simple change of perspective (pun intended) to derive the equivalence between our partitional model and the φ-representation. Suppressing subscripts momentarily, consider an agent associated with the mapping φ and the equivalence relation $x \sim y$ on $X \times X$ defined by $\varphi(x) = \varphi(y)$. When $x \sim y$, the agent starting his search at either point ends up discovering the same local optimum. With respect to the objective of maximizing V, he is indifferent between x and y because they both lead to an identical result. Hence, $\sim$ defines an indifference relation that partitions X into equivalence classes such that each starting point in the same class leads to the same local maximum.

Clearly, the problem-solving ability of an agent represented by φ is uniquely identified with the partition Π induced by φ. In this respect, it is worth noting that Assumption (0.b) implies $x \sim \varphi(x)$ for any x and thus plays the important role of ensuring that φ induces a partition Π that is consistent in the sense defined in Section 3.2.

Vice versa, any partition Π of X uniquely defines a problem-solving mapping φ by the following construction. Let $\sim$ be the equivalence relation on $X \times X$ defined by $x \sim y$ if and only if $y \in \Phi(x)$. Then, for each x in X, the mapping $\varphi(x) = \arg\max_{y \sim x} V(y)$ characterizes the problem-solving ability of the agent endowed with the partition Π. This establishes a formal equivalence between the mapping φ and the partition Π as models of limited problem-solving for an agent.

This formal equivalence allows us to rephrase results from one perspective to the other. For instance, Hong and Page (2004) note that the image $\varphi(X)$ of the mapping is the set of local optima discoverable by the agent. Since the elements in $\varphi(X)$ are in a one-to-one correspondence with the blocks in the partition Π, the cardinality of $\varphi(X)$ is the same as the number of blocks in Π. In a similar vein, it is easy to check that our paper satisfies the three main assumption (Difficulty, Diversity, and Uniqueness) made by Hong and Page (2004).

For instance, consider Difficulty. We use $\Phi = \{\varphi_i : i \in T\}$ or $= \{\Pi_i : i \in T\}$ to denote the team of agents in either perspective. Hong and Page (2004) state that, for any φ in Φ, there exists a solution x such that $\varphi(x) \neq x^*$. That is, for each agent, there exists at least one starting point from which the global optimum x^* is not available; any problem solver has a nut he cannot crack. Hence, the problem is difficult because no agent alone is sure to be always successful. Our Assumption 1 that no agent is endowed with the trivial partition is logically equivalent.

3.5 Teaming up does not ensure success

The uniform model, studied in Section 3.3, assumes that agents are independently chosen according to the uniform distributions over all partitions. This is the simplest way to build a random model of team formation since it assigns the same probability to each available partition. This section studies an alternative randomization, what we call the *urn model.* After describing the construction of this model, we will study how it affects the results obtained under the uniform model.

For a given mapping $f : X_n \to X_n$, the sets $\{x \in X : f(x) = y\}$ form a partition π_f of X_n. If f is chosen uniformly at random from the set of all n^n mappings then Π_f is random, but not uniform. For instance, suppose that $n = 3$. There are $3^3 = 27$ possible mappings, each of which is chosen with the same probability. Three of them generate the trivial partition 123; six of them generate the other four possible partitions 1|23, 2|13, 3|12 and 1|2|3. Therefore, the probability distribution over the set of possible partitions generated by this model assigns probability $3/27 = 1/9$ to the trivial partition and $6/27 = 2/9$ to each of the other four partitions.

This model takes its name from the fact that it can be expressed by using a urn problem: There are n numbered balls that need to be distributed in n numbered urns. It is allowed to have empty urns. Here, balls correspond to solutions and urns to blocks of partitions. For example, suppose that $n = 3$. There are three balls to be distributed in three urns. Therefore, there are $3^3 = 27$ possible ways to do this. The trivial partition (all three balls in the same urn) is generated three times. Hence, under the assumption that each possible way is equally likely, the probability for the trivial partition is $3/27 = 1/9$. Similarly, there are six possible ways to generate the other four possible partitions 1|23, 2|13, 3|12 and 1|2|3. Hence, the urn model assigns $6/27 = 2/9$ probability to each of these partitions.

Note that the urn model generates higher probabilities for partitions with a higher number of blocks. The higher the number of blocks in a partition, the less able the agent in problem solving. Therefore, this models assigns higher probabilities to agents with less ability. As a result, it is less likely to generate the trivial partition. In fact, the probability to draw such partition for a set X_n is $1/n^{n-1}$, whereas it is $1/B_n$ under the uniform model.

3.5.1 Small teams over small solution spaces

To compute the probability that the two agents always discover the global maximum, it suffices to apply the marginal probabilities generated by this model to Table 1. We do so in Table 3.4 where we add the marginal probabilities on the rightmost column and the bottom row, as well as the joint probabilities for the relevant events. The panels are arranged as in Table 3.1: on the left, the case without difficulty; on the right, the case with difficulty. Adding up the joint probabilities, we find that the probability is $41/81 \approx 0.506$ without difficulty and $3/8 = 0.375$ with difficulty.

	123	1|23	2|31	3|12	1|2|3	
123	1/81	2/81	2/81	2/81	2/81	1/9
1|23	2/81		4/81	4/81		2/9
2|31	2/81	4/81		4/81		2/9
3|12	2/81	4/81	4/81			2/9
1|2|3	2/81					2/9
	1/9	2/9	2/9	2/9	2/9	

	1|23	2|31	3|12	1|2|3	
1|23		1/16	1/16		1/4
2|31	1/16		1/16		1/4
3|12	1/16	1/16			1/4
1|2|3					1/4
	1/4	1/4	1/4	1/4	1/4

Table 3.4: Probability of the trivial partition under the urn model with (right) and without (left) difficulty.

Table 3.5 summarizes the probability that the team of $m = 2$ agents is always successful for $n = 2, 3, 4, 5$ under the urn model with difficulty. Note that this probability is decreasing in n. It becomes less and less possible for a team of 2 agents to always find the global optimum as the number of available solutions is increasing. In fact; Theorem 3.5.2, given in the next section, proves that this probability goes to zero when $n \uparrow \infty$.

$n = 2$	$n = 3$	$n = 4$	$n = 5$
0	$6/16 = 0.375$	$138/441 \approx 0.313$	$5400/24336 \approx 0.222$

Table 3.5: Probability that a team of 2 agents is always successful under the urn model.

3.5.2 Small teams over large solution spaces

Let P_{mn} the probability that the meet of $m \geq 2$ partitions of X_n chosen under the urn model with difficulty is the trivial partition. Then, $P_{mn} \to 0$ as $n \uparrow \infty$. Denote by P'_{mn} the probability that the meet is the trivial partition under the urn model without difficulty. Since $P'_{mn} \geq P_{mn}$, it suffices to show that $P'_{mn} \to 0$. The strategy of the proof is the following.

We say that a solution j is *isolated* for an agent i when $\{j\}$ is a singleton block for his partition. Analogously, a solution j is isolated for the team if it is isolated for each agent. Let A^i_j and A_j denote the event that the j-th solution is isolated for Agent i and for the team, respectively. When the meet of the agents' partitions is the trivial partition, no solution j can be isolated for the team. Therefore,

$$P'_{mn} \leq 1 - P\left(\bigcup_{j=1}^{n} A_j\right).$$

We are going to show that $P\left(\bigcup_{j=1}^{n} A_j\right) \to 1$ as $n \uparrow \infty$.

We begin with a few preliminary observations. Given a set X_n of n possible solutions, A_j^i corresponds to the event that the j-th ball ends up alone in one of the n urns; thus

$$P\left(A_j^i\right) = \frac{n(n-1)^{n-1}}{n^n} = \left(1 - \frac{1}{n}\right)^{n-1}$$

Analogously, for $j_1 < j_2 \ldots < j_k$, the probability that $k \leq n$ solutions in X_n are isolated for Agent i is

$$P\left(\bigcap_{s=1}^{k} A_{j_s}^i\right) = \prod_{s=1}^{k}\left(1 - \frac{1}{n-s+1}\right)^{n-s}.$$

To see why, use the following inductive argument. The probability that the j_1-th solution is isolated (or, equivalently, the j_1-th ball ends up alone) is $(1-1/n)^{(n-1)}$. Conditional on this event, the other $n-1$ balls are distributed uniformly in the remaining $n-1$ urns. Hence, the probability that the j_2-th solution is isolated (or, equivalently, the j_2-th ball ends up alone) is $(1-1/(n-1))^{(n-2)}$. And so on.

Finally, since all agents' partitions are identically and independently distributed,

$$P(A_j) = \prod_{i=1}^{m} P(A_j^i) = \left(P(A_j^i)\right)^m$$

and thus, for $j_1 < j_2 \ldots < j_k$, the probability that $k \leq n$ solutions in X_n are isolated for the team is

$$P\left(\bigcap_{s=1}^{k} A_{j_s}\right) = P\left(\bigcap_{i=1}^{m}\bigcap_{s=1}^{k} A_{j_s}^i\right) = \prod_{s=1}^{k}\left(1 - \frac{1}{n-s+1}\right)^{m(n-s)} \tag{3.2}$$

Note that, for any fixed k, $P\left(\bigcap_{s=1}^{k} A_{j_s}\right) \to \mathrm{e}^{-mk}$ as $n \uparrow \infty$.

We are now ready for the main argument. Given $\varepsilon > 0$, choose a sufficiently large integer M so that $(1-\mathrm{e}^{-m})^M < \epsilon$. By the inclusion-exclusion formula, we have

$$P\left(\bigcup_{j=1}^{M} A_j\right) = \left[\sum_{j=1}^{M} P(A_j) - \sum_{1\leq j_1<j_2\leq M} P(A_{j_1}\cap A_{j_2}) + \ldots + (-1)^{M+1}P(\bigcap_{j=1}^{M} A_j)\right]$$

Taking limits on both sides,

$$\begin{aligned}
\lim_n P\left(\bigcup_{j=1}^{M} A_j\right) &= \sum_{j=1}^{M}(-1)^{j+1}\binom{M}{j}\mathrm{e}^{-mj} = -\sum_{j=1}^{M}\binom{M}{j}(-\mathrm{e}^{-m})^j \\
&= 1 - \sum_{j=0}^{M}\binom{M}{j}(-\mathrm{e}^{-m})^j = 1 - \sum_{j=0}^{M}\binom{M}{j}(-\mathrm{e}^{-m})^j \cdot 1^{M-j} \\
&= 1 - (1-\mathrm{e}^{-m})^M > 1 - \varepsilon
\end{aligned}$$

where the last step follows from our choice of M. This concludes the proof.

Note that the proof works unchanged for $m = 1$ as well, so that the conclusion of Theorem 3.5.2 actually holds for $m \geq 1$. Moreover, similarly to Theorem 3.3.2, it holds regardless of whether difficulty is assumed or not.

The reason why the limit probability of always success goes to one under the uniform model but it goes to zero under the urn model is the following: When n is large, the expected number of blocks in a random partition under the uniform model is much less than under the urn model; $n/\log n$ and $(1 - e^{-1})n$, respectively (see Sachkov (1997)). The fewer the number of blocks in partitions, the easier it is to get the trivial partition in the meet. Having a few number of blocks means, in general, having large blocks in the partition. Finally, to reach to the trivial partition in the meet of two partitions with large blocks is of high probability.

Theorem 3.5.2 shows that the results under the uniform model and under the urn model differ substantially. When the solution space is large, a team of 2 agent is always successful under the uniform model, whereas any team (regardless of its size) may fail to solve problem under the urn model. Hence, the assumption on the randomization is crucial.

3.6 Defendit numerus

This section examines another model for randomizing partitions using graph theory. Here, we study the size it requires for a team to be always successful when the solution space is large. The complexity of solution space makes it hard for agents to solve the problem. To capture this idea, we assume that the ability of an agent decreases as the solution space gets larger.

We consider a partition of X_n as a graph G having n vertices and each block of the partition as the connected subgraphs of G. With this setting, the trivial partition correspondences to the connected graph and the finest partition, in which each solution is in a single block, correspondences to the graph having n isolated vertices.

In graph theory, a random graph is obtained by starting with a set of n vertices and adding edges between them at random. Here, we consider the *Erdős-Rényi Model*, $G(n, p)$, for generating random graphs. In this model, a graph is obtained by connecting n vertices randomly and an edge between any two vertices is included in the graph with probability p. The structure of a graph (partition) is determined by p; as this parameter increases the number of blocks in a partition decreases. For instance, when $p = 0$ all vertices are isolated (finest partition) and when $p = 1$ each vertex is connected to another, i.e., the graph is fully connected (trivial partition). As mentioned earlier, we assume that the ability of an agent decreases with n. In other words, we assume that the number of blocks in an agent's partition is increasing as the solution space gets larger. Therefore, p is a decreasing function of n.

In order to have an edge between two specific vertices in the meet of k partitions, it is required that at least one of the k partitions has an edge between the two vertices.

The probability of this event is given by $1-(1-p)^k$. Therefore, the meet of k partitions of X_n is $G(n, 1-(1-p)^k)$. Furthermore, according to the theorem by Palmer (1985, pg 54, Theorem 4.3.1), for $p > \frac{\log n}{n}$ a graph in $G(n,p)$ is almost surely connected. Thus $\frac{\log n}{n}$ is a sharp threshold for the connectedness of $G(n,p)$. This implies that the meet of k partitions of X_n is almost surely the trivial partition if $1-(1-p)^k$ is greater than this threshold. In other words, the meet is almost surely the trivial partition if

$$1-(1-p)^k > \frac{\log n}{n} \tag{3.3}$$

Observe that when calculating the probability of having an edge between two specific vertices in the meet of k partitions we assumed that in all the given k graphs (partitions) the probabilities of having an edge between two vertices are equal to p. However, our inference does not change even if these probabilities are different. To see how, assume that the probabilities are $p_1, p_2, ..., p_k$, i.e., the graphs are $G(n,p_1), G(n,p_2), ..., G(n,p_k)$. Assume further that $p_1 \geq p_2 \geq ... \geq p_k$ and define $p := p_k$. Therefore, the probability of having an edge between two specific vertices in the meet is at least $1-(1-p)^k$ and the meet is almost surely the trivial partition if (3.3) holds.

The smallest integer k that satisfies (3.3) gives the minimum number of partitions that it is required for the meet to be the trivial partition. This correspondences to the minimum number of agent we need in order to find the global optimum.

Example In this example we assume that $p = 1/n$ and calculate the minimum size of a team in order to get the trivial partition in the meet for $n = 10, 10^2, 10^3, 10^4, 10^5, 10^6$.

k	n	p	threshold	$1-(1-p)^k$
3	10	0.1	0.230258509	0.271000000
5	10^2	0.01	0.046051702	0.049009950
7	10^3	0.001	0.006907755	0.006979035
10	10^4	0.0001	0.000921034	0.000999550
12	10^5	0.00001	0.000115129	0.000119993
14	10^6	0.000001	0.000013816	0.000013999

Table 3.6: The minimum size of an always-successful-team

From Table 3.6 we see that when a solution space has 10 elements, a team of 3 agents is almost always successful. As the cardinality of the solution space increases, the ability of an agent decreases, and the size it requires for a team to be always successful increases. It reaches to 14 when the solution space has one million elements.

3.7 Groupthink impairs the power of diversity

This section concerns groupthink, which is a situation in which all members of a team begin to think alike or pretend to think alike. No members are then willing to raise objections or concerns about an issue. In psychological literature, groupthink is given as a possible explanation why individuals might be better than groups in decision making.

In this section, we explore the effect of groupthink on the team performance. To this aim, we continue to use the *Erdős-Rényi Model*, introduced in the previous section. In order to model the idea that members of a team are similar, we assume that the probability of having an edge between any two vertices p is positively correlated among agents.

In order to study the minimum size (k) it requires for a team to be always successful, we follow the same procedure as in the previous section: derive the probability that at least one of the k partitions has an edge between two vertices (P_{kn}) and increase k until the probability crosses the threshold.

We start the analysis by studying two extreme cases: (i) when there is no correlation among the team members (independent case) and (ii) when there is a perfect correlation among team members (complete dependence case). In the independent case, $P_{kn} = 1-(1-p)^k$. In the complete dependence case, all agents act the same. In other words, they are exact replicas of each other. Hence, $P_{kn} = p$. This shows that P_{kn} belongs to the interval $[p, 1-(1-p)^k]$ and is increasing as the correlation decreases. At this point, we appeal to copulas. Without entering in details, we use the family B11 copula in Joe (1997): $C(\mathbf{u};\delta) = \delta C_2(\mathbf{u}) + (1-\delta)C_1(\mathbf{u})$, where $\delta \in [0,1]$ is the dependence parameter, $C_1(\mathbf{u})$ and $C_2(\mathbf{u})$ are the independent and the Fréchet upper bound copula (obtained in the case of complete dependence), respectively. Thus $P_{kn} = \delta p + (1-\delta)\left(1-(1-p)^k\right)$ for $\delta \in [0,1]$. This implies the meet of k correlated partitions of X_n is almost surely the trivial partition if

$$\delta\,(p) + (1-\delta)\left(1-(1-p)^k\right) > \frac{\log n}{n} \tag{3.4}$$

Example Here, we repeat the example of the previous section for $\delta = 0$, 0.25, 0.50, 0.75, 1 with $n = 10$, 10^3, 10^6.

Note that when $\delta = 0$, there is no correlation. Therefore, the first rows in Table 3.7 are the same the first, third and the fifth row of the Table 3.6. As δ increases, P_{kn} decreases and hence the minimum size of an always-successful-team increases. When $\delta = 1$ (complete dependence), all the agents are exact replicas of each other and $P_{kn} = p$. Therefore, no team is always successful. This example illustrates that groupthink harms the performance, but it is still possible to build an always-successful-team unless there is a perfect correlation. This goal can be achieved by increasing the size of the team.

δ	P_{kn}	k
0	0.271000	3
0.25	0.282925	4
0.50	0.254755	5
0.75	0.237830	10
1	0.100000	∞

$n = 10$, $p = 0.1$

δ	P_{kn}	k
0	0.006979	7
0.25	0.006973	9
0.50	0.006961	13
0.75	0.006926	25
1	0.001000	∞

$n = 10^3$, $p = 10^{-3}$

δ	P_{kn}	k
0	0.0001201	12
0.25	0.000122	16
0.50	0.000120	23
0.75	0.000117	44
1	0.000010	∞

$n = 10^5$, $p = 10^{-5}$

Table 3.7: The minimum size of an always-successful-team in case of groupthink

3.8 Conclusions

The power of diversity is explored in this paper. We study a model where a team of agents with limited problem-solving ability face a disjunctive task over a large solution space. The limitations in agents' abilities are formalized using partitions. We provide sufficient conditions under which two heads are better than one. In this case, a team of two agents pool their abilities and always succeed in solving the difficult problem, whereas neither can do it alone. We also provide sufficient conditions for the situation in which teaming up does not guarantee success. When agents are not sufficiently creative, it is also possible for any team, no matter how large, to fail to solve the difficult problem.

We further study situations in which the ability of agents is adversely affected by the complexity of the solution space. As the solution space gets larger, the problem becomes more complex since the number of elements among which the search is done increases. By using graph theory in order to model limited ability, we show that small sized teams will still solve the problem. The size of a team is determined by the cardinality of the solution space, i.e., by the complexity of the problem.

Our final finding is about the effect of groupthink on the performance of a team. When members of a team begin to think and act alike, its performance decreases. In other words, groupthink harms the performance and impairs the power of diversity. We show that a team whose members' abilities are both positively correlated and adversely affected by the complexity of the problem will still be able to solve the problem. However, larger teams are necessary in this case.

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Estratto per riassunto della tesi di dottorato

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Titolo della tesi : Three Essays on Economic Interactions under Bounded Rationality

Abstract: The issues explored in this work concern economic interactions under bounded rationality. Each chapter considers these interactions from different angles. The first chapter characterizes the optimal contract designed by an ordinary profit-maximizing monopoly when facing diversely bounded rational agents. The second chapter analyses the interaction between fully and boundedly rational agents in situations where their interests are perfectly aligned. Finally, the third chapter studies a model where a team of agents with limited problem-solving ability face a disjunctive task over a large solution space.

Estratto: Ogni capitolo di questo lavoro di ricerca considera diversi temi concernenti interazioni economiche in presenza di razionalitá limitata. Il primo capitolo propone il contratto ottimale di un monopolio ordinario il quale massimizza il suo profitto in un contesto in cui gli agenti sono caratterizzati da diversa razionalitá limitata. Il secondo capitolo analizza l'interazione tra agenti con razionalitá piena e quelli con razionalitá limitata nel caso in cui i loro interessi siano perfettamente allineati. Infine, il terzo capitolo analizza un modello in cui un gruppo di agenti con limitata capacitá di risolvere problemi affronta un compito disgiuntivo su uno spazio di soluzione di grandi dimensioni.