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Diversification, Leverage and Systemic Risk

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2 Summary

The Thesis is composed by three papers that investigate from a theoretical perspective the relationship between leverage, diversification and systemic risk. Starting from the folk wisdom that asset diversification enhances financial stability by dispersing credit risks, the papers contribute to the debate shedding light on a critical facet of this strategy.

The first paper "Leverage, Diversification and Default Risk" puts the fundaments for the other papers of the Thesis. It undertakes a theoretical investigation of the following fundamental question. Does diversification, based on modern portfolio theory wisdom, lead to greater safety for leveraged investors? The hypothesis is that diversification can have undesired effects when the initial wealth is leveraged with forms of debt that are, in principle, decoupled from the value of assets in portfolio. In particular, it might exist an optimal diversification level which minimizes the default probability of the investor. In presence of a leveraged investor, a crucial issue is the optimal number of assets which ensures the lowest default probability. Therefore, the object of the analysis is to find the optimal "default-induced" portfolio size. In other words, we aim to determine the optimal number n of securities to hold in a equally weighted portfolio in order to minimize the default risk. Under the standard framework of asset pricing theory in a frictionless, arbitrage-free and complete market, the paper shows that asset diversification is not always beneficial. Beyond a given threshold, diversification may increase the default risk of the leveraged investor.

The second paper "Diversification and Financial Stability" translates the concept of knife-edge dynamics into an analytic form and investigates its impact on the stability of the financial system. According to the knife-edge property, financial interconnection/diversification serves as shock-absorber (i.e., connectivity engenders robustness and risk-sharing prevails) up to a certain tipping point, beyond which they act as shock-amplifier (i.e., connectivity engenders fragility and risk-spreading prevails). The paper considers a financial system composed of leveraged risk-averse financial institutions, which invest in two classes of assets. The first class consists of obligations issued by other banks in the network. The second class represents risky assets external to the financial network, which may include securities on, e.g., real-estate, loans to firms and households and other real-economy related activities. This balance-sheet approach is conjugated with a stochastic setting. In a mean-field approximation, we derive the law of motion of the financial system fragility and we use it to compute its probability to collapse. As first result, the paper shows that banks' default probability increases with diversification in external assets in the case of a downtrend and decreases in the case of an uptrend. In accordance, under the utility function framework, there exists a gap (which increases with diversification) between the banking systems expected utility in uptrend and the expected utility in downtrend. Thus, diversification generates and amplifies the knife-edge property of the financial network. Moreover, for a given level of diversification, the spread between utility levels increases with the magnitude of the trend. A second result shows that by assigning a probability p to the downtrend event and $1-p$ to the uptrend, diversification displays a tradeoff with regard to the banking system's utility. The expected utility exhibits a maximum which corresponds to an intermediate optimal level of diversification. Such a level depends on the probability p of downtrend, on the magnitude of the expected profit (loss) and on the values of the other parameters of the model. Finally, the paper shows that individual banks' incentives favor a financial network that is over-diversified in external assets with respect to the level of diversification which is socially desirable.

The third paper "Leverage Cycle" detects the emergence of systemic risk in presence of a feedback loop between asset prices and leverage caused by an active balance sheet management. The object of the paper is to outline the aggregate consequences of procyclical leverage by modelling the influence of expansions and contractions of balance sheets on asset prices and vice-versa. We introduce a simplified economy of risk-averse interconnected banks managing portfolios of risky assets. Through a risk management practice that adjusts the Value-at-Risk (VaR) to a target level, the institutions actively manage their balance sheet: it shrinks or enlarges according to asset price movements. The paper offers three main contributions to the literature. First, we attempt a microfunded relationship between aggregate balance sheet size and leverage of the financial system. Far from being passive, financial intermediaries adjust their balance sheets actively and do so in such a way that leverage is high during booms and low during busts. Second, we model the macro consequences of procyclical leverage and analyze the reciprocal influence between contraction-expansion of the aggregate balance sheet and asset pricing of risky activity. Finally, the paper theoretically investigates how systemic risk is affected by asset diversification when leveraged institutions manage their balance sheets actively in response to price changes. The presence of a feedback loop between asset price and leverage is supposed to amplified the knife-edge properties of the diversification strategy analyzed in "Diversification and Financial Stability".

Standard textbooks of Investment/Financial Management teach that portfolio diversification help to reduce investment risk. How valid is this "common" knowledge? The Thesis examines the issue on naive (equal weight) diversification. We analytically show that in a idealized frictionless world, maximum diversification is not always desirable. In presence of leverage, diversification can reveal to be a double-edge sword strategy: it reduces the default risk only up to a certain threshold beyond which, default risk increases. Moreover, the Thesis studies the side effects of diversification also in a financial system of interconnected banks. We shed light on the conflict between, on one hand, the individual incentive to reduce, through diversification, the idiosyncratic risks and, on the other hand, the emergence of systemic risk.

The financial stability have raised to the top of the agenda of economic policy discussions because the world and the financial environment have substantially changed since the mid seventies. This issues grew up even more at the dawn of the recent 2007-08 financial crisis. Therefore, the topic of the Thesis is on the same wave of works aiming at rethinking the financial architecture after the "breaks" revealed during the recent crisis.

From our analysis we draw the conclusion that the major problem of an individual institution perspective is a potentially misleading risk assessment for the financial system as a whole. Then, even though it is uneasy to put in practice a system perspective, regulators need to change into a more system oriented supervisory risk assessment policies. Moreover, they also need to face the challenge to design new approaches in order to strengthen countercyclical regulatory instruments that work as cushions in booms and as buffers when times deteriorate.

Diversification, Leverage and Default Risk

Paolo Tasca, Stefano Battiston

Abstract

We model the financial fragility of a leveraged investor who borrows the funds to be invested in a $1/n$ portfolio. The insolvency occurs when the value of the portfolio falls below the borrowed amount. We investigate the optimal number n^* of assets to hold in order to minimizes the default probability. Under the standard framework of asset pricing theory in a frictionless, arbitrage-free and complete market we analytically show that full diversification is not always desirable. Beyond a given threshold n^*, diversification may increase the risk of default.

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1 Introduction

Does diversification, based on modern portfolio theory wisdom, lead to greater safety for leveraged investors? In this paper, we undertake a theoretical investigation of this question. The evidence we present suggests that, in contrast to the recommendations of traditional portfolio theories, in a idealized frictionless world, maximum asset diversification is not always guaranteed to produce superior risk-adjusted performance for leveraged investors.

The motivations for the benefits of diversification are well known. After the seminal papers of Markowitz (1952, 1959) and Tobin (1958), analytic tools have been developed to show that no-perfect positive correlation among risky assets' returns provides an incentive to diversify. [1]

Since the pioneering work of Evans and Archer (1968), the question of optimal diversification has been answered by determining the rate at which risk-reduction benefits are realized as the number of securities in a equally weighted portfolio is increased. This benchmark is the so-called, $1/n$ or *naive* rule. If leverage, costs of transactions, information and management are zero, the benefits derived from diversification increase with the number of securities but at decreasing rate. See e.g.,(Bird and Tippett, 1986; Elton and Gruber, 1977; Johnson and Shannon, 1974; Statman, 1987).

However, inspection of the literature suggests that relatively little attention has been given to the fundamental relationship between diversification and leverage. With this respect, we have to consider that a

[1] In particular, Samuelson (1967) Theorem III and Brumelle (1974) Theorem IV, have provided more general models which show that nonpositive correlation (or negative correlation in a two-parameter model of Markowitz and Tobin) of asset returns provides the strongest incentive to diversify.

leveraged investor is subject to a default risk, by which we mean any type of failure to honor the financial agreements embedded in the borrowing activity.

Our hypothesis is that diversification can have undesired effects when the initial wealth is leveraged with forms of debt that are, in principle, decoupled from the value of assets in portfolio. In particular, it might exist an optimal diversification level which minimizes the default probability of the investor.

Indeed, the performance of an asset allocation strategy depends not only on the *investor profile* (i.e., individual's preferences in investments decisions), but also on the *source of financing* (i.e., internal or external). A leveraged diversified portfolio displays a trade-off between two competing forces acting on the risk side. The first is *leverage* which amplifies the portfolio risk (volatility) that in the traditional capital structure theory translates into higher default risk.[2] The second one is *diversification* through which default risk can be reduced and managed.[3]

In detail, we study the risk profile of the $1/n$ allocation strategy performed by an investor allowed to leverage k times the initial capital E.[4] She borrows the amount $H := kE$ from a unique source of funding by paying a market premium over the riskless rate.[5] Then, the overall wealth ($A := E + kE \equiv E + H$) is invested in an equally-weighted combination of risky assets.

The object of our analysis is to find the optimal "default-induced" portfolio size. Namely, the number n of securities to hold in a equally weighted portfolio in order to minimize the risk of a failure.[6] To this end, we model the default risk and we study how it changes with respect to the degree of asset diversification.

In the literature there are three main approaches to analyze the default (credit) risk : *structural*, *reduced-form* and *incomplete information* approach.[7] Many authors (e.g., Black and Cox, 1976) put forward structural-type models in which the restrictive and unrealistic features of the structural model originated

[2]See e.g., Altman (1984); Harris and Raviv (1991); Jensen and Meckling (1976).

[3]Default risk, and the rewards for bearing it, will ultimately be owned by those who can diversify it best. Default risk refers to the probability that the borrower will not meet her debt obligations. This risk is firm-specific and can be considered as part of financial risks, the impact of which is reduced through the construction of a well-diversified portfolio. A bunch of theoretical and empirical works have investigated the relation between default risk and diversification, (see e.g., Acharya et al., 2006; Jarrow et al., 2005).

[4]There are two main reasons for using the naive $1/n$ rule as a benchmark. First, it is easy to implement because it does not rely either on estimation of the moments of asset returns or on optimization. Second, despite the sophisticated theoretical models developed in the last decades, investors continue to use such simple allocation rule for allocating their wealth across assets (see e.g., Benartzi and Thaler, 2001; Pflug et al., 2012). Moreover, recent studies e.g. DeMiguel et al. (2009) Tu and Zhou (2011), show that the naive $1/n$ investment strategy can perform better than those sophisticated Markowitz-type rules.

[5] It should be noted that investors are generally subject to financing constraints. It is indeed more difficult to diversify debt than assets (e.g., Guiso et al., 1996). For instance, when financing is obtained via short-selling, a number of limitations apply to this practice (e.g., Sharpe, 1991).

[6]Precisely, the default is triggered as soon as, at any time $t \geq 0$, the market value of the assets is lower than the debt's market value (i.e, when $A < H$).

[7]A review of the credit risk literature can be found in Ammann (2001), Bielecki and Rutkowski (2002), Duffie and Singleton (2003) and Lando (2004). A systematic study of the mathematical techniques used in reduced-form models is available in Elliott et al. (2000) and Jeanblanc and Rutkowski (2000).

with Black and Scholes (1973) and Merton (1974) are relaxed.[8] In most of these models, the time of default is given as the first passage time of the value process across an exogenous[9] or endogenous[10] barrier, namely the distress boundary. The paper uses this first passage time approach. The default event corresponds to the (mean) first passage time taken by the asset value process to hit (from above) the debt value, which is a *time-varying barrier*, endogenously given with respect to the model. Since the value of both A and H is described by continuous time processes, we turn to a *fixed-barrier* problem by considering the (mean) first passage time taken by the debt-asset ratio (i.e., leverage) to hit the upper (default) bound of its range of variation. More precisely, we use the log transformation of the leverage – i.e., $\log(H/A)$ which, for the investor view point, denotes a condition of "fragility" when it tends to the upper bound and "robustness" when it tends to the lower bound.

Under the standard assumptions from asset pricing theory, [11] we find that full diversification is not always desirable. In particular, when either the expected value or the variance of the investor's log-leverage are U-shaped with respect to the number of securities in portfolio, there exists an intermediate optimal diversification level (i.e., $1 < n^* < N$) which maximizes the mean-time to default.

The paper is organized in the following way. Section 2 presents the basic framework and assumptions. Section 3 introduces the $1/n$ leveraged portfolio. In Section 4 the stochastic dynamics of investor's log-leverage is derived. Furthermore, we introduce the concept of default risk and compute the (mean) time to default. In Section 5 comparative statics is used to analyze the effect of asset diversification on default risk an the optimal $1/n$ leveraged portfolio is derived. Section 6 concludes the paper.

2 Continuous-time Setting

We introduce a stochastic framework to describe a continuous-time security market where a risk-averse investor can choose between $1, 2, ..., N$ risky securities. The future states of the world are represented by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on the sample space Ω, with a filtration $\mathbb{F} = (\mathcal{F})_{t\in[0,T]}$ representing the

[8]Leland (1994) and Leland Hayne E. (1996) developed structural models that include four additional aspects: debt has an average maturity $T < \infty$; coupon interest is tax deducible; default incurs default costs, the firm may have net cash payouts to security holders.

[9]Zero net worth trigger (e.g., Brennan and Schwartz, 1978; Longstaff and Schwartz, 1995), zero cash-flow trigger, Kim et al. (1993)

[10]Such that the default boundary maximizes the equity value (see e.g., Black and Cox, 1976; Leland, 1994; Mello and Parsons, 1992)

[11]Security prices evolve in time according to a geometric Brownian motion. For simplicity, securities do not pay dividends and are infinitely divisible. They are traded in a single purely competitive market at a given price which does not depend on the investor transaction (i.e., the price-taking assumption holds). The market is frictionless, arbitrage-free and complete.

information up to time t and a probability measure $\mathbb{P}$. [12] The price processes of the risky securities form an Ito process $\mathbf{S} := (S_1, ..., S_N)^{\mathrm{T}}$ in $\mathbb{R}^N$. For the sake of simplicity, the securities pay no income during the time interval $[0, T)$. Moreover, there exists a riskless security with price S_0 based on the riskless short-term interest rate process r_f such that $S_0(t) = S_0(0)exp\left(\int_0^T r_f(s)\,\mathrm{d}s\right)$.[13] Therefore, $S_0(t)$ can be understood as the market value of an account that is continually reinvested at short-term riskless rate. To sum up, the economy is described by the security-price process $\mathbf{X} := (S_0, \mathbf{S})$, with

$$d\mathbf{S}(t) = diag(\mathbf{S}(t))\left[\boldsymbol{\mu}(t)dt + \boldsymbol{\Sigma}(t)d\mathbf{W}^{\mathbb{P}}(t)\right], \tag{1}$$

and

$$dS_0(t) = r_f(t)S_0(t)dt, \quad S(0) > 0, \tag{2}$$

where $\mathbf{W}^{\mathbb{P}} := (W_1^{\mathbb{P}}, ...W_N^{\mathbb{P}})^{\mathrm{T}} \in \mathbb{R}^d$ is ad N-dimensional Wiener process, i.e., a vector of N correlated one-dimensional standard Wiener processes on the given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The term $diag(\mathbf{S}(t))$ denotes the $(N \times N)$-matrix with the vector $\mathbf{S}$ along the main diagonal and zeros off the diagonal. The processes $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$ and r_f are adapted (valued in $\mathbb{R}^N$, $\mathbb{R}^{N\times N}$ and $\mathbb{R}$, respectively) to the information filtration $\mathbb{F}$.[14] The N-vector $\boldsymbol{\mu} := (\mu_1, ..., \mu_N)^{\mathrm{T}} \in \mathbb{R}^N$ contains the expected rates of returns and $\boldsymbol{\Sigma} := [\sigma_{ij}]_{N\times N}$ is the $(N \times N)$-matrix which measures the sensitivities of the risky asset prices with respect to the exogenous shocks so that $\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\mathrm{T}}$ is the $(N \times N)$-matrix containing the variance and covariance rates of instantaneous rates of return. It is assumed that $\boldsymbol{\Sigma}$ is uniformly positive definite for all $t \geq 0$.

The time-independent correlation matrix $\mathbf{Q}$ of $\mathbf{W}^{\mathbb{P}}$, which is assumed to be positive definite, is of the form $\mathbf{Q} := [\rho_{ij}]_{N\times N}$ with $\rho_{ij} = cor\{W_i^{\mathbb{P}}, W_j^{\mathbb{P}}\}$. Of course, the diagonal elements of this matrix represent the correlation of a variable with itself, and are all equal to one. We can write (1) in componentwise as $dS_i(t) = S_i(t)\left[\mu_i(t)dt + \sum_{j=1}^N \sigma_{ij}(t)dW_j^{\mathbb{P}}(t)\right]$, for all $i, j = 1, ..., N$.

2.1 Free-arbitrage pricing

Under the following main assumption, we place the conditions to ensure the free-arbitrage microstructure of the market.

Assumption 2.1. *We assume* (i) *There exists a frictionless market*, (ii) *Divisibility and no short-selling constraint. All the securities are marketable, perfectly divisible and can be held only in positive quantities*, (iii) *Price Taking Assumption. Stock prices are beyond the influence of investors*, (iv) *Only expected*

[12] We recall that a stochastic process $\{W(t, \omega)t \in \mathbb{R}^+, \omega \in \Omega\}$ is called (standard) Wiener process if: (i) $W(0) = 0$, (ii) it has stationary, independent increments, (iii) it admits a Gaussian $N(0, t)$ distribution, $\forall t > 0$, (iv) it has continuous sample paths (non "jumps"). The Wiener process is said to be adapted to filtration $\mathbb{F} = (\mathcal{F})_{t\in[0,T]}$, if $\omega(t)$ is a random variable $\mathbb{F}$-measurable for all $t \geq 0$.

[13] With the property that $\int_0^T |r_f(s)|\,\mathrm{d}s$ is finite almost surely.

[14] Further technical requirements should be imposed, e.g., that the processes r, $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are progressively measurable, that $\mathrm{diag}(\mathbf{S}_t)\boldsymbol{\mu}_t$ is in $\mathcal{L}^1$-process and that $\mathrm{diag}(\mathbf{S}(t))\boldsymbol{\mu}(t)$ is in $\mathcal{L}^2$-process. (We limit the discussion to diffusion processes -no jumps.)

returns and variances matter. Quadratic utility or normally distributed returns, (v) *Not-paying dividend assumption. The return on any stock is simply the change in its market price during the holding period*, (vi) *Homogeneous expectations.*

The "First Fundamental Theorem of Asset Pricing" states that the absence of arbitrage opportunities in a security market is equivalent to the existence of an equivalent martingale measure (EMM) $\mathbb{Q}$ under which all securities' prices relative to some numeraire are martingales. In a more formal way, the following equivalence theorem holds.

Theorem 2.1. *If Assumption 2.1 is satisfied, the following statements are equivalent: (i) there exists a state-dependent discount factor M, the so-called deflator, such that for all time* $s, t \in [0, T]$ *with* $s \leq t$ *and all assets* $i = 1, ..., N$ *we have* $S_i(s) = \mathbb{E}_s^{\mathbb{P}}\left[\frac{M(t)}{M(s)} S_i(t) | \mathcal{F}_s\right]$*, s.t. all prices can be represented as discounted expectations under the physical measure* $\mathbb{P}$*, (ii) there exists a risk-neutral probability measure* $\mathbb{Q}$ *such that all discounted price processes* $Z_i(t) := \frac{S_i(t)}{S_0(t)}$ *are martingales under* $\mathbb{Q}$*, i.e. for all time* $s, t \in [0, T]$ *with* $s \leq t$ *and all securities* $i = 1, ..., N$ *we have* $Z_i(s) = \mathbb{E}_s^{\mathbb{Q}}[Z_i(t)|\mathcal{F}_t] \iff \frac{S_i(s)}{S_0(s)} = \mathbb{E}_s^{\mathbb{Q}}\left[\frac{S_i(t)}{S_0(t)} | \mathcal{F}_s\right]$*, (iii) the financial market contains no arbitrage opportunities.*

The existence of the EMM $\mathbb{Q}$ such that this result holds is formally proven in M.Harrison and D.Kreps (1979) and M.Harrison and S.Pliska (1979). The change of probability between the true and risk-neutral measure is defined by the *Radon-Nikodym derivative* of $\mathbb{Q}$ with respect to $\mathbb{P}$[15], $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\left(-\frac{1}{2}\int_0^t \|\boldsymbol{\lambda}(s)\|^2 - \int_0^t \boldsymbol{\lambda}(s) dW(s)\right)$, where $\|\lambda(s)\|^2 = \mathbb{E}\left[\int_0^s \boldsymbol{\lambda}_u^2 \, \mathrm{d}u\right]$. The process $\boldsymbol{\lambda}$ has the interpretation of the vector $\boldsymbol{\lambda} := (\lambda_1, ..., \lambda_N)^{\mathrm{T}} \in \mathbb{R}^N$ of market prices of risk (corresponding to the shock process $\mathbf{W}$) since it measures the excess rate of return relative to the standard deviation,

$$\boldsymbol{\lambda}(t) = \mathbf{Q}^{-1}\left[\frac{\boldsymbol{\mu}(t) - r_f(t)\mathbf{1}}{\boldsymbol{\Sigma}(t)}\right] := \mathbf{Q}^{-1}\begin{pmatrix} \frac{\mu_1(t) - r_f(t)}{\sigma_1(t)} \\ \frac{\mu_2(t) - r_f(t)}{\sigma_2(t)} \\ \vdots \\ \frac{\mu_N(t) - r_f(t)}{\sigma_N(t)} \end{pmatrix}, \quad \text{for all } t \geq 0. \tag{3}$$

Given that the market is *complete*, under the "Second Fundamental Theorem of Asset Pricing", the EMM $\mathbb{Q}$ is unique and (3) has a unique (uniformly bounded) solution.

In the next Section we derive the $1/n$ leverage portfolio under the following assumption.

Assumption 2.2. (i) *Each* i^{th} *security has the same time-independent variance:* $\sigma_i = \sigma$ *for all* $i = 1, ..., N$ *and for all* $t \geq 0$, (ii) *the variables* r_f *and* λ_i *are real constants (i.e., no time-dependent)*,

Assumption 2.2 (i) does not violate the no-arbitrage condition. Nevertheless, in practice, under the above framework, some statistical arbitrage techniques (e.g., *pair trading*) can be exploited by traders. Explic-

[15] In continuous-time, the deflator $M(t)$ is given by $M(t) := \frac{1}{S_0(t)} \frac{d\mathbb{Q}}{d\mathbb{P}}$.

itly, these trades suffer from some sources of risk. They are not arbitrages and they are only demonstrably correct as both the amount of trading time an the liquidity approach infinity.

3 The $1/n$ Leveraged Portfolio

At time 0, the representative investor leverages k times the initial endowment E to A, s.t. the aggregated wealth (namely, $A := E + kE$) is then invested according to the equally-weighted heuristic. The investor allocates an equal weight of $1/n$ to each of the n securities in portfolio chosen from the homogeneous asset class of N risky securities available for investment (with $1 \leq n \leq N$). The amount of wealth which is debt-financed (i.e., kE) originates from a unique source of financing H (i.e., no diversification on debt-side) s.t. at time 0, $kE \equiv H$. At time t, the market value of debt is

$$H(t) = x_H(t)S_H(t) \ ,$$

with $x_H(t)$ and $S_H(t)$ respectively the quantity and price of debt. While, the asset side reads as

$$A(t) := E(t) + kE(t) = \sum_{i=1}^{n} y_i(t)S_i(t) + k\sum_{i=1}^{n} y_i(t)S_i(t) = \sum_{i=1}^{n} x_i(t)S_i(t) \ ,$$

where $y_i(t)$ is the number of the i^{th} security held in the unlevered portfolio (i.e., initial endowment E) and $x_i(t) = y_i(t) + ky_i(t) \geq 0$ is the number of the i^{th} security held in the leveraged portfolio.[16] Moreover,

$$u_i(t) = \frac{x_i(t)S_i(t)}{A(t)} \equiv 1/n \ , \tag{4}$$

is the proportion of wealth invested in the i^{th} security and $\sum_{i=1}^{n} u_i(t) = 1$ represents the budget equation for all $i = 1, ..., n$. We now define the leverage investor as follows.

Definition 3.1 (**Leverage**)**.** *The investor that at time* 0*, leverages the value of the initial endowment E by the debt amount $H := kE$, is called a leveraged investor with a leveraged index Φ defined as*

$$\Phi(t) := H(t)/A(t) \quad \textit{with } \Omega_\Phi = [a, 1] \ , \textit{ for all } t \geq 0 \ .$$

where Ω_Φ is the domain of Φ and $1 > a > 0$.

In the above definition, a represents a lower bound on the level of leverage. This captures the assumption that the investor does not have the incentive to reduce her debt and thus her leverage below a certain level. While the upper bound fixed at one means that the market value of the investment position cannot decreases below the market value of debt[17] At time 0, the leverage is $\Phi(0) := k/(1 + k)$, with $k := \Phi A/E$.

[16] $A(t)$ is the market value of the portfolio at time t which specifies the investor's position in each security.
[17] Otherwise the investor is forced to close the positions or goes bankrupt.

3.1 Self-Financing Condition

Under the assumption that the debt is a tradable asset, we incorporate short-long sales constraints into the model by restricting the investor to hold non negative net positions in the securities composing the $1/n$ portfolio and a non positive position in the special "security" $_H$.[18] We say that the trading strategy $\mathscr{S}$ (i.e., the pair of investment-borrowing positions), is admissible if it satisfies those constraints. More generally we have the following definition.

Definition 3.2 (**Admissible Strategy**). *Let $\mathscr{S} := (\mathbf{x}, x_H)$ be the pair of investment-borrowing positions so that (i) $\mathbf{x} := (x_1, ..., x_n) \in \mathbb{R}^n_+$ is the vector of risky assets where each component $x_i \in \mathbf{x}$ represents the number of units of the i^{th} security held in portfolio and (ii) $x_H \in \mathbb{R}_-$ indicates the quantity of the borrowing. A simple strategy $\mathscr{S}$ is said to be admissible if on $[0, T]$ it does not require selling short securities $i = 1, ..., n$, i.e., if $x_i \geq 0$ for all $i = 1, ..., n$, and if it does not require holding positive quantities of $_H$, i.e., if $x_H \leq 0$; s.t., $\{A(t)\} > C$ and $\{H(t)\} < -C$ for all $t \geq 0$ and for some constant $C \geq 0$.*[19]

We now impose the self-financing condition which characterizes the leveraged portfolio with no exogenous infusion or withdrawal of money. Namely, we assume that (i) the purchase of a new assets is financed solely by the sale of assets already in the portfolio, (ii) new funding is obtained solely by the repayment of previous debts.

Definition 3.3 (**Self-financing trading strategy**). *Let $\mathbf{x}$ and x_H be adapted-valued processes which are progressively measurable w.r.t. $\mathbb{F}$[20] s.t. both satisfy (P-a.s.) $\sum_{i=1}^n \int_0^t (x_i(s) S_i(s))^2 \, \mathrm{d}s < \infty$ and $\int_0^t (x_H(s) S_H(s))^2 \, \mathrm{d}s < \infty$.[21] Then, $\mathscr{S}$ is said to be a trading strategy. If $dA(t) = \sum_{i=1}^n \int_0^t x_i(s) dS_i(s)$ and $dH(t) = \int_0^t x_H(s) dS_H(s)$ for all $t \in [0, T]$ hold, then $\mathscr{S}$ is said to be self-financing.*

3.2 Investment-borrowing dynamics

Under the properties of admissible self-financing trading strategy $\mathscr{S}$ given in *Definitions 2.3-4*, and considered the $1/n$ heuristic (4), we assume the investor adopts a dynamic strategy defined as follows.

Assumption 3.1 (**1/n rebalanced allocation strategy**). *If the investment long-only position $\mathbf{x}$ of the strategy $\mathscr{S}$ locks the amount of wealth allocated to each i^{th} security in the $1/n$ portfolio, the strategy is called rebalanced and is denoted as $\mathscr{S}_d$.*

[18]To borrow funds, the investor is restricted to hold a non-positive net position in "asset"$_H$ (i.e., $x_H < 0$).

[19]Notice that if we work with the vector $\mathbf{u} := [u_1, ...u_n]$ of wealth proportion invested in each security, then we need to require that $A(t) > C$ for all $t \geq 0$. Otherwise, $\mathbf{u}$ is not well-defined. See eq. (4).

[20]This means the the investor can only use current and past information to trade, (no clairvoyants).

[21]This guarantee that the payoffs of the strategy have finite second moments.

Then, $\mathscr{S}_d$[22] represents a dynamic strategy where the composition of the investment long-only position $\mathbf{x}$ changes over time sot that each $x_i \in \mathbf{x}$ becomes a function of time and asset price S_i

$$x_i(t) = \frac{A(t)}{nS_i(t)}, \quad \forall\ i = 1, ..., n \quad and \quad \forall\ t \geq 0. \tag{5}$$

At each trading date the asset allocation is revised so that after rebalancing, the weights are such that the amount invested in each of the assets is again $1/n$. Generally speaking, the agent holds the portfolio $\mathbf{x}(t)$ during the period $[t-dt, t]$ and liquidates it at time t at the prevailing prices, simultaneously setting up the new long-only portfolio $\mathbf{x}(t+dt)$. Therefore, $\mathbf{x}$ represents an $\mathbb{F}$-predictable "contrarian" asset allocation strategy because the portfolio is rebalanced by selling "winners" and buying "losers" at any time t according to the formula $x_i(t+dt) = \frac{1/n\sum_i^n x_i(t)S_i(t+dt) - x_i(t)S_i(t+dt)}{S_i(t+dt)}$.[23]

Proposition 1 (**Investment-borrowing dynamics**). *Considered the dynamic self-financing trading strategy $\mathscr{S}_d$ in Assumption 2.3, under the condition that (i) the cost S_H of debt follows the dynamics $dS_H(t) = S_H(t)\left[(rf + \sigma\lambda_H)dt + \sigma dW_H(t)\right]$, (ii) the innovation terms $dW_{j=1,...n}(t)$ are correlated Wiener processes: $\mathbb{E}[dW_i(t), dW_j(t)] = \rho_{ij}dt$. Then, the aggregate investment-borrowing process $\{A(t), H(t)\}_{t\geq 0}$, under $\mathbb{P}$, is a continuous process with stochastic differential*[24]

$$\begin{cases} dA(t) = A(t)\left[\mu_A dt + \sigma_A dW_A^{\mathbb{P}}(t)\right] \\ dH(t) = H(t)\left[\mu_H dt + \sigma dW_H^{\mathbb{P}}(t)\right] \end{cases} \tag{6}$$

where $\mu_A = \frac{1}{n}\sum_{i=1}^n \mu_i = r_f + \sigma_A\lambda_A$, $\sigma_A = \sqrt{\frac{\sigma^2}{n} + \frac{n-1}{n}\rho_A\sigma^2}$, $\rho_A = \sum_{i=1}^n\sum_{j\neq i}^n \frac{\rho_{ij}}{n(n-1)}$[25]*, $\mu_H = r_f + \sigma\lambda_H$, $\mathbb{E}[dW_A(t), dW_H(t)] = \rho dt$ and $(dW_A^{\mathbb{P}}, dW_H^{\mathbb{P}}) \sim N(0, dt)$.*

Proof. See Appendix A. □

To avoid arbitrage opportunities in the market, a precise relation between λ_A and λ_H must hold under certain conditions. Indeed, for a positive (negative) debt-asset correlation ρ, the increment of asset value induces a higher (lower) debt value and vice-versa. This is stated in the following *Proposition*.

Proposition 2 (**Assets and debt market prices of risk**). *Given the free-arbitrage investment-borrowing dynamics in (6), the following relations hold*

$$\lambda_A = \lambda_H \Leftrightarrow (\rho = 1 \text{ and } \sigma_A = \sigma).$$

[22]The subindex d stands for "dynamic" or "rebalanced" investment (long-only) position.

[23]The requirement of $\mathbb{F}$-predictability ensures that $\mathbf{x}(t)$, the composition of the long-only portfolio to be held during the period $[t-dt, t]$, will be known at time $t-dt$, as soon as the information $\mathcal{F}_{t-dt}$ arrives. This is natural since $\mathbf{x}(t)$ has to be set up at time $t-dt$.

[24]Notice that λ_H is the market price of risk associated to the Brownian shock $W_H^{\mathbb{P}}$.

[25]It is reasonable to assume that the average correlation among n risky assets is not negative. Particularly when n is sufficiently large. Then, for the sake of simplicity and in line with other studies referring to stock correlation (e.g., Mao, 1970) we concentrate the analysis only on the positive range of correlation values between risky assets in the portfolio. See Table 2 in Appendix B.

Proof. See Appendix A. □

Since the market securities have the same variance as the debt (see *Assumption 2.2* and *Proposition 1*), *Proposition 2* says that if the investment is perfectly positively correlated with the the borrowing position (i.e., $\rho \to 1$) this directly implies that all the securities are perfectly positively correlated among them (i., $\rho_A \to 1$). Therefore, to avoid arbitrage opportunities, the market price of risks have to be the same. Namely, $\lambda_1 = ... = \lambda_N \equiv \lambda_H$.

3.3 How many and which securities ?

The $1/n$ rule raises two questions concerning the selection procedure. First, how many securities should be included in the final long-only position of the dynamic strategy $\mathscr{S}_d$? Second, given that n securities are in the portfolio, which n of the available securities are to be included ?

The first question has been answered for "not-leveraged" naive portfolios by measuring the reduction of risk – mapped by the portfolio's variance (or standard deviation), as the number of securities are expanded (see e.g., Bird and Tippett, 1986; Elton and Gruber, 1977; Evans and Archer, 1968; Johnson and Shannon, 1974; Statman, 1987). Section 5 through Section 4 will try to answer to this question in the case of a "leveraged" naive portfolio. The benefit of diversification will be determined by measuring the reduction of the default risk as the number of securities are expanded.

The second question has been usually answered by averaging the portfolios' standard deviation obtained by *random* selection procedures for increasing portfolio's size ($n = 1, 2, ...$).[26] See e.g.,Elton and Gruber (1977); Evans and Archer (1968). Differently, we do not impose any selection criteria. The investor is allowed to selects the $1/n$ portfolio which best satisfies her preferences with respect to risk and returns. Since N is the population size, for a portfolio with size $n < N$ we have a total of ${}_NC_n$ possible portfolios with the same portfolio size n. [27] The statistical properties that describe the difference between the $1/n$ portfolios of equal size are the mean market price of risk λ_A and the mean pairwise correlation ρ_A. For any portfolio of size $n = 1, 2, ...$, Section 5 explores the value of the investment-borrowing positions (6) for all the values in the range of λ_A and ρ_A in Table 2 in Appendix B. This is equivalent to consider all the combination ${}_NC_n$ of portfolios with size n.

The question of optimal diversification cannot be answered by the simple average variance of equally allocated portfolio returns. One must consider the possibility of alternative risk measures in presence of borrowed funds. The following section addresses this issue.

[26]A part the random selection procedure, the second question can be answered by adopting the simplifying assumptions of Sharpe's diagonal model Sharpe (1963).

[27]For example, for a population of 10 stocks, we have 45 portfolios of size 2 (i.e., $45 = {}_{10}C_2$), 120 portfolios of size 3 (i.e., $120 = {}_{10}C_3$), etc.

4 Log-Leverage and Default Risk

The default of an investor typically results from illiquidity. In this perspective, a leveraged investor faces two inter-related types of liquidity risk: (i) *asset liquidity*, i.e., the ability to sell or unwind positions and (ii) *funding liquidity*, i.e., the ability to meet obligations at due time. Leverage exacerbates both liquidity risks. So, a low initial endowment E w.r.t. the debt value H, implies a high profitability, but also a high liquidity risk. This is symptomatic of a fragile balance between internal and external sources of finance.

4.1 Log-leverage

In the remainder, we characterize the investor's solvency state by looking at the log-leverage

$$\phi(t) := \log\left[\Phi(t)\right] = \log\left[\frac{H(t)}{A(t)}\right] ; \tag{7}$$

where $\Omega_\phi := [-\varepsilon, 0]$ is the domain of ϕ and $-\varepsilon = log(a)$.

We will refer to a situation of "fragility" if ϕ is closed to zero and of "robustness" if ϕ is close to $-\varepsilon$ (i.e., the minimum debt state). This implies that a highly leveraged investor is also highly fragile and thus prone to default. On the other hand, an investor almost entirely self-financed is highly robust. In short,

$$\begin{cases} \Phi \sim 1 \Rightarrow \phi \sim 0 \Rightarrow \text{ "fragility state"} \\ \Phi \sim 0 \Rightarrow \phi \sim -\varepsilon \Rightarrow \text{ "robustness state".} \end{cases}$$

Definition 4.1 (**Default Event**). *The event that at any time $t \geq 0$, the log-leverage ratio $\phi(t)$ is equal to zero is classified as "default event". Namely,*

$$\textit{Defaul Event} \Leftrightarrow \phi(t) = 0 \ , \quad \forall t \geq 0 \ .$$

Via the Ito's Lemma we derive the dynamics of the investor's log-leverage, from the aggregate wealth-borrowing process (6), and eq.(7),

Proposition 3 (**Log-Leverage Dynamics**). *From the investment-borrowing dynamics (6), the log-transformation of leverage defined in 3.1, follows the dynamics*

$$d\phi(t) = \mu_\phi dt + \sigma_\phi dW_\phi^{\mathbb{P}}(t) \ , \tag{8}$$

where $\mu_\phi = \sigma\left(\lambda_H - \frac{\sigma}{2}\right) + \sigma_A\left(\frac{\sigma_A}{2} - \lambda_A\right)$, $\sigma_\phi = \sqrt{\sigma^2 + \sigma_A^2 - 2\rho\sigma\sigma_A}$ and $dW_\phi^{\mathbb{P}} \sim N(0, dt)$.

Proof. See Appendix A □

Eq. (8), describes the dynamics of an arithmetic Brownian motion with linearly growing mean $\mathbb{E}[\phi_t]$ and linearly growing variance $V[\phi_t]$.[28] Clearly, $\mathbb{E}[\phi_t] := \phi_0 + \mu_\phi(t)$ and $V[\phi_t] := \sigma^2_\phi(t)$.[29] In the following sections, we link together the concepts of *diversification*, *leverage* and *default probability*. Indeed, leverage and diversification have adverse effects on default risk. On one hand, in the traditional capital structure theory, leverage induces higher default risk. On the other hand, diversification reduces default risk. To address this issue, we characterizes the default probability with a first passage time approach.

4.2 Mean Time to Default

From the Definition 4.1 of default event and the dynamics of log-leverage in Proposition 3, we describe the default probability as follows.

Definition 4.2 (**Default Probability**). *Consider the probability space* $(\Omega_\phi, \mathcal{A}, \mathbb{P})$ *and the motion of* $\{\phi(t)\}_{t\geq 0}$ *on* $\mathbb{R}$. *Then, the default probability of a leveraged investor is* $\mathbb{P}(D) := \int_0^\infty f_\phi(0,\, t \geq 0 \mid \phi_0,\, 0)\mathrm{d}t$, *where* f_ϕ *is the density function of* $\{\phi(t)\}_{t\geq 0}$ *conditional to the initial value* $\phi(0) := \phi_0 \in (-\varepsilon, 0)$.

In words, $\mathbb{P}(D)$ is the probability that the process $\{\phi\}_{t\geq 0}$, initially starting at an arbitrary level $\phi_0 \in (-\varepsilon, 0)$ exits through the default barrier at zero after time $t \geq 0$. In a stochastic setting, it is common to relate the probability of hitting an absorbing barrier to the first passage time at the same barrier.

Definition 4.3 (**First Passage Time**). *Assuming that the diffusion process* $\{\phi(t)\}_{t\geq 0}$ *starts at* ϕ_0 *and its sample path is continuous, the first passage time of hitting the default barrier zero is defined by* $\tau = \inf\{t \geq 0 : \phi(t) \geq 0\}$. *In the case that zero is never reached, by convention,* $\inf\{\emptyset\} = \infty$[30].

The full characterization of τ depends on the transition probability density function $f_\phi(0,\, t \geq 0 \mid \phi_0,\, 0) = \frac{1}{\sqrt{2\pi\sigma^2_\phi t}} \exp\left(-\frac{(-\phi_0 - \mu_\phi)^2}{2\sigma^2_\phi t}\right)$. The mean first passage time denoted by $\bar{\tau}$, is exactly the expected first hitting time for the stochastic variable ϕ to reach the absorbing default barrier at zero.[31] The transition probability density function $f_\phi(0,\, t \geq 0 \mid \phi_0,\, 0)$ evolves in time according to the (forward) Fokker-Planck equation $\frac{\partial p}{\partial t} = -\frac{\partial}{\partial \phi}(\mu_\phi p) + \frac{1}{2}\frac{\partial^2}{\partial \phi}(\sigma^2_\phi p)$ where $p = f_\phi(0,\, t \geq 0 \mid \phi_0,\, 0)$. By working on the connection between the Fokker-Planck equation and the stochastic differential equation (8), we obtain the expression for $\bar{\tau}$. See Gardiner (2004).

[28] Since both drift and diffusion terms are constants we can write $\phi_t = \phi_0 + \int_0^t \mu_\phi\, \mathrm{d}(s) + \int_0^t \sigma_\phi\, \mathrm{d}W_\phi(s) = \phi_0 + \mu_\phi(t) + \sigma_\phi \lim_{N\to+\infty} \sum_{g=1}^N (W_\phi(t_g) - W_\phi(t_{g-1})) = \phi_0 + \mu_\phi(t) + \sigma_\phi W_\phi(t)$.

[29] Since $W(t)$ is normal, it follows that $\phi(t) \sim N\left(\left(\sigma(\lambda_H - \lambda_A) + \frac{\sigma_A^2 - \sigma^2}{2}\right)t, \sigma^2_\phi t\right)$.

[30] Recall that through (7) we simplify the problem moving from a term structure of threshold values $H(t)$ to a fixed constant barrier at zero.

[31] A standard approach to derive $f_\phi(0,\, t \geq 0 \mid \phi_0,\, 0)$ is to inverse its Laplace transform $\mathbb{E}[e^{-\gamma\tau}]$, where $\gamma < 0$. The relation between the mean first passage time and the Laplace transform of first passage time is established by the equation $\bar{\tau} = -\lim_{\gamma\to 0^+} \frac{\partial \mathbb{E}[e^{-\gamma\tau}]}{\partial \gamma}$.

Lemma 4.4 (Mean Time to Default). *Given the log-leverage dynamics (8), the definition of default probability in 4.2 and first passage time in 4.3, the mean time taken by a leveraged investor to enter into a default state is*

$$\bar{\tau} = \left(\frac{\sigma_\phi^2}{2\mu_\phi^2}\right)\left[\exp\left(\frac{-2(\varepsilon+\phi_0)\mu_\phi}{\sigma_\phi^2}\right) - \exp\left(\frac{-2\varepsilon\mu_\phi}{\sigma_\phi^2}\right)\right] - \left(\frac{\phi_0}{\mu_\phi}\right) . \tag{9}$$

Proof. See Appendix A. □

The extended form of expression (9), shows a direct dependence of $\bar{\tau}$ on various parameters and in particular on the number n of securities in the long position of the strategy $\mathscr{S}_d$.

$$\bar{\tau} = \left(\frac{\sigma^2+\sigma_A^2-2\rho\sigma\sigma_A}{2\left(\sigma\left(\lambda_H-\frac{\sigma}{2}\right)+\sigma_A\left(\frac{\sigma_A}{2}-\lambda_A\right)\right)^2}\right) \times \left[\exp\left(\frac{-2(\varepsilon+\phi_0)\left(\sigma\left(\lambda_H-\frac{\sigma}{2}\right)+\sigma_A^2\left(\frac{\sigma_A}{2}-\lambda_A\right)\right)}{\sigma^2+\sigma_A^2-2\rho\sigma\sigma_A}\right)\right.$$
$$\left. - \exp\left(\frac{-2\varepsilon\left(\sigma\left(\lambda_H-\frac{\sigma}{2}\right)+\sigma_A^2\left(\frac{\sigma_A}{2}-\lambda_A\right)\right)}{\sigma^2+\sigma_A^2-2\rho\sigma\sigma_A}\right)\right] - \left(\frac{\phi_0}{\left(\sigma\left(\lambda_H-\frac{\sigma}{2}\right)+\sigma_A\left(\frac{\sigma_A}{2}-\lambda_A\right)\right)}\right) .$$

where $\sigma_A = \sqrt{\frac{\sigma^2}{n}+\frac{n-1}{n}\rho_A\sigma^2}$.

Lemma 4.4 implies that the mean time to default grows with the inverse of the expected log-leverage, $\mathbb{E}[\phi]$. Precisely, if the expected log-leverage increases (i.e., $\mathbb{E}[\phi] \to 1$) the mean time to default decreases and vice-versa. Moreover, the mean time to default depends on the log-leverage's volatility (measured by the variance $V[\phi_t]$), as follows.

Corollary 1. *(i) The mean time to default decreases with the volatility of the log-leverage, if the investor's log-leverage trend is positive (more fragile investor); (ii) The mean time to default increases with the volatility of the log-leverage, if the investor's log-leverage trend is negative (more robust investor).*

Proof. See Appendix A. □

In words, if the fragility is increasing (i.e., $\mu_\phi > 0$), the mean time to default is closer. Thus, lower volatility is desirable because it reduces the likelihood that some trajectories of $\{\phi(t)\}_{t\geq 0}$ hit the upper default bound (0). Namely, $\partial\bar{\tau}/\partial V[\phi_t] < 0$ for $\mu_\phi > 0$. While, if the robustness is increasing (i.e., $\mu_\phi < 0$), the mean time to default is faraway. Thus, a higher volatility is desirable since it increases the likelihood to reach the lower safe barrier ($-\varepsilon$). Namely, $\partial\bar{\tau}/\partial V[\phi_t] > 0$ for $\mu_\phi < 0$. Moreover, Proposition 3 implies also a precise relation between the variance $V[\phi_t]$ and the debt-asset correlation ρ, which impacts on $\bar{\tau}$ as follows.

Corollary 2. *(i) The mean time to default increases with positive debt-asset correlation (i.e., $\rho > 0$), if the investor's log-leverage trend is positive (more fragile investor); (ii) The mean time to default increases* *with negative debt-asset correlation (i.e., $\rho < 0$), if the investor's log-leverage trend is negative (more robust investor).*

Proof. See Appendix A. □

In a nutshell, Corollary 2 says that in bad times (i.e., when the investor's fragility is raising), positive debt-asset correlation is desirable so that lower market value of assets is coupled with a lower debt value and the volatility of the log-leverage decays. While, in good times (i.e., when the investor's robustness is increasing), negative debt-asset correlation is desirable so that higher market values of assets is coupled with a lower debt value and the volatility of the log-leverage growths.

5 The optimal 1/n Leveraged Portfolio

The aim of this section is to answer to the first question posed in the sub-Section 3.3. Namely, how many securities should be included in the final long-only position of the dynamic strategy $\mathscr{S}_d$ in order to minimize the portfolio's risk. In a idealized frictionless world, we answer to this question by studying the optimal number n which minimizes $\bar{\tau}$ in Lemma 4.4. In this respect, the present section provides the result of a comparative static analysis of (8) supported by a numerical analysis of (9) – in reduced form written as $\bar{\tau} = \bar{\tau}(\phi(n))$, conducted in Appendix B.

Definition 5.1 (**Optimal 1/n Leveraged Portfolio**)**.** *The optimal $1/n^*$ leveraged portfolio is the one that maximizes the mean time to default (i.e., minimizes the default probability)*

$$n^* = \arg\max_n \left[\bar{\tau}(\phi(n))\right]. \tag{10}$$

Proposition 4 (**Optimal Diversification**)**.** *Under the results in Corollary 1-2, in a frictionless free-arbitrage market, the optimal $1/n^*$ leveraged portfolio is*

$$n^* = \begin{cases} \sigma^2(\rho_A - 1)/(\rho_A\sigma^2 - \lambda_A^2) \text{ when } \mathbb{E}[\phi_t] \text{ is U-shaped w.r.t. } n \\ (\rho_A - 1)/(\rho_A - \rho^2) \text{ when } V[\phi_t] \text{ is U-shaped w.r.t. } n. \end{cases} \quad s.t., \bar{\tau} \leq \bar{\tau}^*\ , \forall n \gtrless n^*.$$

Proof. See Appendix A and Cases (a.iii), (b.i) , (b.ii) and (b.iii) in Appendix B (Analysis). □

In words, Proposition 4 says that when $\mathbb{E}[\phi_t]$ or $V[\phi_t]$ are U-shaped w.r.t. n, there exists an optimal intermediate $1/n^*$ heuristic (i.e., $1 \leq n^* \leq N$), which maximizes $\bar{\tau}$ for all $n \gtrless n^*$. See Figure 1 and 3 in Appendix D. Two different critical points emerge $n^* = \sigma^2(\rho_A-1)/(\rho_A\sigma^2-\lambda_A^2)$ and $n^* = (\rho_A-1)/(\rho_A-\rho^2)$, which converge to the same iff $\lambda_A = \sigma|\rho|$. Under this condition, there exists a unique critical point which minimizes both $\mathbb{E}[\phi_t]$ and $V[\phi_t]$. (See Figure 2 in Appendix D). Instead, when both $\mathbb{E}[\phi_t]$ and $V[\phi_t]$ are not U-shaped w.r.t. n, the mean time to default is a concave monotonically increasing function of n s.t., maximum diversification is the best default-induced portfolio strategy (i.e., $n^* \to N$). See Analysis in Appendix B.

In Appendix C we carry out a robustness analysis by solving the maximization problem $\arg\max_n[\bar{\tau}(\phi(n)]$ for the static (i.e., not-rebalanced) allocation strategy $\mathscr{S}_s$.[32] In this case, the investor allocates $1/n$ at the initial date and then holds the portfolio until the terminal date ("buy-and-hold"). In the real world, it is reasonable to think that investors re-adjust the weights $x_i \in \mathbf{x}$ at any arbitrary time interval Δt in order to maintain a certain return-risk profile and minimize the rebalancing costs. Therefore, the two strategies $\mathscr{S}_d$ and $\mathscr{S}_s$ delimit the span of all admissible $1/n$ self-financing trading strategies $\mathscr{S}$. Even though we are in a idealized frictionless world, as seen for $\mathscr{S}_d$ also for $\mathscr{S}_s$ maximum diversification is not always desirable. We detect an optimal intermediate $1/n$ portfolio (with $1 < n < N$) that minimizes the mean time to default. See Appendix C and Figures 4 in Appendix D.

6 Conclusions

Since the pioneering work of Evans and Archer (1968), the optimal number of securities to include in a portfolio in order to minimize the investor's risk has been addressed both theoretically and experimentally by several studies. They model the risk-reduction benefits of naive diversification, where risk is measured by the portfolio's standard deviation. Under the *Efficient Market Hypothesis* of an idealized frictionless world, the authors suggest maximum diversification (a.k.a.,"buy the market portfolio"). Differently, our theoretical findings show the existence of an intermediate optimal diversification level in the case of leveraged naive portfolio. The crucial point to understand our results is the role played by the mean pairwise asset correlation ρ_A and the debt-asset correlation ρ. For negative values of ρ (especially when it is closed to -1), the paper confirms the standard results already known in the literature for the (unlevered) naive diversification. The optimal $1/n^*$ allocation is the one for which $n^* \to N$. Rather, when ρ from negative values moves towards zero and beyond, maximum diversification is not always desirable in terms of time to default. The optimal $1/n^*$ diversification is an intermediate one for which $1 < n^* < N$.[33] The mechanism behind this counterintuitive result is very simple and mainly explained in terms of the log-leverage's variance. As the variance of the assets is rescaled down by the number n of securities held in portfolio, the log-leverage's variance becomes a unimodal (U-shaped) function of n. Eventually, the corresponding time to default assumes an inverted U-shaped form with respect to' n.

[32] The subindex s stands for "static" or "not-rebalanced" investment (long-only) position.

[33] Also in the classic situation investigated since Merton (1974), debts are positively correlated with assets. There, however, debt value is assumed to be a function of firm value and thus diversification would decreases both asset and debt volatility. As a result, by repeating the same exercise in the Merton framework, the non-monotonic effect of diversification on time to distress would not be present.

A Proofs

A.1 Proposition 1

Proof. Let us describe the dynamics of the long-only position $\mathbf{x}$ which belong to the pair of investment-borrowing positions $\mathscr{S} := (\mathbf{x}, x_H)$ in Definition 3.2. The resulting long-only portfolio is $A := \mathbf{x}'\mathbf{S}$. Despite Assumptions 2.2, we start from a generic price equation $dS_i(t) = S_i(t)\left[\mu_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j^{\mathbb{P}}(t)\right]$. Then, A has dynamics

$$dA(t) = \sum_{i=1}^{n} x_i S_i \mu_i dt + \sum_{i=1}^{n} x_i S_i \sigma_{ij} dW_j^{\mathbb{P}}(t) \; . \tag{11}$$

We are interested in the dynamics of the $1/n$ portfolio composed by equally investments in n securities selected from the population of size $N \geq n$. Under the Assumption 3.1, the portfolio is continuasly rebalanced over time so that $x_i(t) = \frac{A(t)}{nS_i(t)}$ for all $t \geq 0$. Then, the drift term of (11) becomes $[x_1(t)S_1(t)\mu_1 + x_2(t)S_2(t)\mu_2 + ... + x_n(t)S_n(t)\mu_n]\, dt = \frac{A(t)}{n}(\mu_1 + \mu_2 + ... + \mu_n)dt + ... = A(t)\mu_A dt$ where $\mu_A = \frac{1}{n}\sum_{i=1}^{n}\mu_i$. The diffusion term is a weighted sum of n Gaussian correlated shocks: $\frac{A(t)}{n}\sum_{i=1}^{n} dW_i(t)$ which is equivalent to $A(t)\frac{1}{n}\sum_{i=1}^{n}\sqrt{dt}\xi_i$ with $\xi_i \sim N(0,1)$. For the variance we obtain

$$\begin{aligned}\sigma_A^2 &= A^2(t)\frac{dt}{(n)^2}\sum_{i=1}^{n} Var(\xi_i) + A^2(t)\frac{dt}{(n)^2}\sqrt{Var(\xi_i)Var(\xi_j)}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\rho_{ij} \\ &= A^2(t)\frac{dt}{n}\bar{\sigma}_\xi^2 + A^2(t)\frac{dt}{(n)^2}\sqrt{\sigma_{\xi_i}^2\sigma_{\xi_j}^2}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\rho_{ij}\end{aligned} \tag{12}$$

where $\bar{\sigma}_\xi^2$ is the average variance among the n's $\sigma^2_{\xi_{i=1,\ldots,n}}$. Now we consider Assumption 2.2, which imposes $\sigma_{\xi_i}^2 = \sigma^2$ for all $i = 1, ..., N$. Then multiplying and dividing the second term of (12) by $n-1$ and taking its square root, we get $\sigma_A = A(t)\sqrt{\frac{\sigma^2}{n} + \frac{n-1}{n}\rho_A\sigma^2}\sqrt{dt}\xi$ where $\rho_A = \sum_{i=1}^{n}\sum_{j\neq i}^{n}\frac{\rho_{ij}}{n(n-1)}$ and $\xi \sim N(0,1)$. Hence (11) becomes $dA(t) = A(t)\mu_A dt + A\sigma_A dW_A^{\mathbb{P}}(t)$, where $\sigma_A = \sqrt{\frac{\sigma^2}{n} + \frac{n-1}{n}\rho_A\sigma^2}$, $\rho_A = \sum_{i=1}^{n}\sum_{j\neq i}^{n}\frac{\rho_{ij}}{n(n-1)}$, $\mu_A = \frac{1}{n}\sum_{i=1}^{n}\mu_i$ and $dW_A^{\mathbb{P}} \sim n(0, dt)$.

Under the *law of one price*, the return of the portfolio (i.e., μ_A) has to obey the no-arbitrage condition. Indeed the portfolio A can be understood as a new security generated from a linear combination of n securities, each with price S_i dependent on a different exogenous factor (with $i = 1, ..., n$). Then, A is affected by a "new" Brownian shock $W_A^{\mathbb{P}}$, which synthesizes the contribution of all $W_i^{\mathbb{P}}$. The extra return over the risk-less rate per unit of volatility loading onto the "new" factor, is represented by the new market price of risk λ_A associated with the Brownian shock $W_A^{\mathbb{P}}$. To show this, let assume the presence of a pricing kernel M with dynamics $dM(t) = -r_f M dt - M\lambda_A dW_A^{\mathbb{P}}(t)$. Where r_f is the risk-free rate, λ_A is the market price of risk associated to the 'new' Brownian shock $W_A^{\mathbb{P}}$. Now, in order to avoid arbitrage opportunities we apply the martingale condition. Hence, with slight abuse of notation, there is no arbitrage if and only if $\mathbb{E}_t^{\mathbb{P}}[d\{AM\}] = 0$ which has the precise mathematical meaning that the drift in the SDE for the product of A and M is zero (hence, the product is a martingale). By Ito's lemma, this condition implies $[\mu_A - rf - \lambda_A\sigma_A]\, dt = 0$ which turns out to express the expected return of portfolio A in a free arbitrage economy $\mu_A = r_f + \sigma_A\lambda_A$. □

A.2 Proposition 2

Proof. We have to check the conditions under which the market prices of risk (i.e., λ_A, λ_H) of the aggregated wealth-borrowing process in (6) are related to each other. In a similar spirit as Merton (1974, 1973), Let's normalize to one the total amount of dollars invested in the aggregate portfolio such that Φ is the fraction of wealth debt-financed and $1-\Phi$ is covered by the initial endowment (See Definition 2.2). The fraction Φ, is a zero-net investment which is achieved by using the proceeds of short-sales (or borrowing) to finance long positions. The condition of zero aggregate investment can be written as $\Phi(dA/A) - \Phi(dH/H) = 0$. Given the investment-borrowing dynamics (6), we can rewrite the condition as $\Phi\left[\mu_A dt + \sigma_A dW_A^{\mathbb{P}}(t)\right] - \Phi\left[\mu_H dt + \sigma dW_H^{\mathbb{P}}(t)\right] = 0$ which hold for any choice of $\Phi > 0$, s.t. $\Phi\left[\mu_A - \mu_H\right] dt + \Phi\left[\sigma_A dW_A^{\mathbb{P}}(t) - \sigma dW_H^{\mathbb{P}}(t)\right] = 0$. Since there are two distinct positions (A and H) and two different sources of risk ($dW_A^{\mathbb{P}}$ and $dW_H^{\mathbb{P}}$) only under certain conditions on their correlation ρ it would be possible to eliminate the risky part of the zero aggregate investment position s.t.

$$\Phi\left[\sigma_A dW_A^{\mathbb{P}}(t) - \sigma dW_H^{\mathbb{P}}(t)\right] = 0\ . \tag{13a}$$

A nontrivial solution $\Phi \neq 0$ to (13a) implies $\sigma_A dW_A^{\mathbb{P}}(t) = \sigma dW_H^{\mathbb{P}}(t)$ and so $\frac{\sigma_A}{\sigma} = \frac{dW_H^{\mathbb{P}}(t)}{dW_A^{\mathbb{P}}(t)}$. Considered that $\mathbb{E}[dW_A^{\mathbb{P}}(t)dW_H^{\mathbb{P}}(t)] = \rho dt$, we have

$$\begin{cases} \frac{\sigma_A}{\sigma} = \frac{dW_H^{\mathbb{P}}(t)}{dW_A^{\mathbb{P}}(t)} \\ dW_H^{\mathbb{P}}(t) = \frac{\rho dt}{dW_A^{\mathbb{P}}(t)} \end{cases} \implies \begin{cases} \sigma_A = \rho\sigma \\ \sigma_A = \frac{\sigma}{\rho} \end{cases} \implies \begin{cases} \rho = 1 \\ \sigma = \sigma_A \end{cases}$$

Since, $\sigma_A = \sqrt{\frac{\sigma^2}{n} + \frac{n-1}{n}\rho_A\sigma^2}$, the solution $\sigma_A = \sigma$ from the condition (13a) imposes $\rho_A = 1$. Moreover, to prevent arbitrage opportunities, the excess return above the short rate r_f must be zero

$$\Phi\left[(\mu_A - r_f) - (\mu_H - r_f)\right] = 0\ . \tag{13b}$$

A nontrivial solution $\Phi \neq 0$ to (13b) is given by $(\mu_A - r_f) = (\mu_H - r_f)$. Together, the two solutions define the free-arbitrage condition on the market prices

$$\frac{\mu_A - r_f}{\sigma_A} = \frac{\mu_H - r_f}{\sigma} \Leftrightarrow (\rho = 1 \text{ and } \sigma_A = \sigma)\,.$$

This means that in equilibrium, the investment-borrowing strategy defined in (6) is riskless with an expected return equal to zero only when (i) the securities in the asset side are perfectly correlated; (ii) the asset portfolio is perfectly correlated with the debt (i.e., $\mathbb{E}[dW_A^{\mathbb{P}}(t)dW_H^{\mathbb{P}}(t)] = 1$). Both borrowing and asset values will be driven by the same shock, which –in order to avoid arbitrage–, have to be reflected in the same market price of risk. Under these two conditions, the market price of risk of assets and the market price of risk of debt converges together such that the expected return on the investment-borrowing position becomes zero, i.e., $\mathbb{E}(dA + dH) = 0$. □

A.3 Proposition 3

Proof. From (7), we compute the partial derivatives with respect to t, H and A: $\phi_t(t,H,A) = 0$, $\phi_H(t,H,A) = \frac{1}{H}$, $\phi_{HH}(t,H,A) = -\frac{1}{H^2}$, $\phi_A(t,H,A) = -\frac{1}{A}$, $\phi_{AA}(t,H,A) = \frac{1}{A^2}$, $\phi_{HA}(t,H,A) = \phi_{AH}(t,H,A) = 0$. Then we use the product rule to compute the square and the cross-product of the single processes $(dH)^2 = \sigma^2 H^2 dt$ and $(dA)^2 = \sigma_A^2 A^2 dt$. Now using (6), by Ito's lemma we obtain

$$\begin{aligned}
d\phi &= \phi_t dt + \phi_H dH + \phi_A dA + \frac{1}{2}[\phi_{HH}(dH)^2 + 2\phi_{AH} dH dA + \phi_{AA}(dA)^2] \\
&= \frac{1}{H}[(r_f + \sigma\lambda)Hdt + \sigma H dW^{\mathbb{P}}] - \frac{1}{A}[(r_f + \sigma_A\lambda_A)Adt + \sigma_A A dW_A^{\mathbb{P}}] + \frac{1}{2}[-\frac{1}{H^2}\sigma^2 H^2 dt + \frac{1}{A^2}\sigma_A^2 A^2 dt] \\
&= \left((r_f + \sigma\lambda_H) - (r_f + \sigma_A\lambda_A) + \frac{\sigma_A^2 - \sigma^2}{2}\right)dt + \sigma dW^{\mathbb{P}} - \sigma_A dW_A^{\mathbb{P}} \\
&= \left(\sigma\lambda_H - \sigma_A\lambda_A + \frac{\sigma_A^2 - \sigma^2}{2}\right)dt + \sigma_\phi dW_\phi^{\mathbb{P}} = (\sigma_A(\sigma_A/2 - \lambda_A) + \sigma(\lambda_H - \sigma/2))\, dt + \sigma_\phi dW_\phi^{\mathbb{P}} \ ,
\end{aligned} \tag{14}$$

where $\sigma_\phi = \sqrt{\sigma^2 + \sigma_A^2 - 2\rho\sigma\sigma_A}$ and $dW_\phi^{\mathbb{P}} \sim N(0, dt)$. □

A.4 Lemma 3.4

Proof. Let us consider the log-leverage process $\{\phi(t)\}_{t\geq 0}$ with dynamics described in (8), and assume that (i) it moves in $\mathbb{R}$; (ii) at time $t = 0$, $\phi(0) := \phi_0$ is at an arbitrary level in $(-\varepsilon, 0) = \{\phi \mid -\varepsilon \leq \phi_0 \leq 0\}$, where $-\varepsilon$ is the reflecting barrier and 0 is the absorbing one. Then, following Gardiner (2004), the mean first passage time taken by $\{\phi(t)\}_{t\geq 0}$ to touch the upper default barrier 0, is the solution of the following equation

$$\begin{aligned}
\bar{\tau} &= 2\int_{\phi}^{0} \mathrm{d}\phi' \int_{-\varepsilon}^{\phi'} \frac{\exp\left(\int_0^{\phi''} \frac{2\mu(\phi''')}{\sigma(\phi''')^2}\,\mathrm{d}\phi'''\right)}{\exp\left(\int_0^{\phi'} \frac{2\mu(\phi''')}{\sigma(\phi''')^2}\,\mathrm{d}\phi'''\right)\sigma(\phi'')^2}\,\mathrm{d}\phi'' \\
&= 2\int_{\phi}^{0} \mathrm{d}\phi' \int_{-\varepsilon}^{\phi'} \frac{1}{\sigma(\phi'')^2}\exp\left(\int_0^{\phi''} \frac{2\mu(\phi''')}{\sigma(\phi''')^2}\,\mathrm{d}\phi''' - \int_0^{\phi'} \frac{2\mu(\phi''')}{\sigma(\phi''')^2}\,\mathrm{d}\phi'''\right)\mathrm{d}\phi'' \\
&= 2\int_{\phi}^{0} \mathrm{d}\phi' \int_{-\varepsilon}^{\phi'} \frac{\exp\left(\int_{\phi'}^{\phi''} \frac{2\mu(\phi''')}{\sigma(\phi''')^2}\,\mathrm{d}\phi'''\right)}{\sigma(\phi'')^2}\,\mathrm{d}\phi'' \ ,
\end{aligned} \tag{15}$$

where $\mu(\phi) = (\sigma(\lambda_H - \sigma/2) + \sigma_A(\sigma_A/2 - \lambda_A))$ and $\sigma(\phi) = \sqrt{\sigma^2 + \sigma_A^2 - 2\rho\sigma\sigma_A}$. The solution of (15) yields (9). □

A.5 Corollary 1

Proof. The first statement to be verified is that the partial derivative of the mean time to default w.r.t. μ_ϕ is positive when $\mu_\phi < 0$ and is negative when $\mu_\phi > 0$. From (9), we get $\partial\bar{\tau}/\mu_\phi = \frac{1}{2\mu_\phi^3}\left(M^\varepsilon(2\varepsilon\mu_\phi + \sigma_\phi^2) - M^\alpha(2\alpha\mu_\phi + \sigma_\phi^2) + 2\phi_0\right)$, where $M := \exp\left(-\frac{2\mu_\phi}{\sigma_\phi^2}\right)$ and $\alpha := \varepsilon + \phi_0 > 0$.

$$\therefore \text{if} \begin{cases} \mu_\phi < 0, & M^\alpha > M^\varepsilon \Rightarrow \frac{\partial\bar{\tau}}{\mu_\phi} > 0 \\ \mu_\phi > 0, & M^\alpha < M^\varepsilon \Rightarrow M^\varepsilon(2\varepsilon\mu_\phi + \sigma_\phi^2) - M^\alpha(2\alpha\mu_\phi + \sigma_\phi^2) + 2\phi_0 < 0 \Rightarrow \frac{\partial\bar{\tau}}{\mu_\phi} < 0. \end{cases}$$

The second statement says that a lower variance $V[\phi_t]$ induces a longer mean time to default. Let's analyze the partial derivative of $\bar{\tau}[\phi_t]$ w.r.t. $V[\phi_t]$. We get $\partial\bar{\tau}/\sigma^2_\phi = \frac{1}{2\mu^2_\phi\sigma^2_\phi}\left(M^\alpha(2\alpha\mu_\phi + \sigma^2_\phi) - M^\varepsilon(2\varepsilon\mu_\phi + \sigma^2_\phi)\right)$.

$$\therefore \text{ if} \begin{cases} \mu_\phi < 0, & M^\alpha > M^\varepsilon \Rightarrow \frac{\partial\bar{\tau}}{\sigma^2_\phi} > 0 \\ \mu_\phi > 0, & M^\alpha < M^\varepsilon \Rightarrow M^\alpha(2\alpha\mu_\phi + \sigma^2_\phi) - M^\varepsilon(2\varepsilon\mu_\phi + \sigma^2_\phi) < 0 \Rightarrow \frac{\partial\bar{\tau}}{\sigma^2_\phi} < 0. \end{cases}$$

To explain why the inequality $M^\alpha(2\alpha\mu_\phi + \sigma^2_\phi) - M^\varepsilon(2\varepsilon\mu_\phi + \sigma^2_\phi) < 0$ holds true in case $\mu_\phi > 0$ is explained as follows. First, the expression can be read also as $\exp\left(\frac{2\mu_\phi}{\sigma^2_\phi}(\varepsilon - \alpha)\right) < \varepsilon/\alpha$. It is immediate to observe that since $\mu_\phi > 0$, also $2\mu_\phi/\sigma^2_\phi > 0$ and for any α and ε s.t. $a > \varepsilon$, the inequality is always verified. In the case the log-leverage is decreasing, higher volatility is desirable because increase the time to default. While, in the case the log-leverage is increasing, lower volatility is desirable because it lengthens the time to default. In particular, let for the sake of simplicity, the assets in portfolio to be uncorrelated (i.e., $\rho_A = 0$). Then, the drift term in (8) becomes $\mu_\phi = \sigma\left(\lambda_H - \frac{\sigma}{2}\right) + \sigma/\sqrt{n}\left(\frac{\sigma/\sqrt{n}}{2} - \lambda_A\right)$. It can either assume positive or negative values. *Ceteris paribus*, μ_ϕ is positive when $n > \left(2\lambda^2_A - 2\lambda_H\sigma + \sigma^2 + 2\sqrt{\lambda^2_A(\lambda^2_A - 2\lambda_H\sigma + \sigma^2)}\right)/(\sigma - 2\lambda_H)^2$. In this case, further diversification would be desirable. While, for a diversification range below the threshold $n < \left(2\lambda^2_A - 2\lambda_H\sigma + \sigma^2 + 2\sqrt{\lambda^2_A(\lambda^2_A - 2\lambda_H\sigma + \sigma^2)}\right)/(\sigma - 2\lambda_H)^2$, the investor will benefit from lower diversification. □

A.6 Corollary 2

Proof. Consider the expression for $V[\phi_t]$ in expanded form. Namely, $V[\phi_t] := \left(\sigma^2 + \frac{\sigma^2}{n} + \frac{n-1}{n}\rho_A\sigma^2 - 2\rho\sigma\sqrt{\frac{\sigma^2}{n} + \frac{n-1}{n}\rho_A\sigma^2}\right)t$. Since $\rho \in [-1, 1]$, the differentiation of $V[\phi_t]$ w.r.t. $\rho > 0$, gives $\partial V[\phi_t]/\partial\rho = -2\sigma\sqrt{\frac{\sigma^2}{n} + \frac{n-1}{n}\rho_A\sigma^2}$ which is always negative. While, the differentiation of $V[\phi_t]$ w.r.t. $\rho < 0$, gives $\partial V[\phi_t]/\partial\rho = 2\sigma\sqrt{\frac{\sigma^2}{n} + \frac{n-1}{n}\rho_A\sigma^2}$ which is always positive. Then, for any $\rho_a, \rho_b \in (0, 1]$ s.t. $\rho_a < \rho_b$, $V[\phi_t](\rho_a) > V[\phi_t](\rho_b)$. While, for any $\rho_a, \rho_b \in [-1, 0)$ s.t. $\rho_a < \rho_b$, $V[\phi_t](\rho_a) < V[\phi_t](\rho_b)$. So, $V[\phi_t]$ is a monotone decreasing function of ρ when $\rho \in (0, 1]$ and it is a monotone increasing function of ρ when $\rho \in [-1, 0)$. Since the Proof of Corollary 1 shows that $\bar{\tau}$ is a monotonic decreasing function of $V[\phi_t]$ when $\mu_\phi > 0$, when $\rho \in (0, 1]$, then $V[\phi_t]$ is monotone decreasing in ρ. So, when $\mu_\phi > 0$, positive debt-asset correlation is desirable since reduces the default risk by increasing the mean time to default. Moreover, the Proof of Corollary 1 shows that $\bar{\tau}$ is a monotonic increasing function of $V[\phi_t]$ when $\mu_\phi < 0$. Then when $\rho \in [-1, 0)$, $V[\phi_t]$ is monotone increasing in ρ. So, when $\mu_\phi < 0$, negative debt-asset correlation is desirable since reduces the default risk by increasing the mean time to default. Then, we conclude that, the default risk is reduced (increased) for (i) positive (negative) debt-asset correlation when the investor's log-leverage is increasing, (ii) negative (positive) debt-asset correlation when the investor's log-leverage is decreasing. □

A.7 Proposition 4

Proof. Let explicitly write the dependency of ϕ_t from n ($\phi_t = \phi_t(n)$) and take its first and second moment i.e., consider $\mathbb{E}[\phi_t(n)]$ and $V[\phi_t(n)]$. Hereafter we study how they change w.r.t. n. Then in Appendix B (Analysis), the results obtained here are jointly analyzed.

Expected value $\mathbb{E}[\phi_t(n)]$ as a function of n. Let us consider $\lim_{n\to+\infty} \mathbb{E}[\phi_t(n)] = \phi_0 + \sigma\sqrt{\rho_A}\left(\frac{\sigma\sqrt{\rho_A}}{2} - \lambda_A\right) + \sigma\left(\lambda_H - \frac{\sigma}{2}\right)$ and $\mathbb{E}[\phi_t(1)] = \phi_0 + \sigma(\lambda_H - \lambda_A)$. The order relation between $\lim_{n\to+\infty} \mathbb{E}[\phi_t(n)]$ and $\mathbb{E}[\phi_t(1)]$ is explained by the following table

$\mathbb{E}[\phi_t(1)] > \lim_{n\to+\infty} \mathbb{E}[\phi_t(n)]$		$\mathbb{E}[\phi_t(1)] < \lim_{n\to+\infty} \mathbb{E}[\phi_t(n)]$	
λ_A	ρ_A	λ_A	ρ_A
$[0, \frac{\sigma}{2})$	$[0, 1)$	$(\frac{\sigma}{2}, \sigma]$	$[0, -4\frac{\lambda_A}{\sigma} + \frac{4\lambda_A^2+\sigma^2}{\sigma^2})$
$(\frac{\sigma}{2}, \sigma)$	$(-4\frac{\lambda_A}{\sigma} + \frac{4\lambda_A^2+\sigma^2}{\sigma^2}, 1)$	$(\sigma, 1]$	$[0, 1]$

for any value of σ^2 and λ_H in Table 2. A priori, it is not well defined the comparison between $\mathbb{E}[\phi_t(1)]$ and $\lim_{n\to+\infty} \mathbb{E}[\phi_t(n)]$. As shown in the previous table, their difference depends on the values assumed by λ_A and ρ_A. The partial derivative of $\mathbb{E}[\phi_t]$ with respect to n is $\partial\mathbb{E}[\phi_t(n)]/\partial n = \sigma(\rho_A - 1)(\sigma_A - \lambda_A)/2\sigma_A n^2$. By imposing the FOC: $\partial\mathbb{E}[\phi_t(n)]/\partial n = 0$ and avoiding boundary solutions (i.e., solutions for $\rho_A = 1$, $\lambda_A = 0$, $\lambda_A = 1$, $n = 1$), the critical point is $n^* = \sigma^2(\rho_A - 1)/(\rho_A\sigma^2 - \lambda_A^2)$.[34] The second partial derivative $\partial^2\mathbb{E}[\phi_t(n)]/\partial n^2$ is always negative in n^*, hence it is a minimum. Under the condition of positive mean correlation between assets in portfolio (i.e., $\rho_A > 0$), the optimal portfolio size takes values in the interval $[1, \infty)$. [35] In particular, the optimal size n^* is lower then infinity when the mean asset correlation is below a certain threshold (i.e., when $\rho_A << \lambda_A^2/\sigma^2$). In this particular set, $\mathbb{E}[\phi_t(n)]$ is a unimodal (U-shaped) function of n which is monotonically decreasing for all $n < n^*$ and monotonically increasing for all $n \geq n^*$. Hence, by *Corollary 1*, the mean time to default is monotonically increasing for all $n < n^*$ and monotonically decreasing for all $n \geq n^*$. While for a mean asset correlation value that converges to the threshold (i.e., for $\rho_A \to \lambda_A^2/\sigma^2$), the optimal size n^* tends to infinity and $\mathbb{E}[\phi_t(n)]$ is monotonically decreasing in n. Therefore, the mean time to default is monotonically increasing in n. In this case, full diversification is optimal. (See Figure 1 and 3 in Appendix).

Variance $V[\phi_t(n)]$ as a function of n. In expanded form, $V[\phi_t(n)]$ reads as $\sigma_\phi^2(t) := \left(\sigma^2 + \frac{\sigma^2}{n} + \frac{n-1}{n}\rho_A\sigma^2 - 2\rho\sigma\sqrt{\frac{\sigma^2}{n} + \frac{n-1}{n}\rho_A\sigma^2}\right)t$. The asymptotic analysis of the function gives $V[\phi_t(1)] = -2\sigma^2(\rho - 1)$ and $\lim_{n\to+\infty} V[\phi_t(n)] = \sigma^2\left[1 + \rho_A\left(1 - \frac{2\rho}{\sqrt{\rho_A}}\right)\right]$ such that for $n \geq 1$ the following order relation holds.

[34] Which holds for the set of values $\lambda_A \in (0, 1)$, $\sigma^2 \in (\lambda_A^2, 1)$ and $\rho_A \in [0, \lambda_A^2/\sigma^2)$.

[35] In fact, $\lim_{\rho_A\to 0} n^* = \frac{\sigma^2}{\lambda_A^2}$ which is equal to 1 for $\sigma^2 = \lambda_A^2$ or to $\frac{1}{\lambda_A^2}$ when $\sigma^2 = 1$. Then, $\lim_{\rho_A\to\lambda_A^2/\sigma^2} n^* = \infty$ for any value of σ and λ_A.

$V[\phi_t(1)] = \lim_{n\to+\infty} V[\phi_t(n)]$		$V[\phi_t(1)] > \lim_{n\to+\infty} V[\phi_t(n)]$		$V[\phi_t(1)] < \lim_{n\to+\infty} V[\phi_t(n)]$	
ρ	ρ_A	ρ	ρ_A	ρ	ρ_A
$[0, 1/2)$	1	$[0, 1/2)$	$[0, 1)$	$(1/2, 1]$	$[0, 1 - 4\rho(\rho - 1)]$
$[1/2, 1)$	$1 - 4\rho(\rho - 1)$ or $\rho_A = 1$	$[1/2, 1)$	$[1 - 4\rho(\rho - 1), 1]$		
1	1				

As reported in Table 1, $V[\phi_t(n)]$ can assume four increasing levels of complexity.

σ^2_ϕ	$\rho_A = 0$	$\rho_A > 0$
$\rho = 0$	Case (i) $\sigma^2 + \frac{\sigma^2}{n}$	Case (ii) $\sigma^2 + \frac{\sigma^2}{n} + \frac{n-1}{n}\rho_A\sigma^2$
$\rho \in [-1, 0) \cup (0, 1]$	Case (iii) $\sigma^2 + \frac{\sigma^2}{n} - 2\rho\frac{\sigma^2}{\sqrt{n}}$	Case (iv) $\sigma^2 + \frac{\sigma^2}{n} + \frac{n-1}{n}\rho_A\sigma^2 - 2\rho\sigma\sqrt{\frac{\sigma^2}{n} + \frac{n-1}{n}\rho_A\sigma^2}$

Table 1: The variance $V[\phi_t(n)]$ can have four possible forms according to the values assigned to ρ and ρ_A.

Case (i) The variance $V[\phi_t(n)]$ is a strictly decreasing function in n s.t. $V[\phi_t(1)] = 2\sigma^2 > \lim_{n\to+\infty} V[\phi_t(n)] = \sigma^2$ and $\partial V[\phi_t(n)]/\partial n = -(\sigma/n)^2 < 0$.

Case (ii) The variance $V[\phi_t(n)]$ is decreasing in n[36] s.t. $V[\phi_t(1)] = 2\sigma^2 \geq \lim_{n\to+\infty} V[\phi_t(n)] = (\rho_A + 1)\sigma^2$ and $\partial V[\phi_t(n)]/\partial n = (\rho_A - 1)\sigma^2/n^2 \leq 0$ for any value of ρ_A, n and σ.

Case (iii) The variance $V[\phi_t(n)]$ is a unimodal (U-shaped) function of n. The partial derivative w.r.t. n is $\partial V[\phi_t(n)]/\partial n = (\sqrt{n}\rho - 1)\sigma^2/n^2$. By imposing the FOC: $\partial V[\phi_t(n)]/\partial n = 0$, we get $n^* = 1/\rho^2$. Otherwise $\partial V[\phi_t(n)]/\partial n < 0$ if $n^* - \delta < n < n^*$ and $\partial V[\phi_t(n)]/\partial n > 0$ if $n^* < n < n^* + \delta$; The hump shape is due to the following general lemma.

Lemma A.1. *Take any generic twice-differentiable function f, of two processes X and Y such that the new process $Z = f(X, Y)$ has variance of the form $\sigma^2_z = \sigma^2_x + \sigma^2_y - 2\rho\sigma_x\sigma_z$. Then, if the variance of either X or Y is re-scaled down by a generic factor $n > 0$, σ^2_z becomes a unimodal (U-shaped) function of n with a global minimum in*

$$n^* = \begin{cases} \frac{1}{\rho^2}\left(\frac{\sigma_y}{\sigma_x}\right)^2 & \text{if } \sigma_y \text{ is re-scaled by } n \\ \frac{1}{\rho^2}\left(\frac{\sigma_x}{\sigma_y}\right)^2 & \text{if } \sigma_x \text{ is re-scaled by } n. \end{cases}$$

Case (iv) The study of the function shows a conflicting role between ρ and ρ_A. The partial derivative with respect to n, is equal to $\partial V[\phi_t(n)]/\partial n = \left(\frac{\sigma^2(\rho_A-1)}{n^2}\right)\left(\frac{\sigma_A-\rho\sigma}{\sigma_A}\right)$ such that $V[\phi_t(n)]$ is (a) decreasing in n if $\rho \leq 0$ or $\rho > 0$ and $\rho_A \geq \rho^2$ or (b) unimodal (U-shaped) with an absolute minimum in $n^* = (\rho_A - 1)/(\rho_A - \rho^2)$ if $\rho > 0$ and $\rho_A < \rho^2$.

Both Cases *(i)* and *(ii)* show that is assets and debt are uncorrelated (i.e., $\rho = 0$), the variance of the process (8) diminishes with n. Differently, Cases (*iii*) and (*iv.b*) show that when assets and debt are positively correlated (i.e.,

[36] For $\rho_A < 1$, $V[\phi_t(n)]$ is strictly decreasing in n.

$\rho > 0$), for any arbitrary time value t, increasing levels of portfolio diversification n, generate a non-constant, non-monotonic behavior of the dispersion around the mean value $\mathbb{E}[\phi_t(n)]$, measured by the standard deviation $\sigma_\phi(t)$. (See Figure 2 in Appendix). □

B Analysis

Parameter	Set of Values
n	$\{n \in \mathbb{N} : n > 0\}$
σ	$\{\sigma \in \mathbb{R} : 0 < \sigma < 1\}$
λ_A	$\{\lambda_A \in \mathbb{R} : 0 \leq \lambda_A \leq 1\}$
λ_H	$\{\lambda_H \in \mathbb{R} : 0 \leq \lambda_H \leq 1\}$
ρ_A	$\{\rho_A \in \mathbb{R} : 0 \leq \rho_A \leq 1\}$
ρ	$\{\rho \in \mathbb{R} : -1 \leq \rho \leq 1\}$
a	$\{a \in \mathbb{R} : a < 1\}$

Table 2: Support of the parameters.

To conclude our analysis about the default-induced optimal level of portfolio diversification, let us consider the jointly impact of both first and second moments of log-leverage on the mean time to default. For the set of values in Table 2, the expected value of the log-leverage $\mathbb{E}[\phi_t(n)]$, can decrease or have a non-monotonic behavior w.r.t. n (see proof *Proposition 4*). In particular, $\mathbb{E}[\phi_t(n)]$ depends on the reciprocal values taken by σ and λ_A s.t., two different cases exist

Case (a) $\mathbb{E}[\phi_t(n)]$ decreases for $\sigma^2 \in (\lambda_A^2, 1]$ and $\rho_A \in \left[\lambda_A^2/\sigma^2, 1\right)$.

Case (b) $\mathbb{E}[\phi_t(n)]$ have a non-monotonic (U-shaped) behavior if $\rho_A \in \left[0, \lambda_A^2/\sigma^2\right)$. Thus, only for sufficiently low levels of mean asset correlation, the expected log-leverage is monotonically decreasing up to an optimal diversification level $n^* = \sigma^2(\rho_A - 1)/(\rho_A\sigma^2 - \lambda_A^2)$ and then monotonically increasing. This behavior may be amplified by the fluctuations around the mean.

Since we are interested to the mean time taken by $\{\phi(t)\}_{t\geq 0}$ to reach the upper distress barrier fixed at zero, we want to consider the impact of the upward variance on $\mathbb{E}[\phi_t(n)]$. For the set of values in Table 2, $V[\phi_t(n)]$ behaves in two different ways (see proof *Proposition 4*). *Case (i)*: $V[\phi_t(n)]$ decreases w.r.t. n when $\rho \in [-1, 0]$. *Case (ii)*: $V[\phi_t(n)]$ decreases w.r.t. n when $\rho \in (0, 1)$ but the mean asset correlation is sufficiently high, namely $\rho_A \in [\rho^2, 1)$. *Case (iii)*: $V[\phi_t(n)]$ is a non-monotonic (U-shaped) function of n, if $\rho \in (0, 1]$ and $\rho_A \in [0, \rho^2)$ with a minimum in $n^* = (\rho_A - 1)/(\rho_A - \rho^2)$. For positive debt-asset correlation (i.e., $\rho > 0$), and sufficiently low mean asset correlation (i.e., $\rho_A < \rho^2$), the fluctuations around the mean exhibit a minimum in n as shown in Figure 2 in Appendix D.

The match of the above results, brings us to the following six likely events:

Case (a.i) $\mathbb{E}[\phi_t(n)]$ and $V[\phi_t(n)]$ are decreasing in n.

Case (a.ii) $\mathbb{E}[\phi_t(n)]$ and $V[\phi_t(n)]$ are decreasing in n.[37]

Case (a.iii) Both $\mathbb{E}[\phi_t(n)]$ and $V[\phi_t]$ display a non monotonic (U-shaped) behavior w.r.t. n, such that (by *Corollary 1*) $\bar{\tau}$ is a unimodal (inverted U-shaped) function of n.[38]

Case (b.i) $\mathbb{E}[\phi_t(n)]$ is a non monotonic (U-shaped) function of n which is monotonically decreasing up to $n^* = \sigma^2(\rho_A - 1)/(\rho_A\sigma^2 - \lambda_A^2)$ and then monotonically increasing. The debt-asset correlation is negative (i.e., $\rho < 0$) and $\bar{\tau}$ is a unimodal (inverted U-shaped) function of n only for values of mean asset correlation very close to the lower bound (i.e., $\rho_A \to 0$). Otherwise, $\bar{\tau}$ is concave monotonically increasing function (vertically shift downwards as $\rho \to -1$). See Figures (1.a), (1.b), (1.c) in Appendix.

Case (b.ii) $\mathbb{E}[\phi_t(n)]$ is a non monotonic (U-shaped) function of n which is monotonically decreasing up to $n^* = \sigma^2(\rho_A - 1)/(\rho_A\sigma^2 - \lambda_A^2)$ and then monotonically increasing. As soon as as $\rho_A \to \lambda_A^2/\sigma^2$, $\bar{\tau}$ changes from a unimodal (inverted U-shaped) function of n to a concave monotonic increasing function of n.

Case (b.iii) $\mathbb{E}[\phi_t(n)]$ is a non monotonic (U-shaped) function of n which is monotonically decreasing up to $n^* = \sigma^2(\rho_A - 1)/(\rho_A\sigma^2 - \lambda_A^2)$ and then monotonically increasing. $\mathbb{E}[\phi_t(n)]$ is a non monotonic (U-shaped) function in n which is monotonically decreasing up to $n^* = \sigma^2(\rho_A - 1)/(\rho_A\sigma^2 - \lambda_A^2)$ and then monotonically increasing. Moreover, the upward variance is U-shaped around $\mathbb{E}[\phi_t(n)]$. Hence, by *Corollary 1*, $\bar{\tau}$ behaves as non monotonic (inverted U-shaped) function of n. [39]

Summarizing the analysis, we get the following result. When either $\mathbb{E}[\phi_t(n)]$ or $V[\phi_t(n)]$ are U-shaped w.r.t. n, $\bar{\tau}$ is a unimodal (inverted U-shaped) function of n.[40] While, when both $\mathbb{E}[\phi_t(n)]$ and $V[\phi_t(n)]$ are not U-shaped w.r.t. n, the mean time to default is a concave monotonically increasing function of n s.t., maximum diversification is the best default-induced portfolio strategy.[41] In those cases, for decreasing levels of debt-asset correlation (namely $\rho \to -1$), the asymptote to which $V[\phi_t(n)]$ converges, is increasing. Therefore, $\bar{\tau}$ becomes shorter (i.e., it shifts downward). This leads to the statement in *Corollary 2*.

Despite its low likelihood to occur in reality, the case of a portfolio with securities highly correlated (i.e., $\rho_A \approx 1$), deserves some attention. In the particular case of assets correlation equal to one, diversification has not potential benefits. This effect is directly captured in our analysis and explained by a mean time to default which is a constant function of n when $\rho_A = 1$. See Figure 3: black line in Appendix D.

C The 1/n not-rebalanced allocation. A Monte Carlo simulation

In the case without rebalancing, the strategy is a "buy-and-hold" portfolio according to which any quantity $x_i \in \mathbf{x}$ composing the trading strategy $\mathscr{S}_s$ is chosen at the beginning of the investment horizon and the position is never

[37]By *Corollary 1*, in *Case (a.i-ii)* $\bar{\tau}$ is a concave increasing function of n (i.e., nondecreasing for all values of n below some n^* and non increasing for all values of n above n^*) which is vertically shift downwards for $\rho \to -1$, see *Case (a.i)*, or for high levels of ρ_A, see *Case (a.ii)*.

[38]Strictly increasing on $[1, n^*]$ and decreasing on $[n^*, \infty]$ with positive asymptote $\bar{\tau} = k > 0$.

[39]Such as $\bar{\tau}$ it is strictly increasing on $[1, n^*]$ and strictly decreasing on $[n^*, \infty]$. The asymptote k, to which $\bar{\tau}$ tends as $n \to \infty$ is shift upward as $\rho_A \to \rho^2$.

[40]Compare *Case (a.iii)*, *Case (b.i)*, *Case (b.ii)* and *Case (b.iii)*.

[41]Compare *Case (a.i)* and *Case (a.ii)*.

rebalanced thereafter. [42] If we define n_d^* (n_s^*) as the optimal number of securities that solves $\arg\max_n(\bar{\tau}(n))$ of the strategy $\mathscr{S}_d$ ($\mathscr{S}_s$), two central questions arise: (i) $\bar{\tau}(n_d^*) \sim \bar{\tau}(n_s^*)$? ; (ii) $n_d^* \sim n_s^*$? In order to answer to these questions, we resort to a Monte Carlo simulation technique in four steps: (i) Numerical approximation of investment position; (ii) Numerical approximation of borrowing position; (iii) Numerical approximation of log-leverage dynamics; (iv) Estimation of expected time to default. For convenience, the time interval $[0, T]$ is assumed to be discretized as $0 = t_1 < t_2 < ... < tn = T$ where the time increments are equally spaced with width $dt = T/n$.[43]

Numerical approximation of investment position The investment dynamics is discretized starting from the price processes of the risky securities $\mathbf{S} := (S_1, ..., S_n)^{\mathrm{T}}$ in $\mathbb{R}^n$. We take the system of equations (1) and integrate from t to $t + dt$

$$\ln\left[\frac{S_i(t+dt)}{S_i(t)}\right] = \int_t^{t+dt}\left(\mu_i - \frac{1}{2}\sigma^2\right)du + \int_t^{t+dt}\sigma dWu_i. \tag{16}$$

Eq. (16) is the starting point for any discretization scheme according to which at time t the value of $S_i(t)$ is known and we want to obtain the next value $S_i(t + dt)$. The simplest way to discretize the process in (16) is the Euler discretization

$$S_i(t+dt) = S_i(t) + \mu_i S_i(t)dt + \sigma S_i(t)(dt^{1/2})\xi_i + O(dt^{3/2}) , \quad \forall i = 1, ..., n \tag{17}$$

where, $\xi_i\sqrt{dt} = W_i(t+dt) - W_i(t)$ and $\xi_i \sim N(0, 1)$. [44] From eq. (17) let compose a long-only portfolio A with fixed weights x_i for all $t \geq 0$. Namely, $A(t) = x_1S_1(t) + x_2S_2(t) + ...$, so that

$$\begin{aligned}
A(t+dt) &= x_1S_1(t+dt) + x_2S_2(t+dt) + ...\\
A(t+dt) &= x_1\left(S_1(t) + \mu_1S_1(t)dt + \sigma S_1(t)(dt^{1/2})\xi_1\right) + x_2\left(S_2(t) + \mu_2S_2(t)dt + \sigma S_2(t)(dt^{1/2})\xi_2\right) + ...\\
&= A(t) + x_1\left(\mu_1S_1(t)dt + \sigma S_1(t)(dt^{1/2})\xi_1\right) + x_2\left(\mu_2S_2(t)dt + \sigma S_2(t)(dt^{1/2})\xi_2\right) + ...
\end{aligned}$$

In vector notation, $A(t + dt) - A(t) = \mathbf{x}^{\mathrm{T}}\Delta\mathbf{S}$.

Numerical approximation of borrowing position Consider the discretized value of the debt cost process in *Proposition 1*

$$S_H(t+dt) - S_H(t) = \mu_H S_H(t)dt + \sigma S_H(t)(dt^{1/2})\xi_i + O(dt^{3/2}) ,$$

where $\mu_H = r_f + \sigma\lambda_H$. Since $H = x_HS_H$, the borrowing position in (6) is redly discretized as

$$H(t+dt) - H(t) = x_H[S_H(t+dt) - S_H(t)]. \tag{18}$$

[42] Differently from the case of $\mathscr{S}_d$, - see eq. (5), the vector $\mathbf{x}$ is neither a function of time nor of asset price $\mathbf{S}$.

[43] Equally-spaced time increments is primarily used for notational convenience, because it allows us to write $t_i - t_{i-1}$ as simply dt. All the results derived with equally-spaced increments are easily generalized to unequal spacing.

[44] Since μ_i and σ are real constants, this discretization coincides with the more sophisticated classical Runge -Kutta methods and can be equivalently be generated from the log-stock prices $\ln S_i(t+dt) = \ln S_i(t) + \left(\mu_i - \frac{\sigma^2}{2}\right)dt + \sigma(dt)^{1/2}\xi_i$ so that $S_i(t+dt) = S_i(t)\exp\left(\left(\mu_i - \frac{\sigma^2}{2}\right)dt + \sigma(dt)^{1/2}\xi_i\right)$.

Numerical approximation of log-leverage dynamics The Euler-Maruyama method applied to the log-leverage dynamics in (8), yields

$$\phi(t+dt) - \phi(t) = \mu_\phi(t)\ dt + \sigma_\phi(t)\ (dt^{1/2})\xi + O(dt^{3/2})\ ,$$

where $\mu_\phi = (\mu_H - \mathbf{x}^{\mathrm{T}}(diag(\mathbf{S})\boldsymbol{\mu})/A + \frac{1}{2A^2}(\mathbf{x}^{\mathrm{T}}((diag(\mathbf{S})\boldsymbol{\Sigma})^2)\mathbf{x}) - 0.5\sigma^2)$ and $\mu_\phi = (\sigma^2 + (\mathbf{x}^{\mathrm{T}}((diag(\mathbf{S})\boldsymbol{\Sigma})^2)\mathbf{x})/(A^2) - 2\rho\sigma\sqrt{(\mathbf{x}^{\mathrm{T}}((diag(\mathbf{S})\boldsymbol{\Sigma})^2)\mathbf{x}))/A}$.

Estimation of expected time to default By using a Monte Carlo method, we simulate a total of R sample paths for $\{\phi\}_{t\geq 0}$ and estimate the expected exit time with the sample average of the time taken by $\{\phi\}_{t\geq 0}$ to cross the default boundary. Let the r^{th} realization of the path of $\{\phi\}_{t\geq 0}$ be denoted as $\{\phi_r\}_{t\geq 0}$ with $r = 1, ..., R$. For each of these paths the random variable τ exits through the default boundary is defined as $\tau_r = \inf\{t \geq 0 : \phi_{rt} \geq 1\}$. The expected default time $\bar{\tau}$ is approximated as the total sum of exit times divided by the number of realizations

$$\bar{\tau} = \frac{1}{R}\sum_{r}^{R} \tau_r(\mathbb{1}_{\phi_t=0}) \approx \left(\frac{\sigma_\phi^2}{2\mu_\phi^2}\right)\left[\exp\left(\frac{-2(\varepsilon+\phi_0)\mu_\phi}{\sigma_\phi^2}\right) - \exp\left(\frac{-2\varepsilon\mu_\phi}{\sigma_\phi^2}\right)\right] - \left(\frac{\phi_0}{\mu_\phi}\right)\ .$$

Comparison between the rebalanced and the not no-rebalanced 1/N heuristic The comparison between the optimal default time and the optimal degree of diversification for $\mathscr{S}_d$ and $\mathscr{S}_s$ suggests the following differences: (i) $\bar{\tau}(n_d^*) \leq \bar{\tau}(n_s^*)$; and (ii) $n_d^* \leq n_s^*$. The intuition behind the first result is straightforward. According to Assumption 3.1, $\mathscr{S}_d$ is a "contrarian" asset allocation which implies re-adjustments of the portfolio weights by selling (buying) the securities that are over (under) performing w.r.t. to the average.[45] Ultimately, a rebalance executed in continuous time negatively affects the overall portfolio performance as asset price noises may be wrongly confused as informative signals. This means that a debt-financed rebalanced counter-trade, induces a higher default risk with respect to a debt-financed fixed-trade which is not-rebalanced (see Figure 4 in Appendix D). We are not arguing that the "contrarian" trading is a riskier strategy than a buy and hold strategy. More deeply, we say that the goodness of the "contrarian" trade depends on the optimal timing of rebalance which investigation goes beyond the scope of the present paper. The second result is based on the fact that, for $\mathscr{S}_d$, diversification exhausts its benefit at a lower number of securities. In fact, higher the number of securities, the greater the likelihood that some of them under-perform or even perform negatively. Eventually, the mean time to default is reduced.

[45] In the case without rebalancing, the portfolio over time assigns higher weight to securities that have done well in the past. So, this can be interpreted as a "momentum" strategy.

D Figures

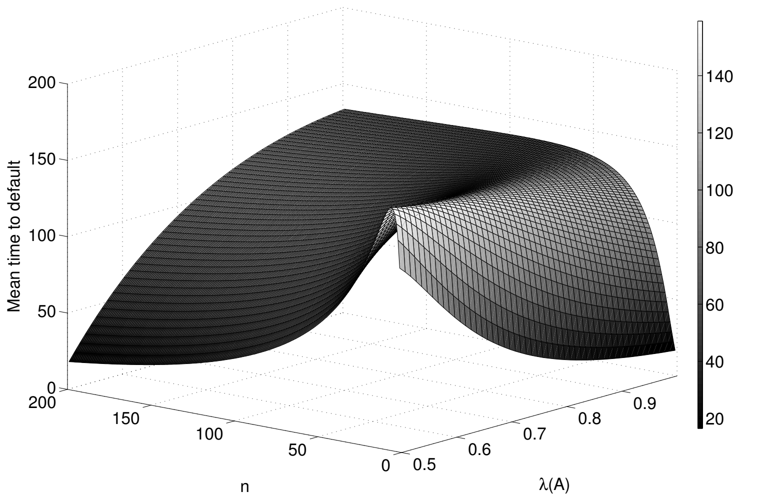


Figure 1: Mean time to default as a function of portfolio size at different levels of λ_A. (Parameters: $\sigma^2 = 0.5$, $\lambda_H = 0.5$, $a = -2$, $\phi_0 = -1$).

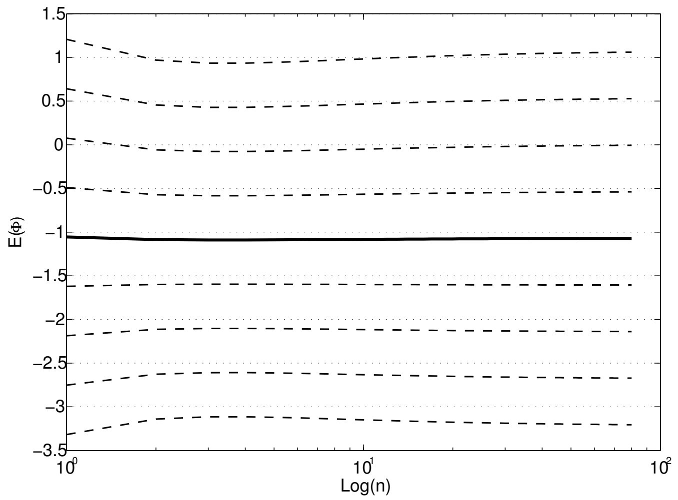
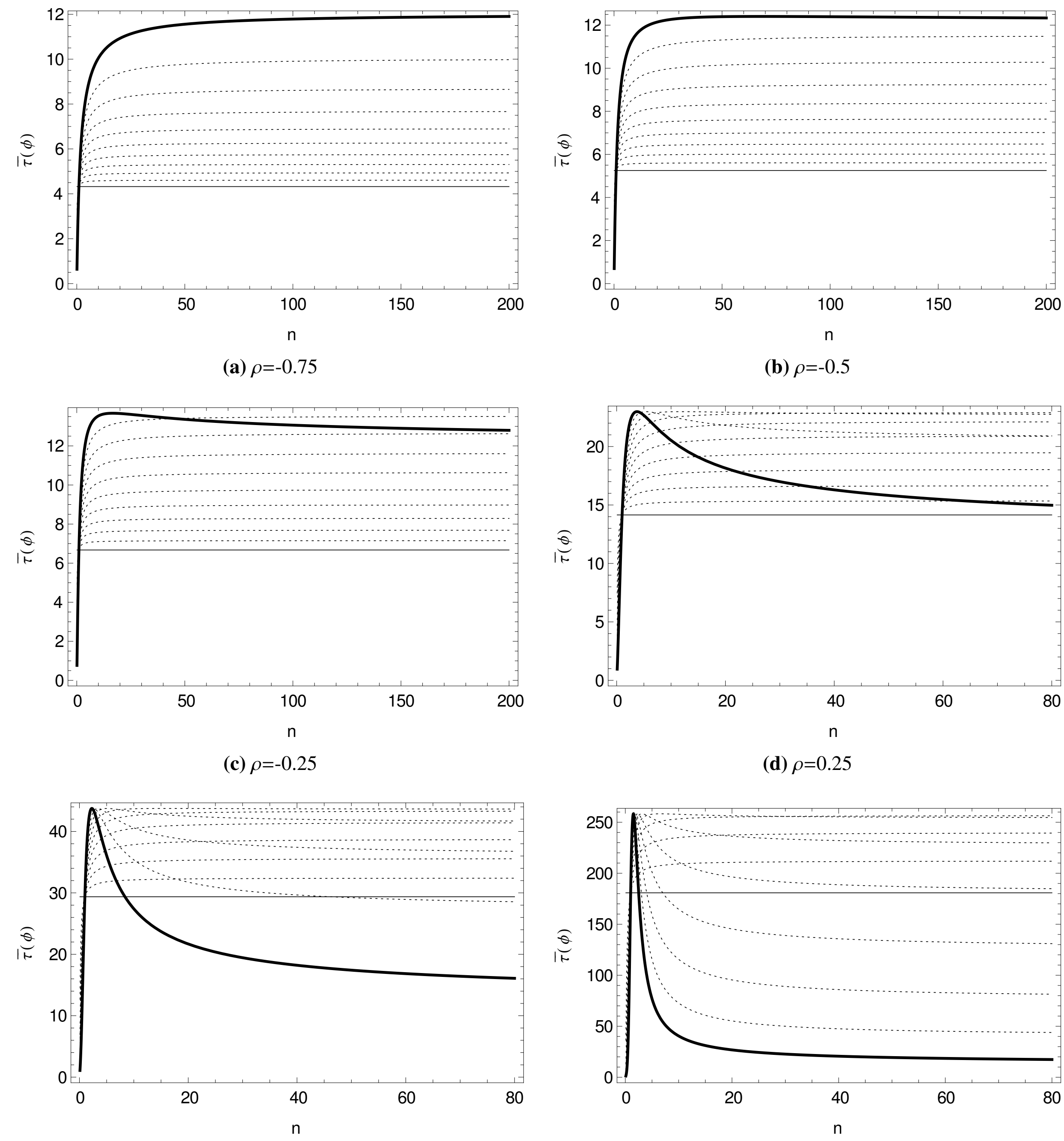


Figure 2: For an arbitrary time t^*, $E[\phi_t]$ is decreasing in N (thick black line). While the fluctuations around the mean measured by $\pm\sigma_\phi$, $\pm 2\sigma_\phi$, $\pm 3\sigma_\phi$, $\pm 4\sigma_\phi$ (dashed lines) decrease up to an optimal level n^* and then increase.



(a) ρ=-0.75

(b) ρ=-0.5

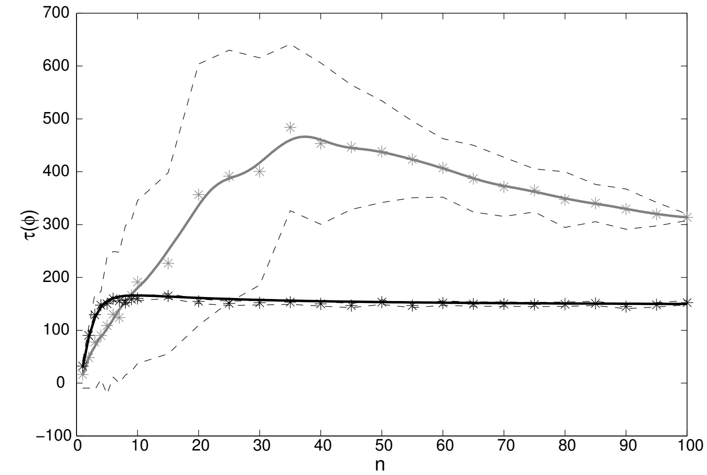
(c) ρ=-0.25

(d) ρ=0.25

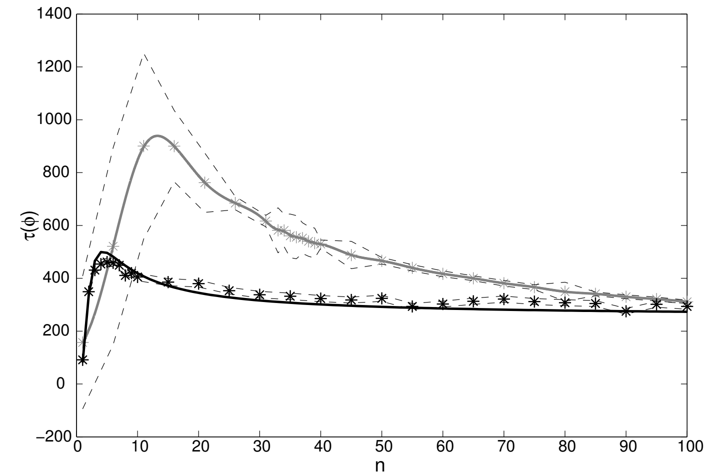
(e) ρ=0.5

(f) ρ=0.75

Figure 3: $-\ \rho_A = 0$ $\cdots\ \rho_A$=$[0.1, ..., 0.9]$ $-\ \rho_A = 1$. Mean time to default as a function of portfolio size at different correlation levels. (Parameters: $\sigma^2 = 0.25$, $\lambda_H = 0.35$, $\lambda_A = 0.5$, $a = -2$, $\phi_0 = -1$).



(a) ρ=0.25



(b) ρ=0.5

Figure 4: Theoretical $\bar{\tau}(n)$ of dynamic strategy (bold black line). Numerical $\bar{\tau}(n)$ of dynamic strategy (black stars). Numerical $\bar{\tau}(n)$ of static strategy (gray stars). Spline (bold gray line). Volatility (dashed line). (Parameters: $\rho_A = 0.1$, $\sigma^2 = 0.5$, $\lambda_H = 0.35$, $\lambda_A = 0.5$, $r_f = 0.03$, $a = -2$, $\phi_0 = -1$).

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Diversification and Financial Stability

Paolo Tasca, Stefano Battiston

Abstract

The recent credit crisis of 2007/08 has raised a debate about the so-called *knife-edge* properties of financial markets. The paper contributes to the debate shedding light on the controversial relation between risk-diversification and financial stability. We model a financial network where assets held by borrowers to meet their obligations, include claims against other borrowers and securities exogenous to the network. The balance-sheet approach is conjugated with a stochastic setting and by a mean-field approximation the law of motion of the system's fragility is derived. We show that diversification has an ambiguous effect and beyond a certain level elicits financial instability. Moreover, we find that risk-sharing restrictions create a socially preferable outcome. Our findings have significant implications for policy recommendation.

JEL Code: G01, G11, G18, G2, G32, G33
Keywords: Systemic Risk, Financial Crisis, Diversification, Default Probability.

1 Introduction

Thus far, many theoretical models of finance pointed towards the stabilizing effects of a diversified (i.e., dense) financial network (e.g., Allen and Gale, 2001). Recently, some studies have started to challenge this view, investigating conditions under which diversification may have ambiguous effects; e.g., Battiston et al. (2009); Brock et al. (2009); Ibragimov et al. (2011); Ibragimov and Walden (2007); Stiglitz (2010); Wagner (2009). In particular, the recent financial crisis is attracting increasing attention on the *knife-edge* properties of the financial system and its optimal architecture (see e.g., Gai and Kapadia, 2007; Haldane and May, 2011; Nier et al., 2007). According to Haldane (2009), financial interconnections serve as *shock-absorber* (i.e., connectivity engenders robustness and risk-sharing prevails) up to a certain tipping point, beyond which they act as *shock-amplifier* (i.e., connectivity engenders fragility and risk-spreading prevails).

The paper translates the concept of knife-edge dynamics into an analytic form and investigates its impact on the stability of the financial system.

In a similar spirit as Shin (2008, 2009), we model a financial network composed of leveraged risk-averse financial institutions (hereafter "banks"), which invest in two classes of assets. The first class consists of obligations issued by other banks in the network. [1] The second class represents risky assets external to the

[1] Although in our simplified economy, the securities in this class are priced depending on the financial health of the obligors, in reality, the debt-claim structure of the financial system is much more intricate. To better understand this view, we need to

financial network, which may include securities on, e.g., real-estate, loans to firms and households and other real-economy related activities, (hereafter "external assets"). The external assets may be in a downtrend with probability p, or in an uptrend with probability $1-p$. In this uncertain world, it is assumed that banks have an incentive to diversify across external assets.[2] Precisely, banks adopt the "naive" heuristic which involves holding an equally weighted portfolio of external assets. Two important assumptions of the model are as follows: first, the up (down) trend is *persistent* i.e., approximately constant during a given period Δt; second, due to information-gathering and transaction costs, the diversification strategy is assumed to be *static* for a given period Δt.[3] These two assumptions imply that, based on the expectation about the future trend of external assets, each bank sets up a diversified portfolio which will be held for a finite arbitrary interval Δt.

As first result, the paper shows that banks' default probability increases with diversification in case of a downtrend and decreases in case of an uptrend.[4] In accordance, under the utility function framework, there exists a gap (which increases with diversification) between the banking system's expected utility in uptrend and the expected utility in downtrend. Thus, diversification generates and amplifies the knife-edge property of the financial network. Moreover, for a given level of diversification, the spread between utility levels increases with the magnitude of the trend.

A second result shows that by assigning a probability p to the downtrend event and $1-p$ to the uptrend, diversification displays a tradeoff with regard the banking system's utility. The expected utility exhibits a maximum which corresponds to an intermediate optimal level of diversification. Such a level depends on the probability p of downtrend, on the magnitude of the expected profit (loss) and on the values of the other parameters of the model.

In addition, we find that individual banks' incentives favor a financial network that is over-diversified in external assets with respect to the level of diversification which is socially desirable. This result is rooted on the asymmetry of expected losses between individual banks and regulator. Whereas individual banks' losses are bounded above,[5] at system level externalities associated with the failure of interconnected

achieve a clear understanding of the building blocks of financial system, mostly in the presence of amplification mechanism stemming from leverage and other factors, (see Trichet, 2009). The banks' asset values depend on the payoffs received from the claims on other banks in which they have invested. But the value of these issuing banks depends, in turn, on the financial health of yet other banks in the system. Moreover, linkages between banks can be cyclical. See Eisenberg and Noe (2001).

[2]Generally, this practice is desirable since it allows banks to promise investors a lower repayment or increase the amount of debts. See Allen et al. (2010).

[3]In the real-world, portfolio adjustments are costly. In particular, the revision costs sometimes could be greater than the increase of the expected rate of return of the revised portfolio. So, portfolio revision is not always convenient.

[4]There is a simple intuition for this effect in terms of the Sharpe ratio which allows to compare different investment returns per unit of risk. When assets have positive expected cash-flows the Sharpe ratio is positive as well. Then it is desirable to combine them in a well diversified portfolio since diversification lowers portfolio's volatility and in doing so, increases its Sharpe ratio. In contrast, when assets have negative expected cash-flows, the Sharpe ratio is negative. Then, a well diversified portfolio is no longer desirable.

[5]In presence of limited liability.

institutions, amplify the expected losses.[6] These externalities reflect the "incremental costs" due to a system collapse which are ultimately borne by taxpayers. In other words, the cost of recovery from a systemic crisis, grows with the number of defaults. Under these conditions, the optimal diversification level for the regulator view point is always smaller than the one desirable by individual banks. These results brings about implications for risk management policies to mitigate systemic risk.

The paper is organized as follows. In Section 2 we introduce the model. Section 3 derives the financial fragility under the mean-field approach. Section 4 computes the systemic default probability. Section 5 studies the optimal diversification policy. In Section 6 we investigate the dichotomy between private incentives of diversification and its social effects. Section 7 concludes and considers some extensions.

2 Model

To keep notations simple as much as possible, whenever it is unambiguous from the context that a variable or parameter are time-dependent, we will omit the time index t.

In a similar spirit as Shin (2008, 2009), we model a system of N financial risk-averse leveraged institutions ("banks"). The balance-sheet identity conceives the equilibrium between asset an liability side, which for the representative bank $i \in \{1, ..., N\}$, is

$$a_i = \bar{p}_i + e_i \ , \tag{1}$$

where $\mathbf{a} := (a_1, ..., a_N)^{\mathrm{T}}$ is the column vector of market values of the banks' assets; $\bar{\mathbf{p}} := (\bar{p}_1, ..., \bar{p}_N)^{\mathrm{T}}$ is the column vector of the book-face values of the banks' obligations (e.g., zero-coupon bonds); [7] $\mathbf{e} := (e_1, ..., e_N)^{\mathrm{T}}$ is the column vector of equity values.

Notably, the market is composed of two investment classes: (*i*) N obligations issued by the banks in the system, and (*ii*) M exogenous investment opportunities non related to the financial sector (external assets). Each bank i selects a portfolio composed of: (*i*) $n \leq N-1$ securities selected from $N-1$ securities issued by the other banks, and (*ii*) $m \leq M$ assets selected from M external assets.

Then, the asset-side in (1) becomes

$$a_i := \sum_j^m \mathbf{Z}_{ij} y_j + \sum_j^n \mathbf{W}_{ij} p_j \ , \tag{2}$$

where

$$p_j := \bar{p}_j / (1 + r_j)^{-t} \tag{3}$$

[6] Either due to direct or indirect exposures.

[7] In particular, $\bar{p}_i$ is the value of the payment at maturity that each obligor-i promised to the lenders. For all the banks in the system, the equation indirectly assumes only one issue of bonds with the same maturity date t.

is the market value of debt issued by bank-j, defined as the discounted value of future payoffs; $\mathbf{y} := (y_1, ..., y_m)^{\mathrm{T}}$ is the column vector of external investments; $\mathbf{Z} := [z_{ij}]_{n\times m}$ is the $n \times m$ weighting matrix of external investments where each entry $z_{i,j}$ is the proportion of activity-j held by bank-i; $\mathbf{W} := [w_{ij}]_{n\times n}$ is the $n \times n$ adjacency matrix where each entry w_{ij} is the proportion of bank-j debt held by bank-i; r_j : is the rate of return used to discount the debt-face value of the obligor-j. In brief, the bank-i manages the following balance-sheet

Assets	Liabilities
$\sum_j^m \mathbf{Z}_{ij} y_j$	$\bar{p}_i$
$\sum_j^n \mathbf{W}_{ij} p_j$	e_i

External assets include e.g., activities related to the real-economy, loans to firms and households, etc., and each of them it is assumed to generates a log-normal cash flows process

$$d\log(Y_j)(t) := dy_j(t) = \mu_j dt + \sigma_j dB_j(t) \quad \forall j = 1, ..., m\ , \tag{4}$$

where $B_j(t)$ is a standard Brownian motion defined on a complete filtered probability space $(\Omega; \mathcal{F}; \{\mathcal{F}_t\}; \mathbb{P})$, with $\mathcal{F}_t = \sigma_y\{B(s) : s \leq t\}$, μ_j is the instantaneous risk-adjusted expected growth rate, $\sigma_j > 0$ is the volatility of the growth rate. We assume correlation across processes $\mathbb{E}(dB_i, dB_j) = \rho_{ij} dt$. The processes in (4) have the relevant property that they preclude instantaneous external assets' distributions from becoming negative at any time $t \geq 0$ and are therefore an economically reasonable assumption.

The results in the next sections hold under the following main assumption of the model.

Assumption 2.1. (i) *There exists a frictionless market where all debt-securities have the same maturity date and seniority*; (ii) *Divisibility and No short-selling constraint. All the securities are marketable, perfectly divisible and can be held only in positive quantities*; (iii) *Price Taking Assumption. Securities prices are beyond the influence of investors*; (iv) *Only expected returns and variances matter. Quadratic utility or normally distributed returns*; (v) *Not-paying dividend-coupon assumption. The return on any security is simply the change in its market price during the holding period*; (vi) *Homogeneous expectations*

2.1 Leverage and Balance-Sheet Management

The objective function of a risk-averse bank is to adopt an investment strategy that, regardless the shape of its utility function with respect to risk and return, it will minimize the risk for a given expected return, or conversely, maximize the expect return for a given level of risk. Following the same line of reasoning adopted in Evans and Archer (1968), we assume the bank i includes some form of *marginal analysis* in its portfolio selection model.

Definition 2.1. ***Management Strategy*** *The bank i is interested in the marginal reduction in portfolio variation resulting from successive increases in the number of securities held in the asset side of its balance sheet –that is, the marginal benefits derived from increased diversification.*

Since the aforementioned strategy is (partially) debt-financed, bank i may face the problem to determine the "optimal" capital structure that minimizes the default risk. To this end, corporate leverage has to be managed and constrained. A standard accounting approach to define leverage is the debt-asset ratio. For the bank-i,

$$\begin{aligned} \phi_i &:= \bar{p}_i / a_i \\ &:= \bar{p}_i / \left(\sum_j^m \mathbf{Z}_{ij} y_j + \sum_j^n \mathbf{W}_{ij} p_j \right) . \end{aligned} \tag{5}$$

Since debts cannot exceeds the asset side, ϕ is bounded between the lower *safe barrier* $\varepsilon \to 0^+$ and the upper *default barrier* 1. Let $a := \varepsilon$ and $b := 1$. Then, if we consider the process $\{\phi\}_{t\geq 0}$ driving the dynamics of ϕ, $\Omega_\phi = \{\phi \in \mathbb{R}^+ \mid a \leq \phi \leq b, \}$ represents the set of values taken by $\{\phi\}_{t\geq 0}$, at any time $t \geq 0$.

The balance-sheet equilibrium between sources of finance and investments cash-flows is a precondition[8] to solvency. In fact, the solvency of a bank is indicated by the positive net worth as measured by difference between assets and liabilities (excluding capital and reserves). Then, the problem of a failing bank normally emerges through illiquidity. In this perspective, the debt-asset ratio is a leverage indicator resuming the potential insolvency of a bank due to a fragile capital structure.

Definition 2.2. ***Bank Fragility***. *The debt-asset ratio ϕ is a leverage index that maps the bank's fragility between zero and one.*

A leverage ϕ which tends to one is symptomatic of a very fragile bank that is prone to be highly sensitive to external shocks which may force the bank towards a bankruptcy regime. On the other hand, a leverage ϕ close to zero describes a bank with a solid balance-sheet equilibrium between internal and external sources of finance.

In line with a large body of theoretical and empirical literature, the rate of return on future payments depends upon the payor's credit-worthiness (e.g. measured by its leverage), the contractual features (e.g., time to maturity) and the market factors. In our simplified economy, the only two factors explaining the internal rate of return, are the risk-free rate r_f and the indicator of fragility ϕ. Thus, for each obligor-j the following equation holds true

$$r_j := r_f + \beta \phi_j . \tag{6}$$

[8]Necessary but not sufficient.

The risk-free rate[9] is fixed over time and is in the common knowledge of every institution. The parameter β in $\mathbb{R} \in (0, 1)$ can be interpreted as the responsiveness of the rate of return to a firm's financial condition.

An extensive form of the balance-sheet management of the bank can be now derived substituting (6) into (3)[10] and then (3)[11] into (5)

$$\phi_i = \bar{p}_i / \left(\sum_j^m \mathbf{Z}_{ij} y_j + \sum_j^n \mathbf{W}_{ij} \bar{p}_j / \Big(1 + r_f + (\beta \phi_j)\Big) \right) . \tag{7}$$

Eq. (7) shows a non-linear dependence of the fragility ϕ_i related to the bank-i, with the fragilities $\phi_{j=1,\dots,n}$ of the j-banks, obligors of i. The following feature of our model is then immediate.

Lemma 2.3. *At a macro level, it can be shown that the fragility of the banking system is the positive root of a system of quadratic expressions in* $\boldsymbol{\phi} := (\phi_1, ..., \phi_n)^{\mathrm{T}}$.

Proof. See Appendix A. □

Differently from Shin (2008) - Eq. (5), *Lemma 2.3* suggests that, under the economically reasonable assumption that banks' leverage impacts on the mark-to-marked values of their debts, the system's balance sheet is described by a quadratic expression in $\boldsymbol{\phi}$.

Since (7) is too complex to analyze directly under arbitrary network structures, in the next section we follow the usual technique of analyzing a mean-field approximation to the given expression (see e.g., Vega-Redondo, 2007).

3 Analysis

An approximate description of the equilibrium properties of the system is obtained by using a mean-field analysis of (7) under the following conditions.[12] According to the managerial practice defined in 2.1, we impose that each bank i composes an equally weighted portfolio of randomly selected:

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[9]The risk-free rate is paid by an asset without any default, timing or exchange rate risk; as such, it is a non-observable theoretical construct. It is usually measured by the rates of return on government securities, which have the lowest risk, in any particular currency.

[10]This substitution makes evident that p_i is an inverse function of ϕ_i. Higher leverage implies a lower market price of the claims and a greater internal rate of return.

[11]Since, p_i in its interval of existence, is a monotonic decreasing function in t, without loss of generality, we consider the time interval in (3) to be $t = 1$.

[12] The mean-field approach is a useful simplification that allows to characterize analytically the behavior of the system under certain assumptions. The main idea is to replace the state variable of each individual with the average state variable across the system. In this approximation, the individual interacts with the "average individual". Clearly, under this condition, some network aspects become unobservable. On the other hand, it has been argued that the financial sector exhibits increasing levels of homogeneity. Many risk management models (e.g., VaR) and investment strategies tend to look and act alike with the

(i) $1/n$ claims against the other banks. Namely, $w_{i1} \approx w_{i2} ... \approx w_{in} \approx 1/n$, $\forall j = 1, ..., n$, such that, the aggregate value of the claims is replaced by the average market value $\sum_j^n \mathbf{W}_{ij} p_j \approx \frac{1}{n} \sum_j^n p_j := p$.

(ii) $1/m$ external asset. Namely, $z_{i1} \approx z_{i2} ... \approx z_{im} \approx 1/m$, $\forall j = 1, ..., m$, such that, the value of investment in external assets is substituted by the average value of external assets in portfolio $\sum_j^m \mathbf{Z}_{ij} y_j \approx \frac{1}{m} \sum_j^m y_j := y$.

Since the total population size of claims and external assets is $N - 1 + M$, for portfolios composed of $n < (N - 1)$ and $m < M$ securities, we have a total of ${}_{N-1}C_n +_M C_m$ possible portfolios with the same size $n + m$. We assume that all the portfolios with the same size, even though comprise different type of securities, have similar distribution properties. Lastly, the book value of promised payments at maturity is assumed to be equal for every bank. Namely, $\bar{p}_i = \bar{p}$ for all $i = 1, ..., n$.

Under the mean-field approximation, banks have portfolios of largely overlapping external assets. Banks engage in similar activities (e.g., mortgages concentrated in specific areas, loans to specific sectors of the economy) and doing so, are, indirectly, linked as they are exposed to the same risk drivers. Therefore, large losses due to exogenous factors hitting a single bank, lead to a chain reaction in the financial network such that when a single bank is close to default, it is likely that several other banks are also close to default. Since at the same time, the financial network of claims and obligations is homogeneous and dense enough [13], the fragility of the individual bank i, i.e., ϕ_i, is very likely to coincide with the average market fragility denoted as ϕ. Namely, $\phi_i \approx \frac{1}{n} \sum_j^n \phi_j := \phi$ for all $i = 1, ..., n$. Then, (7) becomes $\phi = \bar{p}/(y + \frac{\bar{p}}{1+r_f+\beta\phi})$. Solving for ϕ, we get

$$\begin{aligned} &\phi \left(y + r_f y + \beta\phi y + \bar{p} \right) = \bar{p}(1 + r_f + \beta\phi) \\ &... \\ &\phi^2 \beta y + \phi \left(yR + \bar{p}(1 - \beta) \right) - \bar{p}R = 0 \ . \end{aligned}$$

Then, the fragility is the positive root of a quadratic expression in ϕ[14]

$$\phi = \frac{1}{2\beta y} \left[\bar{p}(\beta - 1) - Ry + \left(4\beta\bar{p}Ry + (\bar{p}(1 - \beta) + Ry)^2 \right)^{1/2} \right] \ , \quad R = 1 + r_f \ . \tag{8}$$

Eq. (8), remarks a dependence of ϕ from the aggregated value of external assets/projects y.

Proposition 1. *Let Ω_y be the sample space of y and let $f^- : \Omega_\phi \to \Omega_y$ be a measurable function defined by the inverse of (8) that associates each element ϕ in Ω_ϕ with an element y in Ω_y, s.t., $y \in f^-(\phi)$. Then, the probability space $(\Omega_\phi, \mathcal{A}, \mathbb{P})$ can be mapped into the probability space $(\Omega_y, \mathcal{A}, \mathbb{P})$, s.t. $\Omega_y = \{y \in \mathbb{R}^+ \mid a' \leq y \leq b'\}$ where $a' = \bar{p}(r_f + \beta)/(R + \beta)$ and $b' = \bar{p}(\beta\varepsilon - \varepsilon + 1)/(\beta\varepsilon^2 + \varepsilon)$.*

consequence that the balance sheets' structure of the banks are homogenized. "[...]*The level playing field resulted in everyone playing the same game at the same time, often with the same ball"* (Haldane (2009).

[13] According to the empirical evidence (see e.g., Elsinger et al., 2006; Iori et al., 2006), bank networks feature a core-periphery structure with a dense core of large banks and a periphery of small banks. Our hypothesis about the density of $\mathbf{W}_{n\times n}$ is thus realistic for the banks in the core.

[14] The fragility ϕ takes values in the set $\Omega_\phi = [a, b]$ only under a precise constraint. In particular ϕ is always greater or equal to a. While, ϕ is lower or equal to b under the condition that $p \in \left(0, y(\beta + R)/(\beta + r_f)\right)$.

Proof. See Appendix A. □

Proposition 1, exactly maps together the value of the external assets with the extrema (maximum and minimum) of banking system fragility. The maximum level of financial system robustness – up to its maximum resilience –, occurs when $\phi = a$ or equivalently when $y = b'$. On the contrary, the collapse of the banking system occurs when $\phi = b$ or when $y = a'$. Moreover, it can be shown (through Ito's Lemma), that the process $\{\phi\}_{t\geq 0}$ driving the dynamics of (8) is fully characterized by the underlying process $\{y\}_{t\geq 0}$ which describes the average portfolio of external assets with size m.

Proposition 2. *Given a system of SDEs in (4) and under the mean-field approximation, the average* m*-tuple of external assets follows the dynamics $dy = \mu_y dt + \sigma_y dB$ where B is a Brownian motion.*

Proof. See Appendix A. □

From the definition of ϕ in (5), a downtrend of $\{y\}_{t\geq 0}$ implies an increase of ϕ. While, an uptrend of $\{y\}_{t\geq 0}$ causes ϕ to decrease.[15] In the next sections we will refer to the downtrend, when those assets generate a negative cash-flow (on average). Namely, when $\mu_y < 0$. While, we will refer to the uptrend when they engender a positive cash-flow (on average). Namely, when $\mu_y > 0$. We summarizes this feature of our model as follow.

Assumption 3.1 *The aggregate value of external assets can be in uptrend or in downtrend such to engenders positive or negative cash-inflows for the holder. Both trends are assumed to be persistent i.e., approximately constant during a given period Δt.*

For the remainder of the analysis, all statements refer to mean-field approximations.

4 Systemic Default Probability

4.1 From Individual to Systemic Default

Before to give a formal definition of default probability expressed in terms of leverage, consider that, under the mean-field analysis, the leverage of every bank is assumed to "converge" in distribution to ϕ (i.e., the average leverage over all banks). Therefore, the following assumption enables one to assign a systemic meaning to the results of the next sections.

Assumption 4.1 *Let $\mathbb{P}(\phi_i \in C)$ and $\mathbb{P}(\phi \in C)$ represent the default probability of a single bank and the system, respectively. Since, under the mean-field approximation, $\phi_i \overset{\mathrm{d}}{\simeq} \phi$, for the simplified version of the Continuous Mapping Theorem, $\mathbb{P}(\phi_i \in C) \simeq \mathbb{P}(\phi \in C)$ for all continuity sets $C \subseteq \Omega_\phi$ and all $i = 1, ...n$.*

[15]Thus, the soundness of one bank and the benefit of its diversification in external assets, is conditioned to the up and down trends of the non-financial activities into which the bank invests. Namely, a positive $\mathbb{E}(y_j)_t$ defines an uptrend. While, a negative $\mathbb{E}(y_j)_t$ maps a downtrend.

For the remainder, unless specified differently, the analysis will refer to the systemic default probability, rather than to the failure probability of a single bank.

4.2 Diversification and Systemic Default Probability

Under the mean-field conditions which lead to *Assumption 4.1*, the leverage in (8), takes the meaning of *systemic fragility*. When a large part of banks manages highly leveraged balance-sheets, the system equilibrium is more sensitive to external shocks, such that even a small one may trigger the system towards a distress regime. [16] The systemic default probability (i.e., probability of a financial collapse) is stated in terms of first exit time T which is defined as follows.

Definition 4.1. *Under the results in Proposition 1, the first exit time T, is defined as the first time the process $\{\phi\}_{t\geq 0}$ touches one of the extrema of Ω_ϕ. Equally T is defined as the first time the process $\{y\}_{t\geq 0}$ touches one of the extrema of Ω_y. Namely,*

$$T := inf\{t \geq 0 \mid \mathbb{1}_{\phi_t=b} + \mathbb{1}_{\phi_t=a} \geq 1\} \equiv inf\{t \geq 0 \mid \mathbb{1}_{y_t=b'} + \mathbb{1}_{y_t=a'} \geq 1\}\ . \tag{9}$$

Applying Gardiner (1985), the systemic default probability can be derived either from the dynamics of the process $\{\phi\}_{t\geq 0}$, or equally from the underlying process $\{y\}_{t\geq 0}$.

Proposition 3. ***Systemic Default Probability*** *From Definition 4.1, let $A := \{\phi_T = a\} \equiv \{y_T = b'\}$ be the event exists through the safe barrier and let $B := \{\phi_T = b\} \equiv \{y_T = a'\}$ be the event exists through the default barrier. Then, the systemic default probability is the probability of the event B*

$$\mathbb{P}(B) := \quad \mathbb{P}(\phi_T = b) = \left(\int_a^{\phi_0} \mathrm{d}\phi' \psi(\phi')\right) / \left(\int_a^b \mathrm{d}\phi' \psi(\phi')\right)\ , \tag{10a}$$

$$:= \quad \mathbb{P}(y_T = a') = \left(\int_{y_0}^{b'} \mathrm{d}x \psi(x)\right) / \left(\int_{a'}^{b'} \mathrm{d}x \psi(x)\right)\ , \tag{10b}$$

where $\psi(\phi') = \exp\left(\int_0^{\phi'} -\frac{2\mu_\phi}{\sigma_\phi^2}\,\mathrm{d}\phi\right)$ and $\psi(x) = \exp\left(\int_0^x -\frac{2\mu_y}{\sigma_y^2}\,\mathrm{d}y\right)$. While $\mu_\phi = y\mu_y \frac{\partial f(y)}{\partial y} + \frac{1}{2} y^2 \sigma_y^2 \frac{\partial^2 f(y)}{\partial y^2}$ and $\sigma_\phi = y\sigma_y \frac{\partial f(y)}{\partial y}$.

Proof. See Appendix A. ◻

In words, $\mathbb{P}(B)$ is the probability that the process $\{\phi\}_{t\geq 0}$ initially starting at an arbitrary level $\phi(0) := \phi_0 \in (a, b)$ exits through the default barrier b at any time $t \geq 0$. Equivalently, $\mathbb{P}(B)$ is the probability that the process $\{y\}_{t\geq 0}$ initially starting at an arbitrary level $y(0) := y_0 \in (a', b')$ exits through the default barrier a' at any time $t \geq 0$. Unless otherwise stated, in the remainder when referring to $\mathbb{P}(B)$, we use the expression

[16] Moreover, a status of system fragility is usually accompanied by costs (e.g., higher passive interest rates) which exacerbate the situation of the investors and may even lead to the collapse of the system.

(10b). In particular, we will observe how $\mathbb{P}(B)$ is influenced by: (i) the number m of external assets or projects into which the banking system invests its wealth; (ii) the condition of the non-financial sectors, measured by the instantaneous aggregated expected growth rate μ_y. An asymptotic analysis of (10b), reveals that for an infinite population of external assets (i.e., M tends to infinity), increasing levels of diversification, $\mathbb{P}(B)$ exhibits a bifurcated behavior which depends on the condition of the non-financial sector underlying the external assets. Precisely, $\mathbb{P}(B)$ decreases with diversification in periods of uptrend. While, it increases with diversification in periods of downtrend. Then, the asymptotic behavior of $\mathbb{P}(B)$, yields the following proposition.

Proposition 4. *With respect to (10b), the management strategy defined in 2.1 is desirable in uptrend and is undesirable in downtrend. The degree of (un)desirability increases with the level of diversification.*

Proof. See Appendix A. □

In words, *Proposition 4* says that the benefit of diversification in external assets is influenced by the returns generated by these activities. In periods of uptrend, banking diversification is desirable since the probability of default will asymptotically be reduced to zero as $m \to \infty$. While, in period of downtrend, banking diversification is undesirable since the probability of default will asymptotically goes to one as $m \to \infty$. (See Figure 1 in Appendix). The intuition behind this polarization of probability to "survive" and probability to "fail" is beguilingly simple, but its implications profound. To paraphrase Haldane (2009), in uptrend periods, the system acts as a mutual insurance device with disturbances dispersed and dissipated. Connectivity engenders robustness. In this case, diversification would serves as a shock-absorber. But in downtrend, the system flips the wrong side of the knife-edge. Diversification serve as shock-amplifier, not dampener, as losses cascade.

5 Optimal Diversification Policy

Based on the management strategy in *Definition 2.1*, banks may choose from a finite set of diversification strategies m, with $m = 1, 2, ..., M$. It represents a set of mutually exclusive alternatives s.t. only one of them may be chosen in any period Δt. The "rebalanced" asset allocation would be the following one.

Definition 5.1. ***Asset Allocation***. *Select the number* m *of external investments (assets and/or projects) in which to invest an equal dollar amount. Then, hold the selected assets for an arbitrary period* Δt.

By *Proposition 1*, the payoff from investment is external assets, is bounded.

Definition 5.2. ***Max Profit/Loss***. *The profit (from exogenous investments) of the financial system is bounded between an upper bound at* $y = b'$ *and a lower bound at* $y = a'$. *Then, given the initial aggregate value of the external assets* $y_{t=0} := y_0$, *the gain from investment in external assets is* $\pi^+ := b' - y_0$; *while, the loss is* $\pi^- := y_0 - a'$.

The payoffs resulting from the choice of a diversification strategy m are represented by a random variable Π_m that takes values π in the set $\Omega_\pi = [\pi^-, ..., \pi^+]$. More specifically, given m alternatives $1, 2, ...$ and their corresponding random return $\Pi_1, \Pi_2, ...$, with distribution function $F_1(\pi), F_2(\pi), ...$, respectively, preferences satisfying the von Neumann-Morgenstern axioms imply the existence of a measurable, continuous utility function $U(\pi)$ such that Π_1 is preferred to Π_2 if and only if $\mathbb{E}U(\Pi_1) > \mathbb{E}U(\Pi_2)$. We assume banks are mean-variance (MV) decision makers, such that the utility function $\mathbb{E}U(\Pi_m)$ may be written as a smooth function $V\left(\mathbb{E}(\Pi_m), \sigma^2(\Pi_m)\right)$[17] of the mean $\mathbb{E}(\Pi_m)$ and the variance $\sigma^2(\Pi_m)$ of Π_m or $V\left(\mathbb{E}(\Pi_m), \sigma^2(\Pi_m)\right) := \mathbb{E}U(\Pi_m) = \mathbb{E}(\Pi_m) - (\lambda\sigma^2(\Pi_m))/2$ so that Π_1 is preferred to Π_2 if and only if $V\left(\mathbb{E}(\Pi_1), \sigma^2(\Pi_1)\right) > V\left(\mathbb{E}(\Pi_2), \sigma^2(\Pi_2)\right)$. [18] Then, the maximization problem is

$$\begin{aligned} \max_m \mathbb{E}U(\Pi_m) &= \mathbb{E}(\Pi_m) - \frac{\lambda\sigma^2(\Pi_m)}{2} \\ \text{s.t.:} \quad & m \geq 1 \ . \end{aligned} \tag{11}$$

To simplify notations, let $\mathbb{P}(\mu_y < 0) := p$ and $\mathbb{P}(\mu_y \geq 0) := 1 - p$. Moreover, let $\mathbb{P}(B \mid \mu_y < 0) := q$ be the conditional default probability given a downtrend and $\mathbb{P}(B \mid \mu_y \geq 0) := g$ the conditional default probability given an uptrend. Then, the expected profit $\mathbb{E}(\Pi_m)$ of the financial system is

$$\mathbb{E}(\Pi_m) = p\left[q\pi^- + (1-q)\pi^+\right] + (1-p)\left[g\pi^- + (1-g)\pi^+\right] \ . \tag{12}$$

In Table 1 in Appendix, the expected profit from banking diversification is considered in three different states of nature. (See Table 1 in Appendix). The variance $\sigma^2(\Pi_m)$ of the profit is

$$\begin{aligned} \sigma^2(\Pi_m) = p&\left[q\left(\pi^- - \mathbb{E}(\Pi_m)\right)^2 + (1-q)\left(\pi^+ - \mathbb{E}(\Pi_m)\right)^2\right] \\ &+ (1-p)\left[g\left(\pi^- - \mathbb{E}(\Pi_m)\right)^2 + (1-g)\left(\pi^+ - \mathbb{E}(\Pi_m)\right)^2\right] \ . \end{aligned} \tag{13}$$

5.1 Comparative Statics

Here below we conduct an analytical study of $\mathbb{E}U(\Pi_m)$ for the parameters' range in Appendix (Table 2).

Analysis (i). In this analysis (11) is simplified since two mutually exclusive events are considered. First, the event downtrend which occurs with probability $p = 1$. Second, the uptrend event which occurs with probability $1 - p = 1$. In the downtrend case, $\mathbb{E}U(\Pi_m)$ is maximized for decreasing range of diversification. While, in the uptrend case, $\mathbb{E}U(\Pi_m)$ is maximized for increasing levels of diversification. Then, the following holds true.

[17]To say that V is smooth it is simply meant that V is a twice differentiable function of the parameters $\mathbb{E}(\Pi_m)$ and $\sigma^2(\Pi_m)$.

[18]Only the first two moments matter for the decision maker, so the expected utility can be written as a function in terms of the expected return (increasing) and the variance (decreasing) only, with $\partial V\left(\mathbb{E}(\Pi_m), \sigma^2(\Pi_m)\right)/\partial\mathbb{E}(\Pi_m) > 0$ and $\partial V\left(\mathbb{E}(\Pi_m), \sigma^2(\Pi_m)\right)/\partial\, \sigma^2(\Pi_m) < 0$.

Corollary 1. *Let m_{max} (m_{min}) be the max (min) attainable diversification strategy. Assume banks: (i) invest in external assets with dynamics defined in Proposition 2; (ii) use an allocation strategy as defined in 5.1; (iii) adopt a quadratic utility function of the form in (11). Then, in downtrend, $m_{min} \succ m_{max}$. Conversely, in uptrend, $m_{max} \succ m_{min}$.*

Proof. See Appendix A. □

The *Corollary 1* follows immediately from *Proposition 4*. When external assets pay (with probability one) a negative or positive cash-flow, banks' utility function in (11), respectively decreases or increases with increasing diversification. See results in Table 1, (Case 1 and Case 3) and see Figure 1 in Appendix.

Analysis (ii). In this analysis (11) is maximized w.r.t. m in the case the downtrend and the uptrend are two likely events that might occur with some probability. Then, the following proposition holds.

Corollary 2. *Under the results in Corollary 1, let the event downtrend (uptrend) occur with probability p, $(1-p)$ where $p \in \Omega_P := [0, 1]$. Then, there exists a subset $\Omega_{P^*} \subset \Omega_P$ s.t., to each $p^* \in \Omega_{P^*}$, corresponds an optimal strategy $m^* \in (m_{min}, m_{max})$ that maximizes the MV utility function in (11).*

Proof. See Appendix A. □

In words, when the event downtrend occurs with probability $p < 1$, there exist a specific optimal level of diversification m^* which is a function of p. (See Figure 2 in appendix). *Corollary 2* suggests that only m^* is the optimal diversification strategy that has to be chosen in order to maximize the expected MV utility. Values of $m \gtrless m^*$ are second-best choices. In particular, for m approaching m^* from below, the system increases its robustness, while for m bigger then m^*, the system is beyond its *tipping point* and becomes more fragile. Then the utility function may exhibit a non-monotonic behavior with respect to m. (See Figure 2 in Appendix).

6 Private Incentives vs Social Welfare

Here we discuss the different implications of diversification at system level from the standpoint of the regulator. Let's consider the case of a policy maker that has to include some negative externalities which might be engendered by losses occurred in downtrend periods. Indeed, in an deeply downward trend, the wealth generated by the financial system is below the average-trend and might even be negative due to generalized losses and failures. Because of deadweight costs of systemic failure that exceed the costs of individual failures, the regulator is plausibly considering social costs[19] that might emerge due to losses

[19]Social costs include costs of financial distress and economic distress.

suffered by the financial system. [20] Then, if K is the number of simultaneously crashing banks, treating the policy maker as an expected utility maximizer, it is reasonable to assume the followings.

Assumption 6.1 *The total loss to be accounted by the regulator in downtrend periods, is a monotonically increasing function $f(\cdot)$ of (i) the expected number k of bank crashes given a collapse of at least one bank $\mathbb{E}(K|K \geq 1) = k$, (ii) the magnitude of the loss π^-. Let the function $f(\cdot)$ be defined by $f(k, \pi^-) := k\pi^-$.*

Under this new perspective, the optimal diversification strategy (m^R) from the policy maker point of view does not coincide with the diversification desirable from the financial system point of view (m^*). More precisely,

Corollary 3. *Individual banks' incentives favor a financial network that is over-diversified in external assets w.r.t. to the level of diversification that is socially desirable*

$$m^* \geq m^R \; . \tag{14}$$

Proof. See Appendix A. □

In the presence of limited liability, the expected losses of private firms are bounded. In contrast, at the system level, either due to direct or indirect exposures, the externalities associated with the failure of interconnected institutions amplify the expected losses. Those spillovers reflect the incremental costs due to a system collapse which are ultimately borne by taxpayers. Under this condition, *Corollary 3* says that the optimal level of diversification for the regulator is always smaller than the one desirable at individual level.

7 Concluding Remarks

In this paper, we have combined the balance-sheet approach (Shin, 2008) with a stochastic setting *a' la* Merton (1974). We consider a financial network of risk-averse banks, whose wealth is partially invested in assets external to the financial network, such as mortgages, loans to firms and other activities/projects related to the real-side of the economy. Starting from the law of motion of the value of assets and liabilities, we developed a parsimonious, yet micro-founded, stochastic framework in which the fragility of a bank depends on the fragility of the other banks and on the value of its external assets. Under a mean-field approximation, the systemic failure probability can be derived analytically.

In this setting, we shed light on the conflict between, on one hand, the individual incentive to reduce, through diversification, the idiosyncratic risks of the external assets and, on the other hand, the emergence of systemic risk. In contrast with some optimistic views about diversification, but in line with other recent works (see e.g., Battiston et al., 2009; Ibragimov et al., 2011; Stiglitz, 2010; Wagner, 2009), we find that

[20] The definition of social losses is rather flexible since it depends on the characteristics of the financial system under analysis.

diversification fuels the *double-edged* property of the financial system. Indeed, diversification increases the default probability of the banking system in case external assets pay a negative cash-flow (downtrend) and decreases the default probability in case of positive cash-flows (uptrend).

Two main effects emerge in our model. The first is that the gap between maximal and minimal MV utility is exacerbated with higher degrees of diversification. Moreover, for a given fixed degree, the spread increases the larger is the absolute value of the growth rate (i.e., higher the divergence of returns between the up-down trend. This implies that, for a given probability of occurrence of the down(up) trend, there exists an optimal level of risk diversification which maximize the banking MV utility function.

By including in the analysis social costs due to generalized losses or defaults in the case of a downtrend, we obtain the next final result. Banks' incentives favor a financial network that is over-diversified in external assets w.r.t. to the level of diversification that is socially desirable. Thus, to preserve a stable financial system, diversification should be encouraged in periods of economic boom and constrained in periods of recession. Nevertheless, as diversification of the financial system towards external activities is assumed to be a rigid strategy, the policy should be to adjust the system within a certain diversification range. In line with Ibragimov et al. (2011), our model suggests that policy restrictions to risk-sharing in uptrend periods would limit excessive risk-spreading in downtrend periods. Hence, an important point of our analysis is in recognizing that the objective of the regulator is not to target a specific level of default risk, but rather to manage the tradeoff between the social losses from defaults and the social costs of avoiding defaults.

A Proofs

PROOF OF LEMMA 2.3. The leverage at banking system level can be described starting from (7) which can be rewritten as

$$\bar{p}_i = \phi_i \left(\sum_j^m \mathbf{Z}_{ij} y_j + \sum_j^n \mathbf{W}_{ij} \bar{p}_j / \left(1 + r_f + \beta \phi_j\right) \right)$$

which at system dimension, becomes

$$\begin{aligned} \bar{\mathbf{p}} &= \mathbf{I}\boldsymbol{\phi} \times \left[\mathbf{Z}\mathbf{y} + (\mathbf{R} + \beta \mathbf{I}\boldsymbol{\phi})^{-1} \mathbf{W}\bar{\mathbf{p}}\right] \\ &= \mathbf{\Phi} \times \left[\mathbf{Z}\mathbf{y} + (\mathbf{R} + \beta \mathbf{\Phi})^{-1} \mathbf{W}\bar{\mathbf{p}}\right] \\ &= \mathbf{\Phi}\mathbf{Z}\mathbf{y} + \mathbf{\Phi}(\mathbf{R} + \beta \mathbf{\Phi})^{-1} \mathbf{W}\bar{\mathbf{p}}. \end{aligned}$$

Let define the column vector $\boldsymbol{\delta} := \mathbf{W}\bar{\mathbf{p}}$. Then,

$$\begin{aligned} \bar{\mathbf{p}} &= \mathbf{\Phi}\mathbf{Z}\mathbf{y} + \mathbf{\Phi}(\mathbf{R} + \beta \mathbf{\Phi})^{-1} \boldsymbol{\delta} \\ \bar{\mathbf{p}}\boldsymbol{\delta}^{\mathrm{T}} &= \mathbf{\Phi}\mathbf{Z}\mathbf{y}\boldsymbol{\delta}^{\mathrm{T}} + \mathbf{\Phi}(\mathbf{R} + \beta \mathbf{\Phi})^{-1} \boldsymbol{\delta}\boldsymbol{\delta}^{\mathrm{T}} \\ \bar{\mathbf{p}}\boldsymbol{\delta}^{\mathrm{T}}(\boldsymbol{\delta}\boldsymbol{\delta}^{\mathrm{T}})^{-1} &= \mathbf{\Phi}\mathbf{Z}\mathbf{y}\boldsymbol{\delta}^{\mathrm{T}}(\boldsymbol{\delta}\boldsymbol{\delta}^{\mathrm{T}})^{-1} + \mathbf{\Phi}(\mathbf{R} + \beta \mathbf{\Phi})^{-1} \\ \bar{\mathbf{p}}\boldsymbol{\delta}^{\mathrm{T}}(\boldsymbol{\delta}\boldsymbol{\delta}^{\mathrm{T}})^{-1}(\mathbf{R} + \beta \mathbf{\Phi}) &= \mathbf{\Phi}\mathbf{Z}\mathbf{y}\boldsymbol{\delta}^{\mathrm{T}}(\boldsymbol{\delta}\boldsymbol{\delta}^{\mathrm{T}})^{-1}(\mathbf{R} + \beta \mathbf{\Phi}) + \mathbf{\Phi}. \end{aligned}$$

Let $\mathbf{\Delta} := (\boldsymbol{\delta}\boldsymbol{\delta}^{\mathrm{T}})^{-1}$ and let $\mathbf{K} := \mathbf{Z}\mathbf{y}\boldsymbol{\delta}^{\mathrm{T}}\mathbf{\Delta}$. Then,

$$\begin{aligned} \bar{\mathbf{p}}\boldsymbol{\delta}^{\mathrm{T}}\mathbf{\Delta}\mathbf{R} + \bar{\mathbf{p}}\boldsymbol{\delta}^{\mathrm{T}}\mathbf{\Delta}\beta\mathbf{\Phi} &= \mathbf{\Phi}\mathbf{K}\mathbf{R} + \mathbf{\Phi}\mathbf{K}\beta\mathbf{\Phi} + \mathbf{\Phi} \\ \mathbf{0} &= \mathbf{\Phi}\mathbf{K}\left[\mathbf{R} + \beta\mathbf{\Phi}\right] - \bar{\mathbf{p}}\boldsymbol{\delta}^{\mathrm{T}}\mathbf{\Delta}\beta\mathbf{\Phi} + \mathbf{\Phi} - \bar{\mathbf{p}}\boldsymbol{\delta}^{\mathrm{T}}\mathbf{\Delta}\mathbf{R} \\ &= \mathbf{\Phi}\mathbf{K}\left[\beta\mathbf{\Phi} + \mathbf{R}\right] + \mathbf{\Phi}\left[\mathbf{I} - \bar{\mathbf{p}}\boldsymbol{\delta}^{\mathrm{T}}\mathbf{\Delta}\beta\right] - \bar{\mathbf{p}}\boldsymbol{\delta}^{\mathrm{T}}\mathbf{\Delta}\mathbf{R}, \end{aligned}$$

which is a quadratic expression in $\mathbf{\Phi}$. □

PROOF OF PROPOSITION 1. First let write (8) in compact form as $\phi = f(y)$. Observe that, by definition, the value of external assets cannot be negative, $y \in \mathbb{R}^{+}$. Since the derivative $f'(y)$

$$f'(y) = \frac{-p\left((\beta-1)^2 p + (1+\beta)Ry + (-1+\beta)\sqrt{(-1+\beta)^2 p^2 + 2(1+\beta)pRy + R^2 y^2}\right)}{2\beta y^2 \sqrt{4\beta pRy(p - \beta p + Ry)^2}} < 0$$

is negative for all y and for any value of β, r_f, p in their range of variation (see Table 2 in Appendix), the Inverse Function Theorem implies that f is invertible on $\mathbb{R}^{+}$

$$f^{-1}(\phi) = \frac{p(\phi + \beta\phi + R)}{\phi(\beta\phi + R)}, \quad \text{s.t.} \quad (f^{-1})'(\phi) = \frac{1}{f'(y)}.$$

Moreover, one can show that the inverse f^{-1} is continuous. Given the above result, we obtain the following mapping between the values of ϕ and y at the infimum and supremum of Ω_ϕ:

$$\begin{cases} f(y) = b & \text{iff } f^{-1}(\phi) = \bar{p}(r_f + \beta)/(R + \beta) := a' \\ f(y) = a & \text{iff } f^{-1}(\phi) = \bar{p}(\beta\varepsilon - \varepsilon + 1)/(\beta\varepsilon^2 + \varepsilon) := b' \ . \end{cases}$$

□

PROOF OF PROPOSITION 2. The portfolio composed by the linear combination of m processes described in (4), each of them weighted by $1/m$, is

$$dy(t) = \frac{1}{m}\sum_{j=1}^{m}\mu_j dt + \frac{1}{m}\sum_{i=j}^{m}\sigma_j dB_j(t) = \mu_y dt + ... \tag{15}$$

To derive the diffusion term, it is convenient to think about a sum of m Gaussian correlated error terms weighted by $\frac{1}{m}$. Namely, $\frac{1}{m}\sum_{j=1}^{m}\sigma_j dB_j$ which is equivalent to write $\frac{1}{m}\sum_{j=1}^{m}\sigma_j\sqrt{dt}\xi_j$ with $\xi_j \sim N(0,1)$. Taking the variance yields

$$\begin{aligned}\sigma_y^2 &= \frac{dt}{(m)^2}\sum_{j=1}^{m}Var(\xi_j) + \frac{dt}{(m)^2}\sqrt{Var(\xi_j)Var(\xi_l)}\sum_{j=1}^{m}\sum_{j\neq l}^{m}\rho_{jl}\\ &= \frac{dt}{m}\bar{\sigma}_\xi^2 + \frac{dt}{(m)^2}\sum_{j=1}^{m}\sum_{l\neq i}^{m}\sqrt{\sigma_{\xi j}^2\sigma_{\xi l}^2}\rho_{jl}\end{aligned} \tag{16}$$

where $\bar{\sigma}_\xi^2$ is the average variance among the m's $\sigma^2_{\xi_{j=1,...m}}$. We assume $\sigma^2_{\xi_j}$ is constant, say σ^2 for all $j = 1, ..., m$ and all $t \geq 0$. Then multiplying and dividing the second term of (16) by $m-1$ and taking its square root, we get $\sigma_y := \sqrt{\frac{\sigma^2}{m} + \frac{m-1}{m}\bar{\rho}\sigma^2}$ where $\bar{\rho} = \sum_{j=1}^{m}\sum_{l\neq j}^{m}\frac{\rho_{jl}}{m(m-1)}$. Hence (15) becomes

$$dy(t) = \mu_y dt + \sigma_y dB(t)\ , \quad dB \sim N(0, dt)\ . \tag{17}$$

Tanking all possible ${}_MC_m$ combinations of portfolios into consideration is the same as considering the average value of a portfolio with size m. This is equivalent to adjust the investment position $dy(t)$ in (17) with the following substitutions: $\mu_y \to \frac{1}{M}\sum_{j=1}^{M}\mu_j$, which is the expected return of all external assets in the population; $\bar{\rho} \to \sum_{i=1}^{M}\sum_{j\neq i}^{M}\frac{\rho_{ij}}{M(M-1)}$ which is the expected pairwise correlation of all stocks in the population.

□

PROOF OF PROPOSITION 3. The default probability of the financial system can be expressed in two equal measures: (i) the probability that the process $\{\phi\}_{t\geq 0}$, derived by Ito's lemma from (17), initially starting at an arbitrary level $\phi(0) := \phi_0 \in (a, b)$, exits through the default barrier $b := 1$ after time $t \geq 0$; (ii) the probability that the process $\{y\}_{t\geq 0}$ in (17), initially starting at an arbitrary level $y(0) := y_0 \in (a', b')$, exits through the default barrier $a' := \bar{p}(r_f + \beta)/(R + \beta)$ after time $t \geq 0$. From Gardiner (1985), given the definition of first exit time T in (9), the probability of the event $B = \{\phi_T = b\} = \{y_T = a'\}$ has two equivalent explicit forms

$$\begin{aligned}\mathbb{P}(B) &= \left(\int_a^{\phi_0} \mathrm{d}\phi'\psi(\phi')\right) / \left(\int_a^{b} \mathrm{d}\phi'\psi(\phi')\right) \quad \text{where } \psi(\phi') = \exp\left(\int_0^{\phi'} -\frac{2\mu_\phi}{\sigma_\phi^2}\,\mathrm{d}\phi\right)\ ,\\ &\equiv \left(\int_{y_0}^{b'} dy\psi(x)\right) / \left(\int_{a'}^{b'} \mathrm{d}x\psi(x)\right) \quad \text{where } \psi(x) = \exp\left(\int_0^{x} -\frac{2\mu_y}{\sigma_y^2}\,\mathrm{d}y\right)\ .\end{aligned}$$

Where $\phi_0 \in (a, b)$ with $a = \varepsilon \to 0^+$ and $b = 1$. While, $\mu_\phi = y\mu_y\frac{\partial f(y)}{\partial y} + \frac{1}{2}y^2\sigma_y^2\frac{\partial^2 f(y)}{\partial y^2}$ and $\sigma_\phi = y\sigma_y\frac{\partial f(y)}{\partial y}$ are derived by Ito's Lemma applied to the function $\phi = f(y)$ in (8) by using the dynamics of the process $\{y\}_{t\geq 0}$ in (17) Moreover,

$y_0 \in (a', b')$ and $\sigma_y := \sqrt{\frac{\sigma^2}{m} + \frac{m-1}{m}\bar{\rho}\sigma^2}$, $a' = \bar{p}(r_f + \beta)/(R + \beta)$, $b' = \frac{\bar{p}(\beta\varepsilon - \varepsilon + 1)}{(\beta\varepsilon^2 + \varepsilon)}$. Then, $\mathbb{P}(B)$ has the closed form solution

$$\begin{aligned}\mathbb{P}(B) &= \left(\exp\left(-\frac{2\mu_\phi\phi_0}{\sigma_\phi^2}\right) - \exp\left(-\frac{2\mu_\phi b}{\sigma_\phi^2}\right)\right) / \left(\exp\left(-\frac{2\mu_\phi a}{\sigma_\phi^2}\right) - \exp\left(-\frac{2\mu_\phi b}{\sigma_\phi^2}\right)\right) , \\ &\equiv \left(\exp\left(-\frac{2\mu_y y_0}{\sigma_y^2}\right) - \exp\left(-\frac{2\mu_y b}{\sigma_y^2}\right)\right) / \left(\exp\left(-\frac{2\mu_y a}{\sigma_y^2}\right) - \exp\left(-\frac{2\mu_y b}{\sigma_y^2}\right)\right) .\end{aligned}$$

□

PROOF OF PROPOSITION 4. Let us first provide an heuristic explanation of results in Proposition 3. Consider μ_y to be a random variable $\mu_y : \Omega_{\mu_y} \mapsto \mathbb{R}$ defined on the probability space $(\Omega_{\mu_y}, \mathcal{A}, \mathbb{P})$ where $\Omega_{\mu_y} = [-1, 1]$ with distribution function $F : \mathbb{R} \to \mathbb{R}$ of μ_y as $F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t = \mathbb{P}(\mu_y \leq x)$ [21] To simplify notations, let $\mathbb{P}(\mu_y \leq 0) := p$ and $\mathbb{P}(\mu_y \geq 0) := 1 - p$. Moreover, let $\mathbb{P}(B \mid \mu_y < 0) := q$ be the conditional default probability given a downtrend and $\mathbb{P}(B \mid \mu_y \geq 0) := g$ the conditional default probability given an uptrend. Now, let consider the solution of (17) i.e., $y = y_0 + \mu_y + \sigma_y Bt$ and for simplicity, let $\bar{\rho} = y_0 = 0$. The limit of y for $m \to \infty$, yields a measurable function $y = \mu_y$. Thus, y is also a random variable on Ω_{μ_y}, since the composition of a measurable function is also measurable. Now let bound the variation of y into an arbitrary small subset $\Xi \subset \Omega_{\mu_y}$ such that $y \in [-\xi, \xi]$. Then, for $\mu_y \leq 1$, the probability of y to be equal to the upper bound is $\mathbb{P}(y = \xi \mid \mu_y \leq 1) = 1 - \epsilon$. While, for $\mu_y \geq -1$, the probability of y to be equal to the lower bound is $\mathbb{P}(y = -\xi \mid \mu_y \geq -1) = 1 - \epsilon$. We can now obtain a more formal asymptotic result if we substitute $-\xi$ and ξ with a' and b' respectively. Let rewrite (10b) as $(M^{y_0} - M^{b'})/(M^{a'} - M^{b'})$ where $M = \exp\left(\frac{-2\mu_y}{\sigma_y^2}\right)$. After some arrangements, $\mathbb{P}(B)$ becomes $(M^{y_0 - b'})(M^{a'-b'} - 1) - 1/(M^{a'-b'} - 1)$. Taking the limit for $m \to \infty$ we get that $\mathbb{P}(B \mid \mu_y \geq 0) = 0 - 0 = 0$ and for $\mathbb{P}(B \mid \mu_y < 0) = 0 - (-1) = 1$. Then,

$$\forall \epsilon > 0, \quad \exists \quad \bar{m} > 1 \mid (q - g) > 1 - \epsilon \quad \forall m > \bar{m}$$

□

PROOF OF COROLLARY 1. Based on the "state of nature" of external-assets, Corollary 1 shows the polarization of the expected utility for increasing levels of diversification in those assets. The proof of Corollary 1 moves from the results in Proposition 4. Let first study $\mathbb{P}(B)$ in (10b) as a function of m and for the sake of simplicity, let $\bar{\rho} = \beta = r_f = 0$ and let $\sigma = \bar{p} = 1$. [22]. Then (10b) becomes

$$\mathbb{P}(B) = \left(\exp(-2\mu_y y_0 m) - \exp(-\mu_y m)\right) / \left(1 - \exp(-\mu_y m)\right). \tag{18}$$

When $\mu_y > 0$ ($\mu_y < 0$), (18) is negative (positive) for any parameters' value in the set of Table 2. Its partial derivative w.r.t. m is

$$\frac{\partial\mathbb{P}(B)}{\partial m} = \mu_y \exp\left(\mu_y m(1 - 2y_0)\right)\left(\exp(2\mu_y y_0 m) - 1 + 2y_0 - 2y_0 \exp(\mu_y m)\right) / \left(\exp(\mu_y m) - 1\right)^2 ,$$

which is equal to zero if $\mu_y = 0$, negative if $\mu_y > 0$ and positive if $\mu_y < 0$. Stated otherwise, from the definition of q and g, we have $\frac{\partial q}{\partial m} > 0$ and $\frac{\partial g}{\partial m} < 0$. Let now, study how the utility function changes w.r.t. m in two mutually exclusive cases. Namely, when the external assets are in uptrend or downtrend period for a certainty.

[21]With the usual properties: (i) $F(x) \geq 0$; (ii) $\lim_{x\to+\infty} F(x) = \lim_{x\to+\infty} \mathbb{P}(\mu_y \leq x) = \mathbb{P}(\mu_y < +\infty) = 1$; (iii) $\lim_{x\to-\infty} F(x) = \lim_{x\to-\infty} \mathbb{P}(\mu_y \leq x) = \mathbb{P}(\mu_y < -\infty) = 0$.

[22]By *Proposition 2*, in this simplified case, $y_0 \in [0, 1/2]$

(1) The case $p = 0$ implies a sure uptrend (i.e., $\mu_y > 0$) and the utility function in (11) is simplified as

$$\mathbb{E}U(\Pi_m)_{\mu_y>0} = (g\pi^- + (1-g)\pi^+) - \lambda\left(g\,(\pi^- - \mathbb{E}(\Pi_m))^2 + (1-g)\,(\pi^+ - \mathbb{E}(\Pi_m))^2\right)/2 \tag{19}$$

Then,

$$\frac{\partial \mathbb{E}U(\Pi_m)_{\mu_y>0}}{\partial m} = \left(\pi^- \frac{\partial g}{\partial m} - \pi^+ \frac{\partial g}{\partial m}\right) + \frac{\partial g}{\partial m}\frac{\lambda}{2}\left(\Delta_+^2 - \Delta_-^2\right) ,$$

where $\Delta_+^2 := \left(\pi^+ - \frac{\partial g}{\partial m}(\pi^- - \pi^+)\right)^2$ and $\Delta_-^2 := \left(\pi^- - \frac{\partial g}{\partial m}(\pi^- - \pi^+)\right)^2$. After some rearrangements

$$\frac{\partial \mathbb{E}U(\Pi_m)_{\mu_y>0}}{\partial m} = g\left(\frac{\partial g}{\partial m}\right)(\pi^- - \pi^+) > 0 , \tag{20}$$

which is positive since $\frac{\partial g}{\partial m} < 0$ and $\pi^- - \pi^+ < 0$. Then we conclude that, for $p = 0$, $\mathbb{E}U(\Pi_m)$ is an increasing function of m.

(2) The case $p = 1$ implies a sure downtrend (i.e., $\mu_y < 0$) and the utility function in (11) simplifies as

$$\mathbb{E}U(\Pi_m)_{\mu_y<0} = (q\pi^- + (1-q)\pi^+) - \lambda\left(q\,(\pi^- - \mathbb{E}(\Pi_m))^2 + (1-q)\,(\pi^+ - \mathbb{E}(\Pi_m))^2\right)/2 . \tag{21}$$

Then,

$$\frac{\partial \mathbb{E}U(\Pi_m)_{\mu_y<0}}{\partial m} = \left(\pi^- \frac{\partial q}{\partial m} - \pi^+ \frac{\partial q}{\partial m}\right) + \frac{\partial q}{\partial m}\frac{\lambda}{2}\left(\Phi_+^2 - \Phi_-^2\right) ,$$

where $\Phi_+^2 := \left(\pi^+ - \frac{\partial q}{\partial m}(\pi^- - \pi^+)\right)^2$ and $\Phi_-^2 := \left(\pi^- - \frac{\partial q}{\partial m}(\pi^- - \pi^+)\right)^2$. After some rearrangements,

$$\frac{\partial \mathbb{E}U(\Pi_m)_{\mu_y<0}}{\partial m} = q\left(\frac{\partial q}{\partial m}\right)(\pi^- - \pi^+) < 0 , \tag{22}$$

which is negative since $\frac{\partial q}{\partial m} > 0$ and $\pi^- - \pi^+ < 0$. Then we conclude that, for $p = 1$, $\mathbb{E}U(\Pi_m)$ is a decreasing function of m.

Summarizing, cases (1) and (2) lead to the following result

$$\begin{cases} \mu_y < 0 \Rightarrow \frac{\partial \mathbb{E}U(\Pi_m)}{\partial m} < 0 \\ \mu_y > 0 \Rightarrow \frac{\partial \mathbb{E}U(\Pi_m)}{\partial m} > 0 \end{cases} \therefore \text{if} \begin{cases} p = 0, & V\left(\mathbb{E}(\Pi), \sigma^2(\Pi)\right)_{m_{max}} > V\left(\mathbb{E}(\Pi), \sigma^2(\Pi)\right)_{m_{min}} \iff m_{max} \succ m_{min} \\ p = 1, & V\left(\mathbb{E}(\Pi), \sigma^2(\Pi)\right)_{m_{min}} > V\left(\mathbb{E}(\Pi), \sigma^2(\Pi)\right)_{m_{max}} \iff m_{min} \succ m_{max} \end{cases}$$

where m_{max} (m_{min}) are respectively the max (min) attainable diversification strategies and $\succ$ stands for preference over strategies with $x \succ y$ is read "x i preferred to y". □

PROOF OF COROLLARY 2. The Corollary 2 shows that in the case the downtrend and uptrend events are both likely to occurs with certain probability, there might exists an optimal level of diversification m^* which maximizes the expected utility function $\mathbb{E}U(\Pi_m)$ in (11). To prove it, let's decompose (11) as follow

$$\mathbb{E}U(\Pi_m)_{\mu_y>0} = (1-p)\left[(g\pi^- + (1-g)\pi^+) - \frac{\lambda}{2}\left(g\,(\pi^- - \mathbb{E}(\Pi_m))^2 + (1-g)\,(\pi^+ - \mathbb{E}(\Pi_m))^2\right)\right] \tag{23a}$$

$$\mathbb{E}U(\Pi_m)_{\mu_y<0} = p\left[(q\pi^- + (1-q)\pi^+) - \frac{\lambda}{2}\left(q\,(\pi^- - \mathbb{E}(\Pi_m))^2 + (1-q)\,(\pi^+ - \mathbb{E}(\Pi_m))^2\right)\right] . \tag{23b}$$

Notice that (23a) and (23b) are respectively (19) and (21) used in *Corollary 1* to which a probability p and $1-p$ to occur, has been assigned. In other terms, (11) can be interpreted as a linear combination of (19) and (21). Now, let observe that (23a) is increasing in m, while (23b) is decreasing in m

$$\frac{\partial \mathbb{E}U(\Pi_m)_{\mu_y>0}}{\partial m} = (1-p)g\left(\frac{\partial g}{\partial m}\right)(\pi^- - \pi^+) > 0 \tag{24a}$$

$$\frac{\partial \mathbb{E}U(\Pi_m)_{\mu_y<0}}{\partial m} = pq\left(\frac{\partial q}{\partial m}\right)(\pi^- - \pi^+) < 0 \tag{24b}$$

When the partial derivatives in (24a) and (24b) are equal to each other, the $\mathbb{E}U(\Pi_m)$ is maximized w.r.t. m. The condition to be verified is to find the probability p^* that makes the two equations to be equivalent

$$\text{FOC:} \quad (1-p)g\left(\frac{\partial g}{\partial m}\right)(\pi^- - \pi^+) = pq\left(\frac{\partial q}{\partial m}\right)(\pi^- - \pi^+)\,.$$

Discarding the trivial solution $\mu_y = 0$, the condition is satisfied for all

$$p^* = 1/\left(1 + \frac{q}{g}\left(\frac{\partial q}{\partial g}\right)\right) \in \Omega_{P^*} \subset \Omega_P := [0,1] \tag{25}$$

with $q = q(m^*)$, $g = g(m^*)$. Then, in a more general form, (25) can be written as $p^* = f[g(m^*); q(m^*)]$. Since g, q and f are all one-to-one, for the Inversion Function Theorem are invertible functions. Hence, for a fixed value of p^*, must exists an m^* such that

$$m^* = \left[(g^{-1}; q^{-1}) \circ f^{-1}\right](p^*) \Rightarrow \exists \quad \mathbb{E}U(\Pi_{m^*}) \geq \mathbb{E}U(\Pi_m) \quad \forall m \gtreqless m^*.$$

□

PROOF OF COROLLARY 3. From Assumption 6.1, the total loss accounted by the policy maker in downtrend periods is an increasing function of the expected number k of bank crashes given a collapse of at least one bank and the magnitude of the loss i.e., $k\pi^-$. Then, the expected utility of the regulator $\mathbb{E}U_R(\Pi_m)$ is differently expressed w.r.t. the utility from the bank point of view $\mathbb{E}U(\Pi_m)$ i.e., $\mathbb{E}U_R(\Pi_m) \neq \mathbb{E}U_R(\Pi_m)$. In particular, the changes involve (12) and (13) which are reformulated as follow

$$\mathbb{E}_R(\Pi_m) = p\left[qk\pi^- + (1-q)\pi^+\right] + (1-p)\left[g\pi^- + (1-g)\pi^+\right]$$
$$\sigma_R^2(\Pi_m) = p\left[q\left(k\pi^- - \mathbb{E}(\Pi_m)\right)^2 + (1-q)\left(\pi^+ - \mathbb{E}(\Pi_m)\right)^2\right] + (1-p)\left[g\left(\pi^- - \mathbb{E}(\Pi_m)\right)^2 + (1-g)\left(\pi^+ - \mathbb{E}(\Pi_m)\right)^2\right].$$

Let follow the same reasoning used in Corollary 2 and decompose $\mathbb{E}U_R(\Pi_m)$ as follow

$$\mathbb{E}U_R(\Pi_m)_{\mu_y>0} = (1-p)\left[(g\pi^- + (1-g)\pi^+) - \frac{\lambda}{2}\left(g\left(\pi^- - \mathbb{E}(\Pi_m)\right)^2 + (1-g)\left(\pi^+ - \mathbb{E}(\Pi_m)\right)^2\right)\right] \tag{26a}$$

$$\mathbb{E}U_R(\Pi_m)_{\mu_y<0} = p\left[(qk\pi^- + (1-q)\pi^+) - \frac{\lambda}{2}\left(q\left(k\pi^- - \mathbb{E}(\Pi_m)\right)^2 + (1-q)\left(\pi^+ - \mathbb{E}(\Pi_m)\right)^2\right)\right]. \tag{26b}$$

Let observe that the partial derivative w.r.t. m of (26b) is steeper then the partial derivative w.r.t. m in (24b). In particular,

$$\frac{\partial \mathbb{E}U_R(\Pi_m)_{\mu_y<0}}{\partial m} = pq\left(\frac{\partial q}{\partial m}\right)(k\pi^- - \pi^+) < \frac{\partial \mathbb{E}U(\Pi_m)_{\mu_y<0}}{\partial m} = pq\left(\frac{\partial q}{\partial m}\right)(\pi^- - \pi^+) < 0$$

While
$$\frac{\partial \mathbb{E}U_R(\Pi_m)_{\mu_y>0}}{\partial m} \equiv \frac{\partial \mathbb{E}U(\Pi_m)_{\mu_y>0}}{\partial m} = (1-p)g\left(\frac{\partial g}{\partial m}\right)(\pi^- - \pi^+)$$
since (26a) is not affected by Proposition 5.1 and remains equivalent to (23a). The condition to be verified is to find the probability p^R that makes the two equations to be equivalent
$$\text{FOC:} \quad (1-p)g\left(\frac{\partial g}{\partial m}\right)(\pi^- - \pi^+) = pq\left(\frac{\partial q}{\partial m}\right)(k\pi^- - \pi^+)\,.$$
Discarding the trivial solution $\mu_y = 0$, the condition is satisfied for all
$$p^R = 1/\left(1 + \frac{q}{g}\left(\frac{\partial q}{\partial g}\right)\upsilon\right) \in \Omega_{P^R} \subset \Omega_P := [0,1] \tag{27}$$
with $q = q(m^R)$, $g = g(m^R)$ and $\upsilon = (k\pi^- - \pi^+)/(\pi^- - \pi^+)$. In a more general form, (27) can be written as $p^R = f(\upsilon[g(m^*); q(m^*)])$. Since g, q, υ and f are all one-to-one and hence invertible functions, for a fixed value of p^R, must exists an m^R such that $m^R = \left[(g^{-1}; q^{-1}) \circ \upsilon^{-} \circ f^{-1}\right](p^R)$. Eq.(27) is a decreasing function w.r.t. υ which is a constant function bigger than one as $k > 1$. Then, $p^R < p^*$. This implies that the diversification level m^R which is optimal from the policy maker point of view is lower than the level m^* desirable by the banking system
$$m^R = \left[(g^{-1}; q^{-1}) \circ \upsilon^{-1} \circ f^{-1}\right](p^R) < m^* = \left[(g^{-1}; q^{-1}) \circ f^{-1}\right](p^*)$$

□

B Tables

B.1 Expected Profit from Banking Diversification

Expected profit from banking diversification in the real-sector				
	External-assets trend	p	$\mathbb{E}(\Pi_m)$	$\lim_{m\to+\infty} \mathbb{E}(\Pi_m)$
Case 1	up	0	$g\pi^- + (1-g)\pi^+$	π^+
Case 2	stable	0.5	$\frac{(q+g)\pi^- + (2-q-g)\pi^+}{2}$	$\frac{\pi^+ + \pi^-}{2}$
Case 3	down	1	$q\pi^- + (1-q)\pi^+$	π^-

Table 1: Expected profit from banking diversification in case of up stable and down trend of non-financial assets. First, let remind (from *Proposition 3*), the asymptotic behavior of $\mathbb{P}(y = a')$ for negative and positive values of μ_y which gives $\lim_{m\to+\infty} q = 1$ and $\lim_{m\to+\infty} g = 0$. In Case 1, the expected profit of the banking system increases with diversification because $\lim_{m\to+\infty} g = 0$. In Case 2, we implicitly assume that $\mu_y \sim N(0,1)$ since $\mathbb{P}(\mu_y \leq 0) = F(0) = 0.5$. The expected profit asymptotically, for $m \to +\infty$, converges to $(\pi^+ + \pi^-)/2$. While for $m \to 1$, both q and g converge to one and the expected profit become $\mathbb{E}(\pi) = \pi^- + \pi^+$. In Case 3, the expected profit decreases with banking diversification because $\lim_{m\to+\infty} q = 1$.

B.2 Range of values for Parameters

Parameter	Set of Values
m	$\{m \in \mathbb{N} : m \geq 1\}$
σ	$\{\sigma \in \mathbb{R} : 0 < \sigma < 1\}$
β	$\{\beta \in \mathbb{R} : 0 < \beta < 1\}$
r_f	$\{r_f \in \mathbb{R} : 0 < r_f < 1\}$
μ_y	$\{\mu_y \in \mathbb{R} : -1 \leq \mu_y \leq 1\}$
$\bar{\rho}$	$\{\bar{\rho} \in \mathbb{R} : 0 \leq \bar{\rho} \leq 1\}$
b'	$\{b \in \mathbb{R} : b' > 0\}$
a'	$\{a \in \mathbb{R} : 0 < a' < b\}$

Table 2: Range of values for each expl. variable.

C Figures

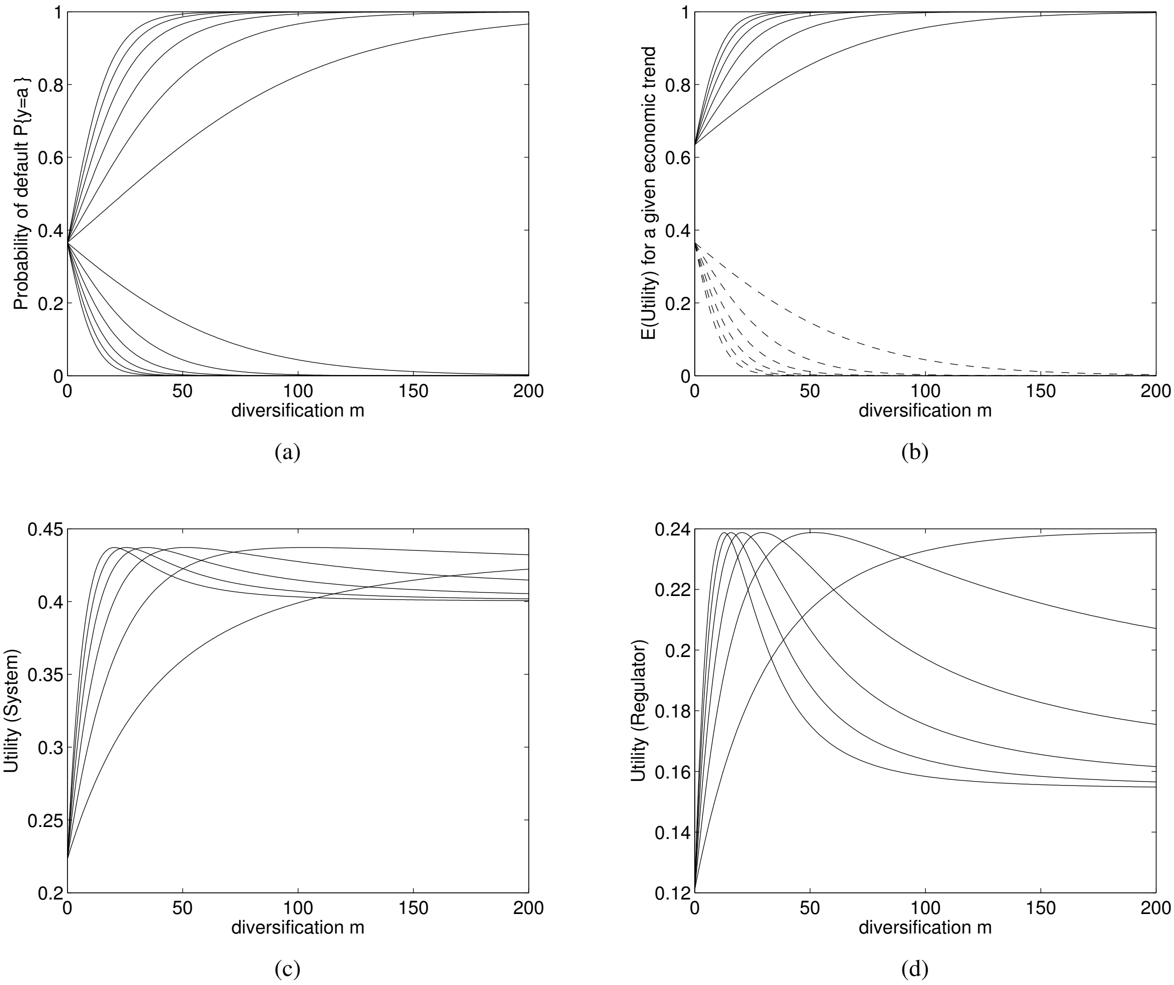

Figure 1: **(Up)**: Default prob. $\mathbb{P}(y = a')$ and Exp. Utility $\mathbb{E}U(\Pi_m)$ of the banking system for different degrees m of diversification. **(a)** $\mathbb{P}(y = a')$ is decreasing in m when $\mu_y > 0$ (lower curves). $\mathbb{P}(y = a')$ is increasing in m when $\mu_y < 0$ (upper curves). **(b)** $\mathbb{E}U(\Pi_m)$ is decreasing in m when $\mu_y < 0$ (lower curves). $\mathbb{E}U(\Pi_m)$ is increasing in m when $\mu_y < 0$ (upper curves). Parameters: $\sigma^2 = 0.35$, $r_f = 0.03$, $\pi^- = \pi^+ = 1, \bar{\rho} = 0.1, m \in [0, 200]$, $|\mu_y| = 0.005, 0.01, 0.015, 0.02, 0.025, 0.03$. **(Down)**: Optimal utility of the banking system vs. Optimal utility of the regulator for different degrees m of diversification and intermediate probability p. **(c)** Exp. Utility of the banking system $\mathbb{E}U(\Pi_m)$. **(d)** Exp. Utility of the regulator $\mathbb{E}U_R(\Pi_m)$. Parameters: $\sigma^2 = 0.35$,$r_f = 0.03$, $\bar{\rho} = 0.1$,$p = 0.4$, $\pi^- = \pi^+ = 1$, $k = 2$, $m \in [0, 200]$, $|\mu_y| = 0.005, 0.01, 0.015, 0.02, 0.025, 0.03$.

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The Knife-Edge of Procyclical Capital Requirements

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Abstract

We introduce a simplified economy of interconnected banks whose assets include claims against other banks and securities external to the network. In response to asset price movements, banks actively manage their balance sheets in line with the risk management practice to adjust the Value-at-Risk to a target level. The contractions and expansions of the balance sheets cause further asset price movements. In this setting of leverage-asset price cycle, banks' incentive to reduce exogenous idiosyncratic shocks through asset diversification reveals to be a double-edge sword strategy. Whereas it mitigates the default risk in presence of positive asset price shocks, it increases the default risk in case of negative shocks. Our findings have significant implications to strengthen countercyclical regulatory instruments.

JEL Code: G01, G11, G18, G2, G32, G33
Keywords: Systemic Risk, Financial Crisis, Leverage, Diversification, Default Probability.

1 Introduction

In a financial system in which balance sheets are continuously marked-to-market, asset price movements have an instantaneous impact on the net worth of the market participants. Since many financial institutions use borrowing to leverage their exposure to risky assets, the impact on their balance sheet is amplified by their degree of leverage.[1] In this context, it has been pointed out that leverage can be procyclical (Adrian and Shin, 2010; Shin and Adrian, 2008). Indeed, sensible price movements – e.g., a devaluation – of an asset class, can trigger the necessity of institutions to reduce their exposure to that asset class, by selling a certain amount. If the contraction of the demand is big enough, it will move prices further down, in a spiral of devaluation. This effect is due to the positive feedback loop between prices and leverage in presence of an active balance sheet management.

The central question addressed in this paper is how *systemic risk* (i.e. the probability of default of a large part of the financial system) depends on the level of asset diversification when risk management practices are influenced by procyclical capital requirements. To this end, by building on Tasca and Battiston (2011), we introduce a simplified economy of interconnected risk-averse financial institutions that hold claims of each other. Moreover, institutions invest also in a number of risky assets linked to activities exogenous to the financial system (hereafter, "external assets"). In line with the risk management rule to adjust the Value-at-Risk (VaR) to a given target level, institutions actively manage their balance sheet in response

[1] Levered market participants include (i) Individual investors who buy stocks or other assets on a margin; (ii) Banks whose capital equity requirements are set by regulatory authorities under the Basel III framework.

to external assets price movements. Our aim is to outline the aggregate consequences of procyclical leverage, by modeling how the practice to expand and contract the balance sheets impact on asset prices and vice-versa.

The main contribution of the paper is to build a model that allows to measure quantitatively systemic risk in a fully dynamic setting where leverage and asset prices are interlinked by a positive feedback loop. We find that such a feedback loop amplifies the *knife-edge* property of the financial system Haldane (2009), i.e., the fact that even when the system appears robust it may be susceptible to sharp disruption once certain critical conditions are reached.

In addition, the paper offers a first attempt to microfound in a dynamic setting the procyclicality of leverage which empirical evidence has been documented so far mainly by Adrian and Shin (2008a,b, 2010, 2011, 2008d); Shin (2008); Shin and Adrian (2008).[2] In this respect, we model the macro-consequences of procyclical leverage and we analyse the mutual influence between contraction-expansion of the aggregate balance sheet and asset pricing of external assets.

This paper is related to several strains of literature. Besides the seminal paper by Allen and Gale (2001), several works have investigated financial contagion on the interbank market (see e.g., Boss et al., 2004; Elsinger et al., 2006; Freixas et al., 2000; Furfine, 2003; Iori et al., 2006). Other works have investigated contagion effects mediated by some common asset (see e.g., Kiyotaki and Moore, 1997, 2002). Moreover, the paper is in line with a body of works that, after the recent financial crisis 2007/8, have investigated the relation between diversification and the systemic financial stability, (Battiston et al., 2009; Brock et al., 2009; Ibragimov and Walden, 2007; Stiglitz, 2010; Tasca and Battiston, 2011; Wagner, 2010). Since the balance sheet approach is central to our paper, the results are also related to the vast literature on the amplification of financial shocks. The literature has distinguished two distinct channels. The first is the increased credit that operates through the borrower's balance sheet, where a greater credit worthiness of the borrower leads to increasing lending (see e.g., Bernanke and Gertler, 1989; Kiyotaki and Moore, 1997, 2005). The second channel operates through the banks' balance sheets, either through the liquidity structure (see e.g., Bernanke and Blinder, 1988; Kashyap and Stein, 2000), or the cushioning effect of the banks equity capital, e.g. Van den Heuvel (2002). Our analysis is closer to the second group of works since we also focus on the bank's balance sheet dynamics. Finally, the possibility that a market populated with VaR constrained traders may have more pronounced fluctuations has been examined by Danıelsson et al. (2002). Mark-to-market accounting may at first appear to be an esoteric question on measurement, but we have seen that it has potentially important implications for financial cycles. Plantin et al. (2008) present a microeconomic model that compares the performance of marking to market and historical cost accounting systems.

To conclude, our findings contributes to a better understanding of the relation between leverage, financial market liquidity and asset pricing (see e.g., Acharya and Pedersen, 2005; Acharya et al., 2011, 2007;

[2] Moreover, Adrian and Shin (2011) offers a two-stage principal-agent model to study the relation between balance sheet leverage and the riskiness of the bank's assets.

Adrian and Shin, 2010; Allen and Gale, 2004; Brunnermeier and Pederson, 2009; Brunnermeier and Pedersen, 2005; Gromb and Vayanos, 2002; Morris and Shin, 2004). The mean field analysis of the model shows that the rate of change of the aggregate balance sheets of the financial system is a proxy for understanding the liquidity and the asset price trend in the market. When external asset prices increase, banks hold "capital surplus" that is utilized by expanding the balance sheets. On the liability side, banks take on more debts, while on the asset side they increase the demand of external assets for a given price (i.e., upward shift of the demand curve).[3] This process increases the liquidity in the market of risky external activities. Conversely, when a shock hits the external assets, the "capital deficit" entails a liquidity shortage in that market. As exemplified during the 2007 sub-prime mortgage crises in the United States, when balance sheets are expanding fast enough, even borrowers that do not have the means to repay are granted credit. However, as soon as a shock hits the market, the balance sheet adjustments produce a downturn in the credit cycle.

The paper is organized as follows. In Section 2 we introduce the model. In Section 3 we analyze the *knife-edge* property of the procyclical ("leverage-cycle") market comparing the results with a standard ("plain-vanilla") market. Section 4 concludes and consider some extensions.

2 Model

In the following, to keep notations simple, whenever it is unambiguous from the context that a parameter is time-dependent, we will omit the index t.

Let time be indexed as $t \in [0, T]$ and consider a financial sector composed by a set $\Omega_n := \{1, ..., n\}$ of risk-averse financial institutions (hereafter, "banks"). In analogy to Tasca and Battiston (2011), the balance-sheet identity of the bank $i \in \Omega_n$ is given by

$$a_i = h_i + e_i \ , \tag{1}$$

where $\mathbf{a} \in \mathbb{R}^n$ is the column vector of the banks' assets market value; $\mathbf{h} \in \mathbb{R}^n$ is the column vector of the book values of banks' obligations (e.g., zero-coupon bonds);[4] $\mathbf{e} \in \mathbb{R}^n$ is the column vector of equity values. It is assumed that, each bank invests in two asset classes composed by: (i) $n - 1$ obligations, each issued by one of the other banks in the system, and (ii) m exogenous investment opportunities non related to the financial sector which belong to the set $\Omega_m := \{1, ..., m\}$ of external assets. Then, the asset-side in

[3] The perverse nature of the demand and supply curves are even stronger when the leverage of the bank is procyclical, i.e., when leverage is high during booms and low during busts.

[4] In particular, h_i is the value of the payment at maturity that each obligor i has promised to the lenders. The equation implicitly assumes that, for all the banks in the system, there is only one type of debt with the same maturity date t and same seniority.

(1) reads as

$$a_i := \sum_l Q_{il} p_l + \sum_j W_{ij} \tilde{h}_j , \tag{2}$$

where, $\mathbf{Q} := [Q_{il}]_{n\times m}$ is the $n \times m$ weighting matrix of external investments where each entry Q_{il} is the quantity of external asset l held by bank i; p_l is the price of the external assets l; $\mathbf{W} := [W_{ij}]_{n\times n}$ is the $n \times n$ adjacency matrix where each entry W_{ij} is the quantity of the debt issued by the obligor j held by bank i; [5] $\tilde{h}_j$ is the market value of debt issued by bank j, defined as the discounted value of future payoffs $\tilde{h}_j := h_j \left[1 + r_j\right]^{-t}$. We assume that external asset prices follow the processes[6]

$$\frac{dp_l}{p_l} = \mu_l dt + \sigma_l dB_l , \quad \forall l \in \Omega_m . \tag{3}$$

2.1 Leverage

From (1) and (2), the ith bank's leverage is defined as its debt-asset ratio

$$\begin{aligned}
\phi_i &:= h_i / a_i && \text{(4a)}\\
&= h_i \Big/ \left(\sum_l Q_{il} p_l + \sum_j W_{ij} \tilde{h}_j \right) && \text{(4b)}\\
&= h_i \Big/ \left(\sum_l Q_{il} p_l + \sum_j W_{ij} h_j \left[1 + r_j\right]^{-t} \right) \in (0, 1]. && \text{(4c)}
\end{aligned}$$

As in Tasca and Battiston (2011), we assume that the rate of return on claims depends upon the obligor's credit-worthiness (measured by its leverage) so that, for each obligor j the following relation holds true[7]

$$r_j := r_f + \beta \phi_j , \tag{5}$$

where r_f is the risk free rate. Substituting (5) into (4c) we obtain a system of coupled equations where the bank i's leverage is a positive function of the leverage of all the other js banks interacting with i:

$$\phi_i = h_i \Big/ \left(\sum_l Q_{il} p_l + \sum_j W_{ij} h_j \left[1 + r_f + \beta \phi_j\right]^{-t} \right), \quad \forall i, j = 1, ..., n. \tag{6}$$

[5] Borrowing from the graph theory terminology, the interbank market can be described as a network with a set of vertices (i.e., nodes) composed of banks and a set of edges given by the mutual interchanged borrowing-lending positions.

[6] Where B_l is a standard Brownian motion defined on a complete filtered probability space $(\Omega; \mathcal{F}; \{\mathcal{F}_t\}; \mathbb{P})$, with $\mathcal{F}_t = \sigma_l\{B(s) : s \leq t\}$, μ_l is the instantaneous risk-adjusted expected growth rate, $\sigma_l > 0$ is the volatility of the growth rate and $d(B_l, B_k) = \rho_{lk}$.

[7] The parameter $\beta \in [0, 1]$ can be understood as the responsiveness of the rate of return to the firm financial condition.

2.2 Target Leverage. An Example.

In this section, we will focus on a single bank and we will thus suppress the index referring to the ith bank.

The role of VaR in explaining banks' dynamic balance sheet decisions with target leverage is discussed in Adrian and Shin (2008a,b, 2010); Shin (2008); Shin and Adrian (2008). The concept of VaR[8] has been widely adopted among financial institutions in their risk management practices.[9] Since the risk management of a bank is intimately tied to its leverage, we assume that the bank adjusts its balance sheet so that the market equity e is set equal to its VaR. In other words, the VaR is the equity capital that a bank must hold in order to stay solvent with probability c. [10]

In the following we link the balance sheet management of the bank to the balance sheet adjustments the bank makes in order to stay close to a target value of the ratio VaR to equity capital. We denote by V the VaR per dollar of assets held by a bank. If the bank maintains equity capital e to meet total VaR, then we have $e = V \times a \Rightarrow a - h = V \times a \Rightarrow h = a \times (1 - V)$. This implies that the bank has a target level of leverage $\phi^* = h/a = (1 - V)$.

In a falling market, decreasing external asset prices increase the leverage (i.e., $\phi > \phi^*$). In this case, the bank desires to shrink its balance sheet by selling (a portion of) the external assets and pay back (a portion of) its debts with the proceeds.[11] In contrast, in a raising market, external asset prices increase and leverage decrease (i.e., $\phi < \phi^*$). In this case, the bank desires to expand its balance sheet by taking on more debts and to invest the new funds in external activities.

In this section, we illustrate with a (discrete-time) numerical example the mechanisms behind the behavior of a bank that manages its balance sheet so to maintain a constant leverage ratio ϕ^*. In the next

[8]The VaR at confidence level c relative to some reference value $\bar{a}$ (e.g., the face value of assets), is defined as the smallest non-negative number VaR such that

$$\mathbb{P}(a < \bar{a} - VaR) \leq 1 - c \equiv \mathbb{P}(\bar{a} - a > VaR) \leq 1 - c \equiv \mathbb{P}(loss > VaR) \leq 1 - c \,,$$

where a is the marked-to-market asset value of the bank as defined in (2) at some give time horizon. In words, the VaR can be seen as the "approximately worst case" loss that the bank may suffer, so that anything worse than this "approximately worst case" has a probability of occurrence less than some benchmark $1 - c$. Notice that, the confidence level c is set high enough to target a given credit rating.

[9]Moreover, it has been enshrined in the regulatory framework for capital since the 1996 Market Risk Amendment of the Basel Accord, and in the Basel II-III Regulations.

[10]The term "economic capital" is sometimes used interchangeably with the bank's value at risk. The most common way to quantify economic capital is by Value-at-Risk. For example, the Basel II Accord specifies that, for credit portfolio risk, the economic capital is equal to $VaR_{0.9998} - \mathbb{E}[\ell]$ where $VaR_{0.9998}$ is computed within the so-called "internal ratings" framework. The rationale for reducing $VaR_{0.9998}$ by the amount of expected losses is that $\mathbb{E}[\ell]$ is typically charged to the client, along with management fees and the like.

[11]The "debt overhang" problem noted by Myers (1977) makes it difficult or impossible to raise new equity when a bank is in trouble. Thus the usual strategy to satisfy capital requirements is to sell assets. This forced selling can generate the positive feedback loop (leverage-asset price) discussed further below.

section, we generalize the example to model an accounting rule. Let the balance sheet be structured as follows (see Table 1 in Appendix Section A). The asset side is composed of $\tilde{h}$ claims against other banks plus a bundle Q of external assets with market price p. The liability side is given by the book value of debts h and the equity e. The time is indexes as $t = 0, 1, 2$. At time t=0, assume the bank holds 80 units of external asset with price $p(0) = 1\$$ and 20\$ of claims. The liability side is split between debts for 60\$ (i.e., $h(0) = 60\$$) and equity for 40\$ (i.e., $e(0) = 40\$$). Therefore, the leverage ratio ϕ is equal to 0.6 (i.e., debts/tot. assets = 60\$/100\$). Let the target leverage ϕ^* coincide with ϕ at time t=0 (i.e., $\phi = \phi^*$). From (4b), to ϕ^* it corresponds an equilibrium price p^* which keeps the leverage equal to the target level $p^* = \frac{h-\tilde{h}\phi^*}{\phi^* Q(0)}$. At time t=0, since $\phi = \phi^*$, also $p(0)$ is exactly equal to p^*. Namely, $p(0) = \frac{60\$-20\$\times 0.6}{0.6\times 80} = 1\$$. We now assume there is a negative shock on price so that $p(1) < p(0)$. As an example, suppose p decreases by 25% so that $p(1) = 0.75\$$. Therefore, the new market value of external assets is 60\$ (i.e., $80 \times 0.75\$ = 60\$$). As a consequence, the balance sheet changes as follows. At time t=1, the asset side comprises 60\$ of external assets and 20\$ of claims against other banks. On the liability side, since the value of debt stays constant at 60\$, equity reduces to 20\$. Then, the leverage ϕ increases to 0.75, (i.e., 60\$/80\$) which is greater than the target one ($\phi = 0.75 > 0.6 = \phi^*$). Under the assumption that the bank adopts a dynamic balance sheet approach, the leverage ratio has to be adjusted back to the target value ϕ^*. In order to do so, it is assumed that the bank sells part of its external assets in the market and with the proceeds pays back part of its debts. As a result, at time t=2 the asset side is composed of 30\$ of external assets and 20\$ of claims. While, the debts decrease to 30\$ and the equities remain at 20\$ such that leverage goes back again to 0.6 (i.e., $\phi = 30\$/50\$ = 0.6 = \phi^*$). This accounting technique indicates to sell 40 units of external assets at $p(1) = 0.75\$$ so that at time t=2, their amount in the balance sheet reduces to $30\$ = 40 \times 0.75\$$. With the proceeds of 30\$ (i.e., $60\$ - 30\$$), the bank partially pays back its debts that diminish to 30\$. Thus, a decrease in the price of external assets leads to a decreased holding worth 30\$. After the sales, the leverage is back down to the target level of 0.6.

Vice-versa, the mechanism works in the opposite direction when the shock on prices is positive, $p(1) > p(0)$. (The corresponding numerical example is given in Appendix Section B). Notice that forced sale can also be generated by trading strategies. For example, Gennotte and Leland (1990) show that portfolio insurance hedging strategies, with “synthetic derivatives”, require to sell a stock as its market value falls. (An example is sketched in the Appendix Section B).

2.3 Target Leverage. An Accounting Rule

The fact that the bank has a target level of leverage implies that facing a movement in asset prices it has either to expand or to shrink both assets and liabilities by a given amount. As explained in the previous section, we assume that a reduction in liabilities is carried out by paying back part of the debt with the proceedings obtained from selling part of the asset that was shocked negatively. Similarly, facing an increase of the asset price the bank takes on more debt and invests the funds in the same asset that was positively shocked. The balance sheet identity (1) imposes that variations on the asset side must equal those on the liability side.

All together these conditions imply (the derivation is reported in Appendix Section C) that the bank has to resize its balance sheet entries by a factor, smaller or larger than one. For a generic bank i, the rate of change of the debt side and of the quantity of external asset is governed by the following equations

$$\frac{dh_i}{h_i} = \left(\frac{\varepsilon_i}{\phi_i}\right)\left(\frac{\phi_i^* - \phi_i}{1 - \phi_i^*}\right) , \tag{7}$$

$$\frac{dQ_{il}}{Q_{il}} = \left(\frac{\varepsilon_i}{\alpha_{il}}\right)\left(\frac{\phi_i^* - \phi_i}{1 - \phi_i^*}\right) . \tag{8}$$

The parameter $\varepsilon_i \in [0, 1]$ measures the reactivity of the ith bank in pursuing the target level of leverage ϕ_i^*; $\alpha_{il} \in [0, 1]$ is the fraction of the external asset l over the total asset value. Accounting constrains: $Q_i = \sum_l Q_{il}$; $dQ_i \geq -Q_i$; $dh_i \geq -h_i$.

2.4 Leverage-Cycle

Motivated by the literature on the leverage-asset price cycle described (also empirically) in Adrian and Shin (2008a, 2010, 2008d), we now model how the accounting rule described by Equation (8) impact on the price of the external asset. We assume a simple relation between the price of external assets in (3) and the aggregated demand for those assets. More precisely, the expected price return is assumed to change linearly with the rate of change of the demand for the asset,

$$\mathbb{E}\left(\frac{dp_l}{p_l}\right) = \gamma\left(\frac{dQ_l}{Q_l}\right) , \tag{9}$$

where the parameter $\gamma \geq 0$ captures the responsiveness of the price to changes in quantity; $Q_l = \sum_i Q_{il}$. Equation (9) implies that we can rewrite the price dynamics as follows

$$\frac{dp_l}{p_l} = \gamma\left(\frac{dQ_l}{Q_l}\right)dt + \sigma_l dB_l , \quad \forall l \in \Omega_m . \tag{10}$$

Indeed, a sensible deviation of the price p_l from p_l^* entails a change in the relationship between price and quantity of external assets. This corresponds to a shift in the demand curve, due to an income effect, because when $e_i > e_i^*$ ($e_i < e_i^*$) the bank has greater (smaller) amount of funds to allocate to external assets (See Table 3 in Appendix Section A and also Appendix Section D).

More precisely, when prices rise, the upward adjustment of leverage entails purchase of risky assets. The greater demand for risky assets tends to put upward pressure on the prices and the equilibrium moves from the original level A to A' (See Figure 4 in Appendix Section D). The mechanism works exactly in reverse in downtrend.[12]

Then, in summary, the dynamics of the financial market is described by the following $3 \times n + m$ dimensional system of equations

$$\begin{cases} \frac{dQ_{il}}{Q_{il}} = \left(\frac{\varepsilon_i}{\alpha_{il}}\right)\left(\frac{\phi_i^* - \phi_i}{1-\phi_i^*}\right) dt \\ \frac{dh_i}{h_i} = \left(\frac{\varepsilon_i}{\phi_i}\right)\left(\frac{\phi_i^* - \phi_i}{1-\phi_i^*}\right) dt \\ \frac{dp_l}{p_l} = \gamma\left(\frac{dQ_l}{Q_l}\right) dt + \sigma_l dB_l \\ \phi_i = h_i / \left(\sum_l Q_{il} p_l + \sum_j W_{ij} \frac{h_j}{(1+r_f+\beta\phi_j)^t}\right) \end{cases} \tag{11}$$

2.5 Plain-Vanilla Vs. Leverage-Cycle Market

In our model (11), the accounting rule based on the target leverage generates changes in the demand which are transmitted to asset prices and back to the leverage. The intensity of the feedback loop between leverage and asset prices depends on two factors. First, the elasticity, ε, by which banks adjust their leverage to the target one, as described in sub-Section 2.3 and Appendix Section C. Second, the responsiveness, γ, of asset prices to shifts in the aggregate demand which depends on the number (n) of banks targeting their leverage according to the asset price variations (i.e., synchronization effect) as discussed in Appendix Section D. [13]

Based on those mechanisms, we distinguish between a standard economy and a leverage-cycle economy as follows.

Definition 2.1 (**Plain-Vanilla Market**). *In a standard plain-vanilla market banks invest in external assets but do not apply target leverage accounting rules at any time along their business activity, i.e.,* $\phi(t) \nrightarrow \phi^*$ *for all* $t > 0$ *and* $\varepsilon \to 0$. *Their strategies are not synchronized, i.e.,* $n \to 1$. *Therefore, it is a market where a cycle between leverage and asset prices is absent, i.e.,* $\gamma \to 0$.

Definition 2.2 (**Leverage-Cycle Market**). *In a leverage-cycle market, banks apply stringent target leverage accounting rules in order to continuously target their leverage to the reference level. Their strategies are synchronized. This market entails a cycle between leverage and asset prices. Banks very promptly*

[12] When assets prices decline, the downward adjustment of leverage to the target level, entails (forced) sale of risky assets. The lower demand for risky assets put downward pressure on their prices and the equilibrium moves from A to A''. In practice, while most market participants base their trading on their view of asset fundamentals relative to price, an important subset of investors must sell even if they believe market fundamentals don't warrant selling. In our model, such forced selling is related to general declines in asset prices and thus correlated across banks. This type of forced selling can lead to price declines which in turn force further selling and further price declines, a positive feedback situation that can lead to extreme market volatility and "crashes".

[13] The synchronization effect means that the change of individual demands to any asset price shock assumes the same sign/direction.

apply the accounting rule, i.e., $\varepsilon \rightarrow 1$. *Any exogenous price shock triggers a synchronized response, i.e.,* $n >> 1$ *and* $\gamma >> 1$.[14]

Since there is no closed form solution for the system (11), in the next sections we study its properties by means of numerical analyses.

3 Analysis

The objective function of risk-averse banks, is to increase the asset value per unit of risk. This business goal is achieved through diversification among the investment opportunities offered by the non-financial sector represented by the matrix $\mathbf{Q}$. More precisely, the diversification strategy is defined as follows.

Definition 3.1 (1/m Diversification Strategy). *The* 1/m *heuristic involves the composition of an equally weighted portfolio composed of* m *external risky assets available for investment.*

According to the diversification strategy defined above, each bank $i \in \Omega_n$ manages a portfolio of external assets $p_i(t) = \sum_l Q_{il}(t)p_l(t)$ s.t., $\tilde{Q}_{il}(t) := \frac{Q_{il}(t)p_l(t)}{\sum_j Q_{il}(t)p_l(t)} \approx 1/m$ for all $l \in \Omega_m$, and for all $t \geq 0$. To keep the proportion $1/m$ unchanged over time, it is assumed that the bank adopts a "balanced" portfolio so that the quantities of external assets $Q_{i1}, ..., Q_{im}$ are functions depending on time and the asset prices $p_1, ..., p_m$. For the sake of simplicity, let $\sigma_l = \sigma$ for all $l \in \Omega_m$. Then, $p_i(t)$ follows the dynamics[15]

$$\frac{dp_i}{p_i} = \bar{\mu}dt + \bar{\sigma}dB(t) \ , \quad \forall i \in \Omega_n \tag{12}$$

3.1 Leverage

To analyze the equilibrium property of Equation (6), we carry out a mean-field analysis as in Tasca and Battiston (2011). The idea is to study the behavior of the system (11) around the state of "homogeneity" where the economic meaning still holds valid. We make the following assumptions. The book value of promised payments at maturity is assumed to be equal for every bank. Namely, $h_i = h$ for all $i \in \Omega_n$. Under the condition that the individual bank is' leverage does not deviate too much from the average one, we replace the leverage of the individual bank with the mean leverage of the system, $\phi_i \approx \frac{1}{n}\sum_j^n \phi_j := \phi$ for all $i \in \Omega_n$. Agents have similar market power, i.e., their portfolio of external assets have similar size $p_i := \sum_l Q_{il}p_l \approx \frac{1}{n}\sum_j p_j := p$. Agents have similar nominal exposure to each other, i.e., $\tilde{h}_i :=$

[14]Notice that the mechanisms of the leverage-cycle market highlight an *homogeneity* of banks business and risk management strategies.

[15]With $\bar{\mu} = \frac{1}{m}\sum_l \mu_l$, $\bar{\sigma} := \sqrt{\frac{\sigma^2}{m} + \frac{m-1}{m}\bar{\rho}\sigma^2}$ where $\bar{\rho} = \sum_l \sum_{k \neq l} \frac{\rho_{lk}}{m(m-1)}$ and $dB \sim N(0, dt)$.

$\sum_j W_{ij}\tilde{h}_j \approx \frac{1}{n}\sum_i \tilde{h}_i := \tilde{h}$. Then, Equation 6 simplifies as $\phi = h/(p + \frac{h}{1+r_f+\beta\phi})$.[16] Solving for ϕ, we get $\phi\left(p + r_f p + \beta\phi p + h\right) = h(1 + r_f + \beta\phi)$ which after some passages becomes $\phi^2\beta p + \phi\,(pR + h(1-\beta)) - hR = 0$, where $R = 1 + r_f$. Then, ϕ is the positive root of a quadratic expression[17]

$$\phi = \frac{1}{2\beta p}\left[h(\beta-1) - p + \left(4\beta hp + (h(1-\beta)+p)^2\right)^{1/2}\right] , \tag{13}$$

where, without loss of generality, the risk free rate is set equal to zero (i.e., $r_f = 0$). In a context of dynamic balance sheet management with banks targeting their leverage ϕ to a reference value ϕ^*, a precise relation between ϕ, p and h emerge. Starting from an accounting approach, the next sections model such interdependence.

3.2 Leverage, Diversification and Default Risk

Similar to Merton (1974), a bank enters into a default regime whenever the market value of its assets falls below the book value of its debt.

Definition 3.2 (**Default Event**). *The event that at any time $t \geq 0$, the leverage ϕ is equal or bigger than one is classified as "default event". Namely,*

$$\text{Default Event} \Leftrightarrow \phi(t) \geq 1 \Leftrightarrow h(t) \geq a(t), \quad \forall t \geq 0. \tag{14}$$

In particular, the expected default time ($\bar{\tau}$) defined in Appendix E, is a positive real function of ϕ. Therefore, according to the $1/m$ diversification strategy in (3.1), the bank optimization problem is the following one.

Definition 3.3 (**Bank's optimization problem**). *Find the optimal number* m $\in \Omega_m$ *of external assets to compose an* 1/m *diversification strategy in order to maximize the expected default time ($\bar{\tau}$)*

$$m^* = \arg\max_m(\bar{\tau}(m)).$$

The previous sub-Section 2.5 distinguishes between two types of financial markets. Namely, the standard plain-vanilla market and leverage-cycle market. Considered that banks try to maximize $\bar{\tau}$ w.r.t. to the number of external assets in portfolio, the key question is the following: "Is diversification always desirable in terms of default risk mitigation?" Moreover, "Does diversification produce the same risk-reduction effects in both markets?" In the remainder, we try to answer those question by maximizing $\bar{\tau}$ w.r.t. the control variables reported in Table 4 in Appendix Section A

$$\bar{\tau} = \bar{\tau}(m, n, \phi, \phi^*, \sigma, \varepsilon, \gamma, p^*, p_i(0)) .$$

[16]Since, $\tilde{h}$ in its interval of existence, is a monotonic decreasing function in t, without loss of generality, we consider the time interval in eq. (6) to be $t = 1$.

[17] The fragility ϕ takes values in the set $\Omega_\phi = (0, 1]$ only under a precise constraint. In particular ϕ is always greater to 0 since debts are supposed to be held in positive amount. While, ϕ is lower or equal to 1 under the condition that $h \in (0, p)$.

3.3 Comparative Statics of Expected time to Default

In equilibrium, banks' leverage ϕ equals the target level ϕ^* and external assets price p equal $p^* := h^* \times \left(\frac{1}{\phi^*} - \frac{1}{1+\beta\phi^*}\right)$.[18] Any exogenous shock that deviates the asset price from p^*, moves the economy out of the equilibrium. In the following, we denote by p_k the new price after the shock and we say that the economy is in a "good" ("bad") state if p_k is grater (lower) than p^*. Otherwise, the market is in a "normal" state. A numerical analysis (the procedure is described in Appendix Section F and the set of parameters is reported in Table 5 in Appendix Section A) is carried out both for the plain-vanilla and the leverage-cycle market. We test the effect of both positive and negative price shocks (as summarized in Table 6 in Appendix Section A).

The goal is to study how the expected time to default changes with the diversification level in external assets in presence of an initial price shock.

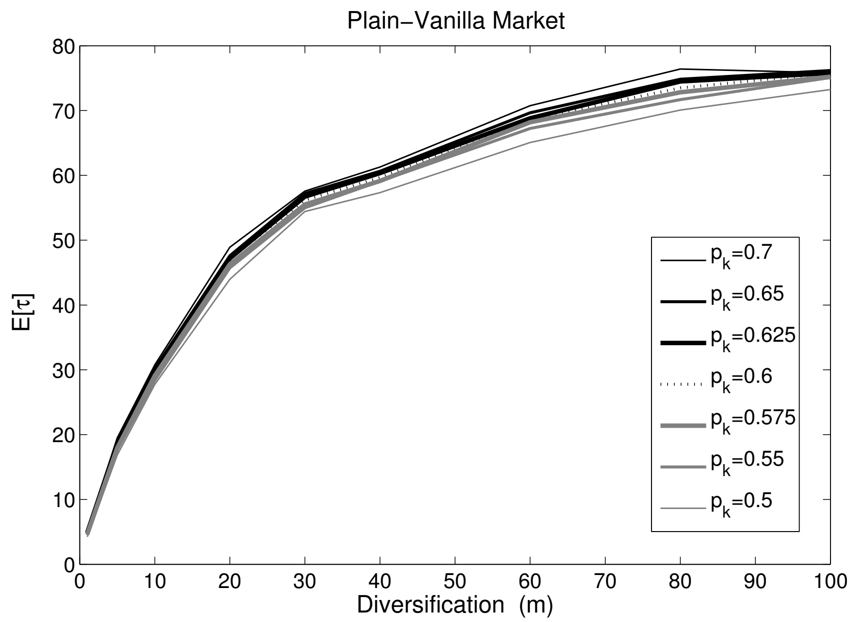

Figure 1: Variation of the expected time to default w.r.t the number of external assets in balance sheet.

Plain-Vanilla Market This market benefits from diversification. The expected default time $\bar{\tau}$ gradually increases with m both in good and bad states. We recall that a plain-vanilla market, by definition, is free of procyclical accounting rules and free of any kind of loop between leverage and asset prices. [19] Therefore,

[18] The equilibrium p^* is derived from (13).

[19] The external shocks on banks' assets are not amplified by some managerial practices.

this market resembles one where the expected return on external assets fluctuate around zero. As already shown in Tasca and Battiston (2011), for these kind of markets, diversification plays its traditional role of risk mitigation tool. It reduces the variance of the assets in the balance-sheet. In doing so, it reduces the amplitudes of the shocks on the leverage ϕ and eventually lengthens $\bar{\tau}$, (see Figure 1).

Leverage-Cycle Market This market exhibits two opposite reactions with respect to the $1/m$ heuristic. The expected default time $\bar{\tau}$ increases with m in the good states. In contrast, $\bar{\tau}$ decreases with m in the bad states. The *accounting rule* and the *positive feedback loop* leverage-assets price, are the two "gears" which allow to understand the reaction of the system to external shocks. In the first place, the accounting rule amplifies the exogenous shocks on the banks' assets. Then, the positive feedback loop transfers the shocks into a price trend. [20] Under this market structure, *diversification* is just the lubricant for the gears. More precisely, diversification polarizes the effect of positive/negative shocks, (See Figure 2). As in Tasca and Battiston (2011), the polarization is a clear symptom that the market exhibits a *knife-edge* dynamics. When a positive shock moves the prices up such that $p_k > p^*$, leverage is below the target level (i.e., $\phi < \phi^*$). In this case, diversification is *desirable*. It engenders financial robustness. Ultimately, $\bar{\tau}$ increases, (black lines in Figure 2). When a negative shock moves the prices down such that $p_k < p^*$, leverage is below the target level (i.e., $\phi > \phi^*$). In this case, diversification is *undesirable* because engenders financial fragility. Ultimately, $\bar{\tau}$ decreased, (grey lines in Figure 2). There is a simple intuition for this effect in terms of the widely used Sharpe ratio. When assets have positive expected cash-flows the Sharpe ratio is positive as well. Then, it is desirable to combine the assets in a well diversified portfolio since diversification lowers portfolio's volatility and in doing so, increases the Sharpe ratio as well. In contrast, when assets have negative expected cash-flows, the Sharpe ratio is negative. Then, a well diversified portfolio is no longer desirable.

Coherently with the results obtained so far, we now offer a formal definition and a quantitative dimension of the *knife-edge* dynamics.

Definition 3.4 (**Knife-edge dynamics**). *A system exhibits a knife-edge dynamics if under different economic states (i.e., good vs. bad) it displays a dichotomy (i.e., bipolarization) of the mean default time (or default probability).*

The *knife-edge* in numbers Let $\bar{\boldsymbol{\tau}}_{\boldsymbol{G}} := [\bar{\tau}_G(1), ..., \bar{\tau}_G(m)]$ and $\bar{\boldsymbol{\tau}}_{\boldsymbol{B}} := [\bar{\tau}_B(1), ..., \bar{\tau}_B(m)]$ be the m-dimensional vectors of expected default times in the good and bad states, respectively. Then, we call

[20] Indeed, the accounting rule (described in sub-Sections 2.2, 2.3), prescribes to increase (decrease) those assets in portfolio (with an elasticity coefficient ε) the value of which has been positively (negatively) shocked. The effects of these managerial practice are then transmitted to asset prices through an upward (downward) shift of the demand curve for those assets. In particular, γ is the transmission coefficient which depends on the number n of banks targeting the leverage according to the asset price variations (synchronization effect). As result, prices are shot up (down).

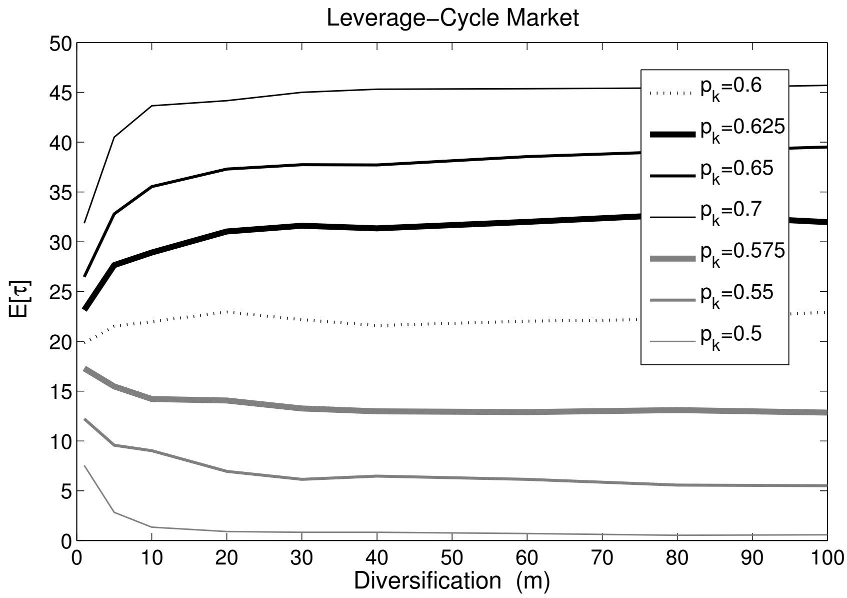

Figure 2: Variation of the expected time to default w.r.t the number of external assets in balance sheet.

knife-edge ($\boldsymbol{\kappa}$) the element-wise ratio between the two vector $\bar{\boldsymbol{\tau}}_{\boldsymbol{G}}$ and $\bar{\boldsymbol{\tau}}_{\boldsymbol{B}}$

$$\boldsymbol{\kappa} = \bar{\boldsymbol{\tau}}_{\boldsymbol{G}}./\bar{\boldsymbol{\tau}}_{\boldsymbol{B}}, \tag{15}$$

with $\boldsymbol{\kappa} \in \mathbb{R}^m$. For a sufficiently large set Ω_m,[21] we say that the market exhibits a *knife-edge* dynamics if the following condition is verified

$$\frac{\kappa(m) - \kappa(1)}{m - 1} > 0 + \xi \ , \tag{16}$$

where ξ is some constant in $\mathbb{R}^+$. If the condition is verified, we say that the market has a *knife-edge* property which is stressed with increasing levels of risk-diversification. In this respect, the plain-vanilla market does not exhibit the *knife-edge* property (black lines in Figure 3). While, the leverage-cycle market obeys to the condition posed in (16) and the *knife-edge* is amplified with the number m of external risky assets (gray lines in Figure 3).

[21] The set Ω_m of natural numbers is called relatively large if the number of elements of Ω_m is greater than the least element of Ω_m.

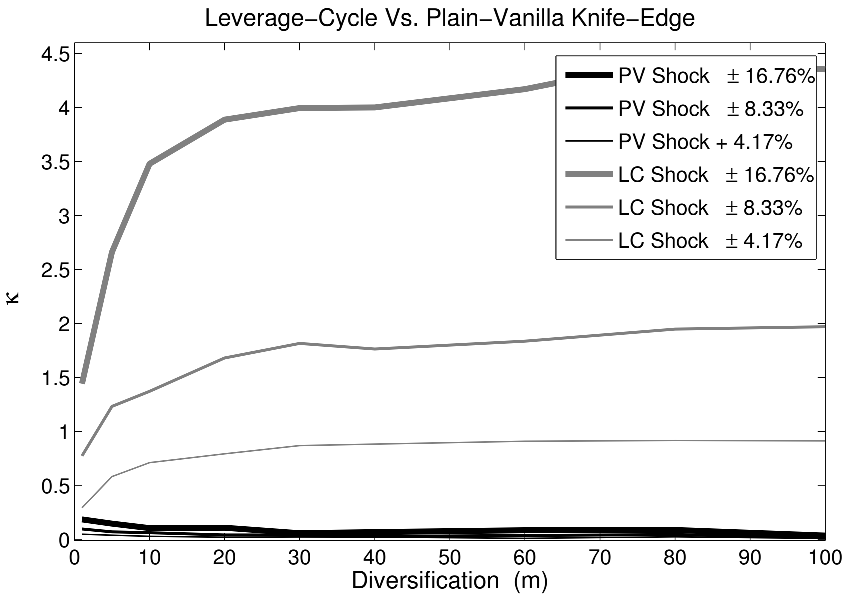

Figure 3: Log-scale of κ. Detection of the *knife-edge* property. Since κ is almost invariant in m, the plain-vanilla market, does not present the *knife-edge* property (black lines). While, the leveraged-cycle market exhibits a strong *knife-edge* property (gray lines). The value of κ is increasing in m.

4 Concluding Remarks

The paper sheds light on the *knife-edge* effects of the procyclical capital adequacy regulation. In an abstract model, we combine a balance sheet approach with a stochastic setting in order to investigate the emergence of systemic risk in the presence of a positive feedback mechanism between banking leverage and asset prices.

Our model is in line with the work of Tasca and Battiston (2011) which analyzes the double-edge-sword nature of banking diversification in external assets when their value moves up or down with a certain probability. We build on the same balance-sheet approach by introducing two novel key ingredients. The first is the accounting rule, which translates shocks to asset prices into changes in demand. The strictness with which the rule is applied is controlled by the parameter ε, measuring the reactivity of the banks. The second element, is the impact of changes in demand onto changes in asset prices. This transmission mechanisms is controlled by the parameter γ which measures the responsiveness of prices to changes in demand. Together, these two elements generate a *positive feedback loop* that can amplify an initial change in price into a spiral of devaluation or over valuation. When both the reactivity of the banks and the responsiveness of the prices are high (i.e., $\varepsilon \to 1$ and $\gamma > 1$), we are in a leverage-cycle market. In this setting, we show the emergence of a tension between (1) the individual incentive to target a ratio of VaR to economic capital and reduce the idiosyncratic risks through asset diversification and, (2) a resulting homogenized banking system where banks adopt the same business and risk management tools.[22]

From an individual firm perspective the application of the accounting rule and asset diversification looks sensible strategies. Crucially, every bank is acting perfectly rationally from its own individual point of view. Rather, from a systemic point of view, these strategies generate an undesired result.[23] When banks start moving in a synchronized way, they can potentially amplify the effects of exogenous asset price shocks. Diversification reduces the fluctuations and thus enhances the gap between good and bad states.

The problem of procyclical effects of capital adequacy regulation is a classical conflict between the micro- and macro-perspective on financial stability. It has extensively been discussed in the case of capital adequacy regulation. From the early days of the Basel accord concerns about these procyclical effects of capital adequacy regulation have been raised and the issue has been widely discussed. See Blum and Hellwig (1995). In this paper, we contribute to this literature by showing that a procyclical market displays a *knife-edge* property in terms of polarization of default risk at increasing levels of asset diversification. This analysis is based on the assumption that banks can not easily issue new equity.[24] Diversification reveals its double face of good or bad strategy which can, at the same time, contains or propagates the default risk. The tipping point that separates benefits from losses, is the market clearing price p^* which is the price ensuring the equivalence between ϕ and ϕ^*. If an exogenous shock positively (negatively) impacts the asset price with respect to the equilibrium level p^*, the default risk (measured by the expected time to failure) is mitigated (exacerbated) for increasing number m of external assets in portfolio. From our analysis we draw the conclusion that the major problem of an individual institution perspective is a potentially misleading risk assessment for the financial system as a whole. Even though it is uneasy to put in practice a system perspective, regulators need to adopt more system-oriented supervisory risk assessment policies. Moreover they also need to face the challenge to design new approaches in order to strengthen countercyclical regulatory instruments that work as cushions in booms and as buffers when times deteriorate.

[22]These features are in line with those outlined by several authors, e.g., Haldane (2009) to describe the lack of diversity emerged in the financial system during the recent years.

[23]In our model, γ is high when the number of banks, applying the same accounting rule - or in other terms, “mimicking” the same risk management strategy, increases.

[24]The effect of capital scarcity leading to a “credit crunch” has been observed and investigated in the literature by Bernanke et al. (1991), Calomiris and Wilson (1998) among others.

A Tables

Bank balance-sheet

Time	Total Assets	Liabilities	Leverage
t=0	$\tilde{h} = 20\$$ $Q(0)p(0) = 80 \times 1\$ = 80\$$	$h(0) = 60\$$ $e(0) = 40\$$	$\phi = 0.6$
t=1	$\tilde{h} = 20\$$ $Q(0)p(1) = 80 \times 0.75\$ = 60\$$	$h(1) = 60\$$ $e(1) = 20\$$	$\phi = 0.75$
t=2	$\tilde{h} = 20\$$ $Q(2)p(1) = 40 \times 0.75\$ = 30\$$	$h(2) = 30\$$ $e(1) = 20\$$	$\phi = 0.6$

Table 1: Contraction of the balance sheet due to a negative asset price shock.

Bank balance-sheet

Time	Total Assets	Liabilities	Leverage
t=0	$\tilde{h} = 20\$$ $Q(0)p(0) = 80 \times 1\$ = 80\$$	$h(0) = 60\$$ $e(0) = 40\$$	$\phi = 0.6$
t=1	$\tilde{h} = 20\$$ $Q(0)p(1) = 80 \times 1.25\$ = 100\$$	$h(1) = 60\$$ $e(1) = 60\$$	$\phi = 0.5$
t=2	$\tilde{h} = 20\$$ $Q(2)p(1) = 104 \times 1.25\$ = 130\$$	$h(2) = 90\$$ $e(1) = 60\$$	$\phi = 0.6$

Table 2: Expansion of the balance sheet due to a positive asset price shock.

Impact of exogenous price shocks on the bank demand curve			
Positive shock: $p_l > p_l^*$ $\implies$	$\begin{cases} \phi_i < \phi_i^* \\ e_i > e_i^* \end{cases}$	$\implies$	Upward shift of demand curve
Negative shock: $p_l < p_l^*$ $\implies$	$\begin{cases} \phi_i > \phi_i^* \\ e_i < e_i^* \end{cases}$	$\implies$	Downward shift of demand curve

Table 3: Variation of the demand curve of risky external assets as the income and leverage deviate from the target levels.

Parameter	Description	Set of Values
m	Number of external assets hold in portfolio	$\{m \in \mathbb{N} : m \geq 1\}$
n	Number of banks investing in the same portfolio of external assets	$\{n \in \mathbb{N} : m \geq 1\}$
ϕ	Leverage measured by the debt/asset ratio	$\{\phi \in \mathbb{R} : 0 \leq \phi \leq 1\}$
ϕ^*	Target leverage which is assume to be fixed over time	$\{\phi^* \in \mathbb{R} : 0 < \phi^* < 1\}$
σ	Common variance of external assets	$\{\sigma \in \mathbb{R} : 0 < \sigma < 1\}$
ε	Elasticity to which banks adjust their leverage to the target one	$\{\varepsilon \in \mathbb{R} : 0 \leq \varepsilon \leq 1\}$
γ	Asset price response to a shift in the aggregate demand curve	$\{\gamma \in \mathbb{R} : \gamma > 0\}$
$\bar{\rho}$	Mean-pairwise correlation between external assets	$\{\bar{\rho} \in \mathbb{R} : 0 \leq \bar{\rho} \leq 1\}$
p^*	Equilibrium market price which equals the leverage to the target level	$p^* = h^* \left(\frac{1}{\phi^*} - \frac{1}{1+\beta\phi^*}\right)$
α	Fraction of external risky assets over the total amount of assets	$\{\alpha \in \mathbb{R} : 0 \leq \alpha \leq 1\}$
p_k	Asset price level after an exogenous shock	$\{p_k \in \mathbb{R} : p_k > 0\}$
r_f	Risk-free rate	0
a_d	Intercept of the linear individual demand function	$\{a_d \in \mathbb{R} : a_d > 0\}$
b_d, b_s	Slope of the demand/supply function	$\{b_d, b_s \in \mathbb{R} : b_d, b_s > 0\}$
a_s	Intercept of the supply function	$< a_d$

Table 4: Range of values for each expl. variable. Notice that p^* is derived from the more general form of ϕ given in (13).

Parameters Values and Market Characteristics													
m	ϕ^*	σ	$\bar{\rho}$	p^*	p_k	β	α	N	dt	T	ε	γ	
1-100	0.5	0.5	0.3	0.6	0.6	0.1	≈ 1	2000	0.01	50000	0.1	1	Plain-Vanilla Market
											1	10	Leverage-Cycle Market

Table 5: Range of values used in the simulations.

Scenarios	Description	Level of price shock
Good State (1)	Strong positive price shock	$p_k = 0.7 > p^*$ (+16.67%)
Good State (2)	Normal positive price shock	$p_k = 0.65 > p^*$ (+8.33%)
Good State (3)	Weak positive price shock	$p_k = 0.625 > p^*$ (+4.17%)
Normal State	No-price shock	$p_k = 0.6 \equiv p^*$ (±0%)
Bad State (1)	Strong negative price shock	$p_k = 0.575 < p^*$ (−16.67%)
Bad State (2)	Normal negative price shock	$p_k = 0.55 < p^*$ (−8.33%)
Bad State (3)	Weak negative price shock	$p_k = 0.5 < p^*$ (−4.17%)

Table 6: Scenarios of possible price shocks. (Equilibrium price $p^* = 0.6$).

Delta Hedging of a long stock position

					Replication Strategy	
					Short sales of stock	Investment in the risk-free bond
$T-t$	S	P	$\Delta_{(put)}$	Change in $\Delta_{(put)}$	$N \times (\Delta'_{(put)}S' - \Delta_{(put)}S)$	$N \times (P' - P + \Delta_{(put)}S - \Delta'_{(put)}S')$
	(1)	(2)	(3)	(4)	(5)	(6)
12	50	7.2054	-0.326355	–	-16317	23523
11	51	6.69078	-0.318389	+0.00796603	+80	-594
10	52	6.15795	-0.309693	+0.00869578	+213	-667
9	53	5.60473	-0.300000	+0.00965049	+122	-755
8	54	5.02884	-0.289126	+0.0109163	+289	-865
7	55	4.42764	-0.276500	+0.0126274	+405	-1007
6	54	4.35517	-0.296022	-0.0195229	-778	+705
5	53	4.25611	-0.318816	-0.0227937	-912	+813
4	52	4.11900	-0.346173	-0.0273579	-1104	+967
3	51	3.92395	-0.380484	-0.0343103	-1404	+1208
2	50	3.63121	-0.427120	-0.046636	-1951	+1658
1	49	3.13802	-0.504015	-0.0768953	-3341	+2848

Table 7: Portfolio Insurance: Synthetic Protective Put.

B Two examples of the procyclicality

Target Leverage of dynamic balance sheet management Let's suppose at time t=0, the bank targets the leverage ϕ to $\phi^* = 0.6$. The balance sheet is outlined in Table 2 in Appendix Section A. Then, due to an exogenous shock, let p to raise by 25% such that $p(1) = 1.25\$$. Then, the total assets increase to 100\$ ($80 \times 1.25\$ = 100\$$). On the liability side, the equities adjust to 60\$, while debts stay constant at 60\$. So, the new leverage ϕ becomes 0.5 (i.e., 60\$/120\$) which is lower than the target one. In order to re-adjust up its leverage to the target level, the bank desires to increase its debts and invest the funding in external assets.[25] The bank takes 30\$ of additional debts to finance 24 units of external assets at $p(1) = 1.25\$$ such that at time t=2, the external assets amount to 130\$. Namely, $(80+24)\times 1.25\$ = 130\$$. The new balance sheet is the followings. Assets comprises 20\$ of claims against other banks and 130\$ of external assets. While, the liability side is given by 90\$ of debts (i.e., 60\$ + 30\$) and 60\$ of equities which value remains unchanged. After the purchase of new assets, leverage ϕ goes back to the target level of 0.6 (i.e., 90\$/150\$ = 06). (See Table 2 in Appendix Section A).

Prociclycality of dynamic portfolio strategies A part from the situation of target leverage sketched in Section 2, the analysis of derivatives allows us to extend the range of those cases when the procyclical balance sheet management is at work. For example, we can think at a bank which holds a portfolio composed by a relatively

[25] As alternative solution, a bank could decrease its equity value for example by buying back its on equities from the market.

large number of shares invested in equities x at price S, issued by a firm X with a performance over the market average. Once X is unexpectedly hit by an important shock, and this information is disclosed, it is more likely that the market reacts positioning itself on the sell side of the book. The final effect will be a downturn of x. Since X performed well till the negative event, it is likely that (i) the market does not react immediately to the news and (ii) x will rise after X will recover from the negative shock. In this context, the bank is likely to keep holding its long position on x, but in the same time it will try to lookout for hedge strategies which effectively insure the portfolio value i.e. keep it from dropping below a certain acceptable level. One type of hedge that can be used to protect stock holdings is the "insurance hedge". An insured hedge only attempts to protect an investor against the effects of a ruinous decline in the value of the portfolio being hedged, while enabling to participate in upside appreciation. In effect, the insured hedge attempts to prevent the value of a portfolio from falling below some floor value, without materially inhibiting its upside performance. The classic portfolio insurance strategy is the so called "protective put". This strategy simply requires the hedger to purchase a sufficient number of put options to fully cover the number of shares held long, with the strike price of the option set equal to the floor value below which the hedger does not want the value of the price per share to fall. In practice, if the floor is set equal to the current stock price, the strategy is implemented as follows. The hedger buy an at-the-money put options on stock x at price P with strike price $K = S$. The option gives the right to sell at[26] a certain future time, the external assets at the price of K (i.e., the floor). In this way, the bank locks in the gains and have the "right to sell the stock" for K. And like any type of insurance, the bank have to pay a premium, (the cost P of the option).

The value of a put option for a non-paying dividend underlying stock derives from Black and Scholes (1973); Merton (1973)

$$P = \Phi(-d_2)Ke^{-rs} - \Phi(-d_1)S \tag{17}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}dy = 1 - \frac{1}{\sqrt{2\pi}}\int_{-x}^{\infty} e^{-\frac{y^2}{2}}dy$ is the cumulative probability distribution function for a standardized normal distribution; $d_1 = \frac{ln(S/K)+(r_f+0.5\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}$; $d_2 = \frac{ln(S/K)+(r_f-0.5\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}} = d_1 - \sigma\sqrt{\tau}$; S is current stock price; K is the strike price; σ is the stock volatility; $T-t$ is the time to maturity and r_f is the risk-free interest rate.

Some portfolio insurance strategies have been created that do not require the purchase of put options. Instead, a "synthetic put", whose payoff pattern behaves like a put with a given strike price, is created. Combining this synthetic put with a long position in the stock being hedged will produce a synthetic protective put strategy. It is possible to synthesize a protective put strategy without using any options at all via "dynamic hedging". For a single stock portfolio, this involves taking position in the underlying stock such that the delta of the position is maintained equal to the delta of the required option. In order to synthetically engineer the payoff of an European put option on a single underlying risky asset, the bank must ensure at any given point of time t, a proportion $\Delta_{(put)}$ of the underlying stock in the portfolio has been sold and the proceeds invested in cash (earning the risk-free rate).[27] Precisely, if the relation option/stock is 1:1 (e.g. one put option per one share), the canonical payoff replication strategy consists in short-selling $N \times \Delta_{(put)}$ shares of stock and buy the riskless bond B. The total value of such a

[26] "At" is replaced by "within" for American options.

[27] If there is more than one underlying asset, the option-based portfolio insurance can be extended using the same basic principle. The payoff from the multi-asset option may be synthetically replicated by a portfolio consisting of positions in all the underlying stocks together with a riskless bond. The technicality associated with this approach imply to uniquely determine the proportions to be invested into each of the underlying stocks and cash. Thus, one must know the deltas of the multi-asset option i.e., the partial derivatives of the option price with respect to each of the underlying assets.

trade, corresponds to the value of N puts

$$N \times P = B - N \times \Delta_{(put)} S \tag{18}$$

where

$$\Delta_{(put)} = -\Phi(-d_1) = \Phi(d_1) - 1 < 0 \ . \tag{19}$$

As a result, this trade is procyclical. For any $dS > 0$, the bank should buy $N\Delta_{(put)}$ of stocks and sell risk-free asset.[28] The dynamic hedging is a "momentum" strategy. This trade just increases or decreases the risky asset price and the related volatility. Hence, if more then one bank adopts the same strategy, the final result consist on feedback loop on S with an increment of the credit spread in a downturn market.[29] See Table 7 in Appendix Section A. Dynamic hedging is based on the principle that when a stock gets close to its floor price, the investor moves out of the stock and into cash. As the price moves away from the floor, the investor moves into the stock and out of cash. [30] By doing this, the hedger reduces the probability that the portfolio will fall below the floor (because, at the floor, the portfolio will be all cash); yet as the value of the portfolio rises above the floor, she becomes more fully invested in stocks so that she can fully participate in the upside potential.[31]

In the following, we give a numerical example of a dynamic hedging strategy for a single-stock portfolio. Let's assume that the bank holds $N = 1000$ shares of not-paying dividend stocks x at current price $S = 50\$$ and want to protect itself against market downturns in the next twelve months by locking the price to 50$. To immunize the portfolio, the bank decides to set up a synthetic put strategy. Let the time to maturity of the "imaginary put" be $T - t = 1$ (i.e., 12 months to expiration), $r_f = 0.1$, $\sigma = 0.5$, the strike price $K = 50\$$. Then applying (17)and (19) we get $\Delta_{(put)} = -0.326355$ and $P = 7,20524\$$, respectively. From (18), the bank short-sell $N \times \Delta_{(put)}$ shares of x at 50$ and invests $B = N \times P + N \times \Delta_{(put)} \times S = 1000 \times 7,20524\& = B - 1000 \times (-0.326355) \times 50\$$ in the riskless bond s.t. $B = 23.523\$$. Let's assume the bank rebalance its delta-hedging every month.[32] At $T - t = 0.916667$ (i.e., 11 months to maturity), let the new stock price S' be 51$. From (17) and (19), the new put price is $P' = 6.69078$, and the new delta is $\Delta'_{(put)} = -0.318389$. To adjust the hedge, the bank as to buy $N \times (\Delta'_{(put)} S' - \Delta_{(put)} S) = 80\$$

[28]The opposite trade is done in a falling market.

[29]This exactly simplifies what happened in the financial crash of October 19, 1987 (when the market fell more than 20% in one day) as explained by the Brady Report (1988). There were no significant news events of sufficient importance to explain the magnitude of the price fall. The Brady Report therefore focused on internal market causes rather than external events. In particular, the Brady Report centered attention on a number of large institutions following "price insensitive strategies" such as portfolio insurance. It describes enormous waves of portfolio insurance selling driving down prices. This led to a vicious circle, as the selling engendered further price declines which in turn led to further portfolio insurance selling, etc. A small initial price decline snowballed into a substantial meltdown.

[30]As the value of the original portfolio declines, the delta of the put becomes more and more negative meaning that a higher and higher proportionate allocation must be made to the risk-free asset. Conversely, as the value of the original portfolio increases, the delta of the put option becomes less and less negative implying that funds must be withdrawn from the risk-free asset and used to buy more of the underlying stock.

[31]In our case, the floor is fixed at K (i.e., the strike price of the "imaginary put"). As the value of the portfolio rises above K, the bank increases exposure to x. As the value of the portfolio falls toward K, it reduces exposure to x.

[32]It is likely to be given that in most practical settings the presence of transaction costs rule out frequent rebalancing of the replicating portfolio.

shares of x. On the same time the bank must sell

$$\begin{aligned}
&N \times \left[(P' - P) + (\Delta_{(put)} \times S) - (\Delta'_{(put)} \times S')\right] \\
&= 1000 \times [(6.69078 - 7.20524) + (-0.326355 \times 50\$) - (-0.318389 \times 51\$)] \\
&= -594\$
\end{aligned}$$

of riskless bonds. Now, let assume that at time $T - t = 05$ (i.e., 6 months to maturity), S declines from 55$ to 54$. From (17) and (19), the new put price is $P' = 4.35517$, and the new delta is $\Delta'_{(put)} = -0.296020$. To adjust the hedge, the bank as to sell $N \times (\Delta'_{(put)} S' - \Delta_{(put)} S) = 778\$$ shares of x. On the same time the bank must buy

$$\begin{aligned}
&N \times \left[(P' - P) + (\Delta_{(put)} \times S) - (\Delta'_{(put)} \times S')\right] \\
&= 1000 \times [(4.35517 - 4.42764) + (-0.296022 \times 54\$) - (-0.2765 \times 55\$)] \\
&= 705\$.
\end{aligned}$$

of riskless bonds. This is a procyclical trade. For example at time $T - t = 11$, to hedge the position, when S rises from 50$ to 51$, the bank has to buy assets for 80$ (see column 5 in Table 7 in Appendix Section A) and sell bonds for 594$ (see column 6 in Table 7 in Appendix Section A). While, when at time $T - t = 6$, S drops from 55$ to 54$, to hedge the position the bank has to sell assets for 778$ (see column 5 in Table 7 in Appendix Section A) and buy bonds for 705$ (see column 6 in Table 7 in Appendix Section A).

C An Accounting Rule Based on Target Leverage

In this section we derive the accounting rules (7)-(8). As in the example of sub-Section 2.2, the underlying assumption is that banks adjust their balance sheets only acting on the quantity of external assets, while the amount of claims against other banks is unaffected by the accounting rule. In formulas: $\sum_j W_{ij}(0)\tilde{h}_j(0) = \sum_j W_{ij}\tilde{h}_j$ for all $t \geq 0$ and all $i, j \in \Omega_n$. The bank i starts at time t=0 its activity with a target leverage, $\phi_i(0) = h_i(0)/\left[\sum_l Q_{il}(0)p_l(0) + \sum_j W_{ij}\tilde{h}_j\right] \equiv \phi_i^*$. At time t=1, one external asset is shocked. The new price $p_l(1)$ entails a leverage deviation from the target level. Namely, $\phi_i(1) = h_i(0)/\left[\sum_l Q_{il}(0)p_l(1) + \sum_j W_{ij}\tilde{h}\right] \neq \phi_i^*$. While the quantities h_i, $\tilde{h}_j$, Q_{il} remain unchanged as they are at time t=0. After the price shock, the bank adjusts its balance sheet and at time t=2 the leverage is back to the target level. The bank achieves the goal by changing the quantity of external investments in the asset side, i.e., $\sum_l Q_{il}(0) \rightarrow \sum_l Q_{il}(2)$, and debts in the liability side, i.e., $h_i(0) \rightarrow h_i(2)$ so that the new leverage at time t=2 is equal to the target level. Namely, $\phi_i(2) = h_i(2)/\left[\sum_l Q_{il}(2)p_l(1) + \sum_j W_{ij}\tilde{h}_j\right] \equiv h_i(0)/\left[\sum_l Q_{il}(0)p_l(0) + \sum_j W_{ij}\tilde{h}_j\right] \equiv \phi_i^*$. Rewriting the equation of ϕ at time t=1 we have $\sum_j W_{ij}\tilde{h}_j + \sum_l Q_{il}(0)p_l(1) = h_i(0)/\phi_i(1) \Rightarrow \sum_j W_{ij}\tilde{h}_j = h_i(0)/\phi_i(1) - \sum_l Q_{il}(0)p_l(1)$ that we substitute into the equation of ϕ at time t=2 as follows $\phi_i(2) = h_i(2)/\left[\sum_l Q_{il}(2)p_l(1) + h_i(0)/\phi_i(1) - \sum_l Q_{il}(0)p_l(1)\right] = h_i(2)/\left[h_i(0)/\phi_i(1) + \Delta Q_{il}(2)p_l(1)\right] \equiv \phi_i^*$. In the last expression ΔQ_{il} indicates the change in quantity of asset l on the balance sheet of bank i. We can then write

$$\frac{h_i(0)}{\phi_i(0)} + \Delta Q_{il}(2)p_l(1) = \frac{h_i(2)}{\phi_i^*}. \tag{20}$$

Since positive (negative) variations of external asset amounts are used to decrease (increase) the value of debts on the liability side, there exists the following relation between change in debt value and change in asset value

$$h_i(2) = h_i(0) + \Delta Q_{il}(2) p_l(1)\ . \tag{21}$$

Then, replacing (21) into (20) yields $\frac{h_i(0)}{\phi_i(0)} + \Delta Q_i(2) p_l(1) = \frac{h_i(0)}{\phi_i^*} + \frac{\Delta Q_{il}(2) p_l(1)}{\phi_i^*}$, from which, with little arrangements we obtain

$$\Delta Q_{il}(2) p_l(1) = \frac{h_i(0)}{\phi_i(0)} \left(\frac{\phi_i^* - \phi_i(0)}{1 - \phi_i^*} \right). \tag{22}$$

In the case the bank is modifying the quantity of more than one asset at the same time, the equation simply becomes

$$\sum_{l \in S} \Delta Q_{il}(2) p_l(1) = \frac{h_i(0)}{\phi_i(0)} \left(\frac{\phi_i^* - \phi_i(0)}{1 - \phi_i^*} \right), \tag{23}$$

where S is the set of assets the quantity of which are modified. The rate of change of the demand is

$$\frac{\sum_{l \in S} \Delta Q_{il}(2) p_l(1)}{\sum_{l \in S} Q_{il}(0) p_l(1)} = \frac{a_i(0)}{\sum_{l \in S} Q_{il}(0) p_l(1)} \left(\frac{\phi_i^* - \phi_i(0)}{1 - \phi_i^*} \right), \tag{24}$$

and depends on ratio between the assets affected by the accounting rule and the total assets of the bank (all the external assets plus the claims on the other banks). The equation above specifies only a condition for the changes in quantity across the assets in the set S. There maybe several allocations of the various $\Delta Q_{il}(2)$ with $l \in S$ that satisfy the condition. Thus, the allocation chosen by the bank needs to be further specified. In our analysis of the model, we have focused on the case in which assets can be treated as a single one for what concerns the accounting rule. This is justified in the hypothesis that they belong to the same homogeneous asset class.

In the case a single asset is affected, the rate of change of the demand becomes

$$\frac{\Delta Q_{il}(2)}{Q_{il}(0)} = \frac{1}{\alpha_{il}} \left(\frac{\phi_i^* - \phi_i(0)}{1 - \phi_i^*} \right) \tag{25}$$

where $\alpha_{il} = Q_{il}(0) p_l(1) / a_i(0)$ is a fraction measuring relative value of the external asset l w.r.t. the total assets held by the bank. Because the proceedings/cost from the variation in quantity of external assets are used to expand/decrease the liabilities, a variation $\Delta Q_{il}(2)$ is followed by a variation $\Delta h_i(2)$. The relative change in debts is easily obtained from Equation 21,

$$\frac{\Delta h_i(2)}{h_i(0)} = \frac{1}{\phi_i(0)} \left(\frac{\phi_i^* - \phi_i(0)}{1 - \phi_i^*} \right) \tag{26}$$

In presence of market frictions,[33] we assume that the bank reacts with a certain (in)elasticity $\varepsilon_i \in (0, 1]$ to deviations in leverage ϕ_i from the target level. Heuristically, in continuous time the accounting rule implies the following (rates of) change, respectively, for the demand for external assets and the amount of nominal debts

$$\frac{dQ_{il}}{Q_{il}} = \left(\frac{\varepsilon_i}{\alpha_{il}} \right) \left(\frac{\phi_i^* - \phi_i}{1 - \phi_i^*} \right) dt \tag{27}$$

$$\frac{dh_i}{h_i} = \left(\frac{\varepsilon_i}{\phi_i} \right) \left(\frac{\phi_i^* - \phi_i}{1 - \phi_i^*} \right) dt\ . \tag{28}$$

where $\alpha_i = \sum_l \alpha_{il}$. The boundary conditions are: (i) $\frac{dQ_i}{Q_i} = \sum_l \left(\frac{dQ_{il}}{Q_{il}} \right)$ s.t. $\frac{dQ_{il}}{Q_{il}} = 0$ for some $l \in \Omega_m$; (ii) $dQ_{il} \geq -Q_{il}$ for all $l \in \Omega_m$ and $i \in \Omega_n$, and (iii) $dh_i \geq -h_i$ for all $i \in \Omega_n$.

[33]I.e., market and (or) firm structures that prevent banks to adjust instantly their balance sheets with the accounting rule.

D The Cycle between Leverage and External Assets Prices

To avoid redundancy of notation, in this section we suppress the subindex referring to the *i*th bank and the index referring to the *l*th external asset.

For simplicity, let the external asset market equilibrium price p^* and the quantity Q^*, be expressed as the result of the canonical model of linear demand-supply curves

$$\begin{cases} p = (b_d Q + a_d) & \text{(individual demand curve)} \\ p = b_s Q + a_s & \text{(supply curve)} \end{cases}$$

where $b_d < 0$; $b_s > 0$; $a_d > 0$ and $a_s < 0$.

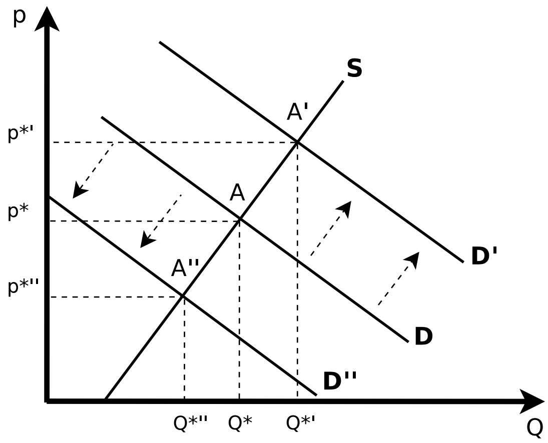

Figure 4: Variation of the demand curve of risky external assets as the income and leverage deviate from the target levels.

Individual demands are assumed to be equal for all market participants and for convenience of tractability, let $|a_d| = |a_s|$, which impose $|b_s| > |b_d|$ since prices cannot be negative. We know that in the market there are n banks investing in external assets with aggregate price p. The aggregated market demand is therefore the sum of individual demands $p/n = (b_d Q + a_d)$. Given these conditions, and solving for Q and p, the equilibrium quantity and the corresponding market clearing price are $Q^* = (a_s - na_d)/(-b_s + nb_d)$ and $p^* = n(a_d b_s - a_s b_d)/(b_s - nb_d)$, respectively. Now, let assume a variation of the quantity demanded for external assets occurs, i.e., a variation of the "y"-intercept of the demand curve from a_d to a'_d. *Ceteris paribus*, if $a'_d > a_d$ ($a'_d < a_d$), the demand curve is upward (downward) shifted to $p/n = (b_d Q + a'_d)$, see Figure 4.[34] The new equilibrium quantity becomes $Q^{*'} = (a_s - na'_d)/(-b_s + nb_d)$ and the market clearing price is $p^{*'} = n(a'_d b_s - a_s b_d)/(b_s - nb_d)$. The rates of change

[34]While a_s, b_s and b_d remain unchanged.

of the equilibrium quantity and the market clearing price, between *pre* and *post* demand shift, are

$$\delta Q^* := \frac{Q^{*'} - Q^*}{Q^*} = (a_s b_d(-1+n) + a'_d(-b_s + b_d)n + a_d(b_s - nb_d))/((a_s - a_d)(b_s - nb_d))$$
$$\delta p^* := \frac{p^{*'} - p^*}{p^*} = -((b_s(a_s b_d(-1+n) + a'_d(-b_s + b_d)n + a_d(b_s - nb_d)))/((a_d b_s - a_s b_d)(b_s - nb_d))) .$$

The rate of change of the aggregated quantity can also be written in terms of the rate of change of the individual demand

$$\delta Q^* = \eta \delta_q$$

where $\delta_q := (a_d - a'_d)/(a_s - a_d)$ is the change in the demand induced by a single individual agent, while $\eta := (a_s b_d(-1+n) + a'_d(-b_s + b_d)n + a_d(b_s - nb_d))/((a_d - a'_d) * (b_s - nb_d))$ is a parameter linking the aggregate rate of change of the demand to the market size n. Namely, $\partial\eta/\partial n > 0$. Equivalently, the rate of change of the equilibrium price can be written as

$$\delta p^* = \eta \delta_p$$

where $\delta_p := -((b_s(a_d(b_s - b_d) + a'_d(-b_s + b_d)))/((b_s - b_d)(a_d b_d - a_s b_s)))$ represents the change in the equilibrium price in response to an individual demand variation. Since $\delta_p = \zeta \delta_q$ where $\zeta = ((a_s - a_d)b_s)/((-a_d)b_s + a_s b_d)$ is a parameter always bigger than one,

$$\delta p^* = \eta\zeta\delta_q = \gamma\delta_q\ , \qquad \gamma := \eta\zeta.$$

Now, if we assume that the shift of the demand curve is due to an income effect (See Table 3 in Appendix Section A) which imposes a bank to apply the accounting rule (8), then in equilibrium $\delta_q := \left(\frac{\varepsilon}{\alpha}\right)\left(\frac{\phi^* - \phi}{1-\phi^*}\right)$ and

$$\delta p^* = \gamma\left(\frac{\varepsilon}{\alpha}\right)\left(\frac{\phi^* - \phi}{1 - \phi^*}\right).$$

Let the deterministic, predictable part of dp/p in (12) be expressed by the rate of change of the market clearing price δp^*. Then, $\frac{dp}{p} = \gamma\delta_q dt + \sigma dB$ which is equivalent to

$$\frac{dp}{p} = \gamma\left(\frac{\varepsilon}{\alpha}\right)\left(\frac{\phi^* - \phi}{1 - \phi^*}\right)dt + \sigma dB\ .$$

The expression encompasses the leverage-asset price cycle. The drift can be written also as $\gamma\varepsilon\left(\frac{\Delta_\phi}{1-\phi^*}\right)$ where $\Delta_\phi := \phi^* - \phi$. Its partial derivative w.r.t. Δ_ϕ is positive, which is consistent with the example in the Section 2.2. When $\Delta_\phi > 0$, the current leverage is lower than the target one, since the price of risky assets have been positively shocked up. If banks follow the accounting rule (8), the demand curve upward shifts and the price level increases. When $\Delta_\phi < 0$, the current leverage is higher than the target level. This happens because the price of risky assets have been negatively shocked down. Again, if banks follow the accounting rule (8), the demand curve shifts downward and the prices level decreases. The feedback loop leverage-asset prices is strengthen with the number of banks.[35] In other terms, the reaction δp^* to δ_q is amplified if the number n of banks investing in the same bundle of external assets increases.[36]

[35]The partial derivative $(\partial\eta)/(\partial n)$ is always positive for any value of b_s, b_d, a_d and a_s reported in Table 4 in Appendix Section A.

[36]The effect is even stronger, if the demand-supply curves are described by non-linear functions.

E Expected time to Default

Leverage in (13) takes values between zero and one. When leverage is closed to the upper *default barrier* at one, the bank is fragile and the default probability is closed to one. On the contrary, when the leverage approach the lower barrier at zero, the bank is robust and the probability of a default closed to zero. Therefore, the default risk of a bank depends on the probability that, her initial leverage touches its upper default bound fixed at one. More formally the default probability $\mathbb{P}(1, t \mid \phi_0, 0)$ is defined as the probability that the process $\{\phi\}_{t\geq 0}$, initially starting at an arbitrary level $\phi_0 \in (0, 1)$ exits through the default barrier 1 at any time $t \geq 0$. In a stochastic setting, it is common to relate the default probability to the first time to default (τ).

Definition E.1 (**Expected Time to Default**). *The first time to default (τ), is defined as the first time the process $\{\phi\}_{t\geq 0}$ touches the upper bound of the set $\Omega_\phi := (0, 1]$. Namely, $\tau = \inf\{t \geq 0 : \phi(t) \geq 1\}$ with $\inf\{\emptyset\} = \infty$ if one is never reached. The full characterization of τ is its transition probability density function $f_\phi(1,\, t \mid \phi_0,\, 0)$.*[37] *Then, the expected time to default is*

$$\bar{\tau} := \int_0^\infty t f_\phi(1,\, t \mid \phi_0,\, 0) dt \tag{29}$$

which is the expected first hitting time for $\{\phi\}_{t\geq 0}$ to reach the absorbing default barrier.

F Numerical Analysis

In this section, we show how to obtain the expected exit time from the system (11), via a simulation of the SDEs. The first step is to simulate the standard Brownian motion. Then we look at stochastic differential equations, after which we will look at the expected exit time.

Simulation of Brownian Motion We start considering a discretized version of Brownian motion. [38] We set the step-size Δt and let W_ℓ denote $W(t_\ell)$ with $t_\ell = \ell \Delta t$. According to the properties of the Brownian motion, we find

$$W_\ell = W_{\ell-1} + dW_j \quad \ell = 1, ..., L. \tag{30}$$

Here L denotes the number of steps that we take with $t_0, t_1, ..., t_L$ being a discretization of the interval $[0, T]$, and dW_ℓ is a normally distributed random variable with zero mean and variance Δt. Expression (30) can be seen as a numerical recipe to simulate Brownian motion.

[37] A standard approach to derive $f_\phi(1,\, t \mid \phi_0,\, 0)$ is to inverse its Laplace transform $\mathbb{E}[e^{-\lambda\tau}]$, where $\lambda < 0$.

[38] We recall that a stochastic process $\{W(t, \omega)t \in \mathbb{R}^+, \omega \in \Omega\}$ is called (standard) Wiener process if: (i) $W(0) = 0$, (ii) it has stationary, independent increments. Namely, for all $0 < t_0 < t_1 < ... < t_k$, the increments $W(t_k) - W(t_{k-1}), ..., W(t_1) - W(t_0)$ are independent; (iii) for all $t > s$ the increment $W(t) - W(s)$ has the normal distribution with expectation 0 and variance $t - s$, i.e., $\mathbb{P}(W(t) - W(s) \in \Gamma) = \frac{1}{\sqrt{2\pi(t-s)}} \int_\Gamma \exp^{\frac{-y^2}{2(t-s)}} \mathrm{h}y, \quad \Gamma \in \mathbb{R}$. The Wiener process is said to be adapted to filtration $\mathbb{F} = (\mathcal{F})_{t\in[0,T]}$, if $\omega(t)$ is a random variable $\mathbb{F}$-measurable for all $t \geq 0$.

Simulation of the SDEs We use the *Euler-Maruyama method*. First, we discretize the dynamics of the two process $\{p\}_{t\geq 0}$ and $\{h\}_{t\geq 0}$. Then, we substitute the value of p and h in the discrete time version of (13) which read as[39]

$$\phi_\ell = \frac{1}{2\beta p_\ell}\left[h_\ell(\beta-1) - p_\ell + \left(4\beta h_\ell p_\ell + (h_\ell(1-\beta)+p_\ell)^2\right)^{1/2}\right] . \tag{31}$$

The dynamics of debts and price have the general forms,

$$\frac{dp}{p} = \bar{\mu}(\phi, p, \phi^*, \gamma, \varepsilon)dt + \bar{\sigma}dW(t) , \quad p(0) = p_0 , \tag{32}$$

$$\frac{dh}{h} = \mu(\phi, h, \phi^*, \varepsilon)dt , \quad h(0) = h_0 , \tag{33}$$

which are discretized as follows

$$p_\ell = p_{\ell-1} + \bar{\mu}(\phi_{\ell-1}, p_{\ell-1}, \phi^*, \gamma, \varepsilon)\Delta t + \bar{\sigma}(p_{\ell-1})dW_\ell ,$$
$$h_\ell = h_{\ell-1} + \mu_h(\phi_{\ell-1}, h_{\ell-1}, \phi^*, \varepsilon)\Delta t .$$

Here ϕ_ℓ, p_ℓ and h_ℓ are the approximation to $\phi(\ell\Delta t)$, $p(\ell\Delta t)$ and $h(\ell\Delta t)$. While, Δt is the step size, $dW_\ell = W_\ell - W_{\ell-1}$ and L is the number of steps we take.

Estimation of expected time to default The expected exit time can be approximated in the following manner. We start by simulating a sample path of $\{p\}_{t\geq 0}$, $\{h\}_{t\geq 0}$ and $\{\phi\}_{t\geq 0}$. While simulating the sample path of $\{\phi\}_{t\geq 0}$, we observe whether it crosses the default boundary fixed at one. Since the boundary is absorbing, the sample path can no longer exits through one. This gives us an approximation to the exit time of the current sample path. By using a Monte Carlo method, we estimate the expected exit time of the system. We simulate a total of N sample paths within $[0, T]$. The time interval is set sufficiently large so that all the trajectories of $\{\phi\}_{t\geq 0}$ exit through 1. Then, we take the sample average of the time taken by those N trajectories to touch the upper bound. Let, the rth realization of the path of $\{\phi\}_{t\geq 0}$ be denoted as $\{\phi_r\}_{t\geq 0}$ with $r = 1, ..., N$. For each of these paths the random variable τ exits through the default boundary is defined as $\tau_r = \inf\{t \geq 0 : \phi_{rt} \geq 1\}$ with $r = 1, ..., N$. The expected default time $\bar{\tau}$ is approximated as the total sum of exit times divided by the number N of paths

$$\bar{\tau} = \frac{1}{N}\sum_{r}^{N} \tau_r(1\!\!1_{\phi_{rt}=1}) \approx \int_0^\infty t f_\phi(1,\, t \mid \phi_0,\, 0)dt . \tag{34}$$

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[39] In particular, we do not need to derive the dynamics of $\{\phi\}_{t\geq 0}$ by Ito's Lemma from the dynamics of $\{p\}_{t\geq 0}$ and $\{h\}_{t\geq 0}$. We use directly the expression (13) which is the closed form solution of ϕ in terms of p and h.

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Estratto per riassunto della tesi di dottorato

L'estratto (max. 1000 battute) deve essere redatto sia in lingua italiana che in lingua inglese e nella lingua straniera eventualmente indicata dal Collegio dei docenti.

L'estratto va firmato e rilegato come ultimo foglio della tesi.

Studente: ____ matricola: ____________________

Dottorato: ____

Ciclo: ______________

Titolo della tesi[1] :____

Abstract:

La Tesi esamina da un punto di vista teorico la relazione tra leva finanziaria, diversificazione e rischio sistemico. Partendo dalla convinzione popolare che la diversificazione innalza la stabilita` finanziaria attraverso la diluzione dei rischi, collaboriamo al dibattito facendo luce su un aspetto critico di questa strategia. Consideriamo dapprima un investitore in leva. Sotto le condizioni standard della teoria moderna della finanza, dimostriamo che la massima divesificazione puo` aumentare il rischio di fallimento. Successivamente, consideriamo un sistema finanziario composto da banche con mutua esposizione finanziaria tra di loro e che investono in attivita` esterne al sistema. Dimostriamo che la diversificazione in attivita` esterne oltre un certo livello puo` dimuire la stabilita` finanziaria: effetto 'filo di rasoio'. Alla fine troviamo che in presenza di regolamenti di capitale che creano un ciclo tra leva e prezzi, la diversificaione amplica l`effetto 'filo di rasoio'.

The Thesis investigates from a theoretical perspective the relationship between leverage, diversification and systemic risk. Moving from the folk wisdom that asset diversification enhances financial stability by dispersing credit risks, we contribute to the debate shedding light on a critical facet of this strategy. First a representative leveraged investor is considered. Under the standard framework of asset pricing theory in a frictionless, arbitrage-free and complete market, we show that maximum diversification may increase the default risk. Then, we consider a financial system of interconnected banks whose assets include also securities outside the financial network. We show that diversification in external assets displays a knife-edge effect: beyond a certain range, it may induce financial instability. Finally, we found that in presence of procyclical capital requirements that create a positive feedback loop between leverage and asset prices, the knife-edge property is amplified.

Firma dello studente

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1 Il titolo deve essere quello definitivo, uguale a quello che risulta stampato sulla copertina dell'elaborato consegnato.