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Three Essays in Continuous-time Macro-finance

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Introduction

> *"What we know about the global financial crisis is that we don't know very much."*
>
> \- Paul Samuelson, *1999*

Over the last decades, several episodes of financial instability associated to undeniable consequences on the real economy, making clear that financial intermediaries do not stand as a veil between lenders and borrowers. On the contrary, intermediaries' supply of financial services, such as monitoring productive firms rather than diversifying idiosyncratic risks, has proved determinant for both real and financial cycles.

The purpose of this doctoral dissertation is to study, from a theoretical standpoint, some of the mechanisms through which financial intermediaries' activities jointly channels real and financial economic cycles, and so affect households' welfare.

This work consists of three papers, organized in chapters, on continuous-time general equilibrium models in macro-finance. Each chapter relates to its peers as they all belong to the literature that studies how financial frictions over heterogeneous classes of agents may channel time-varying financial leverage, risk premiums, and the overall macroeconomic dynamics.

Chapter 1, based on a joint work with Pietro Dindo, reviews some important topics of continuous-time methods as they relate to the core mechanisms prominent in the new-born macro-finance literature. The contents span across three sub-sections that, by progressively adding frictions and heterogeneity among economic actors, develop the modelling environment that acts as a baseline for further developments in the second and third chapters, respectively. The focus is on the role of market incompleteness and agents' heterogeneity at determining long-run equilibrium dynamics.

Chapter 2 is based on a joint work with Pietro Dindo and Loriana Pelizzon. The paper aims at studying the inter-dependence between financial intermediaries' risk-pooling activity, economic macro-dynamics and, in turn, households' welfare. To do so, it develops a suitable DSGE model of a productive economy. The economic environment is populated by heterogeneous households and a homogeneous financial intermediaries (financial sector). The model features financial frictions in the form of restricted market participation (on the households' side) and pooling/intermediation costs (on the intermediaries' side).

Due to their idiosyncratic risk exposure, households are willing to purchase risk-free liabilities issued by the financial sector that, accordingly, pools risky claims issued by different firms within its assets. In equilibrium, exogenous systematic shocks change the relative size of the financial sector jointly with the amount of idiosyncratic risks it is willing to pool, making its leverage state-dependent and counter-cyclical. In the same fashion, risk-free interest rates paid on its liabilities issuances are pro-cyclical. Within this framework, the paper investigates the relationship between financial sector capitalization (size) and households' welfare. On the one hand, when the financial sector is too small, its supply of risk mitigation instruments is scarce and costly (risk free rates are extremely low, even negative). Nonetheless, those states associates to a higher growth rate of households' consumption. On the other hand, when the financial sector is too large, it destroys resources after the payment of intermediation costs. Therefore, households benefit the most when the financial sector is neither too small nor too big.

Chapter 3 studies how banks resolution regimes may affect households' welfare in the short and in the long-run. It does so within the framework of a DSGE model of a productive economy populated by homogeneous households, banks, and firms. The model introduces financial frictions by assuming that: a) Banks benefit of a cost advantage at monitoring capital producing firms; b) At the moment of default, banks may recapitalize by issuing new equity at a fixed cost. In equilibrium, it is individually optimal for each bank to be recapitalized by its own shareholders up to a certain threshold, where the marginal value of her equity (market-to-book value) equals the recapitalization cost. In the same fashion, banks optimally pay out dividends when their market-to-book value shrinks below one. However, as banks are homogeneous and uniformly exposed to a unique common source of systematic risk, their recapitalizations (and dividends payouts) are always systemic. As a consequence, as the whole banking sector is jeopardized, we show that a bailout resolution that tops up banks' individual optimal recapitalization policies may improve long-run welfare, even when all actors of the same type are homogeneous, and there does not exist idiosyncratic insurable risk. This happens because, in a perfectly competitive environment, banks (and households) fail at internalizing the positive effect of banks' aggregate capital (size) over equilibrium prices (pecuniary externality).

Three Essays in Continuous-time Macro-finance

Chapter 1

Continuous-time Macro-finance: Theory and Methods[1]

"In precisely built mathematical structures, mathematicians find the same sort of beauty others find in enchanting pieces of music, or in magnificent architecture."

- Kiyoshi Itô, *1998*

Abstract

This paper reviews some important topics of continuous-time methods as they relate to the core mechanisms prominent in the new-born macro-finance literature. The contents span across three sub-sections that, by progressively taking into consideration financial frictions and heterogeneity among economic actors, develop the modelling environment that act as a baseline for our own models in Chapter 2 and 3, respectively. The focus is on the role of market incompleteness and agents' heterogeneity at determining long-run equilibrium dynamics. Each topic is framed by several seminal contributions in the macro-finance literature.

Keywords General Equilibrium, Financial Frictions, Macro-finance, Stochastic Optimal Control.

JEL Classification C60, D5, G0.

[1]Based on a joint work with Pietro Dindo.

1.1 Introduction

In recent years, especially after the 2007-2008 global financial crisis, the traditional New Keynesian framework revealed to be inadequate for explaining the possibility of occasionally acute and persistent changes in financial and real macroeconomic variables such as asset prices, interest-rates, output, and consumption. To the aim of modelling the mechanisms behind those patterns, a new stream of literature arose, mainly employing continuous-time methods for the development of a novel class of macro-finance general equilibrium models.[2] Since the seminal contribution of Merton (1975), continuous-time dynamic models with Brownian uncertainty have been of widespread interest in economic theory and mathematical finance, and several versions of the so-called differential approach has been used in numerous fields spanning from economics growth to asset pricing (a technical summary is, for example, in Merton and Samuelson, 1992; Wälde, 2011).

While the first generation of models developed by this literature focus on the more traditional representative agent setting with complete markets, and account for financial frictions and market incompleteness from the partial equilibrium standpoint only, the new born continuous-time macro-finance complements the conventional environment by featuring, all at once, financial frictions, market incompleteness, and heterogeneity among agents.

Right, but why continuous-time? Broadly speaking, continuous-time methods have both advantages and shortcomings with respect to their discrete-time counterpart. On the one hand, the former formulation grants greater analytical tractability and mathematical elegance. Moreover, its solution methods, mainly based on solving systems of differential equations, bypass several technical issues concerning recursive methods in discrete-time, that are often tackled my linear approximation around a deterministic steady state. Therefore, continuous-time modelling naturally allows to fully capture non-linear dynamics characterising the outcome of general equilibrium macroeconomics models. On the other hand, the main drawback is that the continuous-time formulation may lead to less intuitive implications, as real world data and inputs of computer languages are naturally expressed in discrete-time.

The purpose of this chapter is to review some relevant aspects of continuous-time methods as they relate to the core mechanisms that are prominent in the most recent macro-finance literature. The contents range across three core sections that develop, progressively building over each other, a more and more complex economic environment. The focus will be on the joint role of market incompleteness, agents' heterogeneity, and long-run equilibrium dynamics.

[2]For a very much comprehensive review of the the most important macroeconomic models with financial frictions see Brunnermeier et al. (2012).

The content of this chapter develops as follows. First, Section 1.2 and 1.2.2 characterize a toy model of a continuous-time endogenous growth economy with AK production technology, complete financial markets, and a representative agent. The follow-up sub-sections introduce several extensions. In order: adjustment costs (1.2.2.1); uninsurable idiosyncratic risk - incomplete financial markets (1.2.2.2). Sections 1.2.3 and 1.2.4 discuss the models micro-foundation and the relationship between agents' optimal strategies and asset pricing, respectively.

Section 1.3 acts a pass-though before moving to more advanced models where heterogeneity is persistent in the long-run. In particular, it introduces a few recent application featuring institutional heterogeneity among classes of agents with focus on: extreme financial frictions (1.3.1); restricted market participation and risk-free bonds (1.3.2); the role of money as a risk-free asset (1.3.3).

Finally, Section 1.4 studies several important (advanced) models (some journal published, others still at working paper stage) that focus on different macro-financial phenomena such as: aggregate bank capital and credit dynamics (Klimenko et al., 2016, Section 1.4.1); intermediary asset pricing (He and Krishnamurthy, 2013, Section 1.4.2); amplification of exogenous systematic shocks (Brunnermeier and Sannikov, 2014, Section 1.4.3); money and macro-prudential policies (Brunnermeier and Sannikov, 2016a, Section 1.4.4).

1.2 Representative Agent Models in Continuous-time

This section introduces the benchmark growth model that acts as the backbone of most of the more advanced specifications we discuss along the Chapter.

First, Section 1.2.1 considers a AK productive technology in a standard neo-classical Real Business Cycle (RBC) model, where uncertainty uniquely comes from exogenous aggregate shocks affecting the depreciation rate of physical capital dynamics. For the purpose of this work, we focus on the case of CRRA preferences, due to their greater analytical tractability.[3] Second, Section 1.2.2 reformulates the model in general equilibrium setting, discusses its main implications, and lays the foundation of further generalizations. Third, Section 1.2.2.1 introduces *technological illiquidity* between output (consumption) and productive goods (physical capital). Forth, Section 1.2.3 considers the situation where agents are exposed to uninsurable idiosyncratic risk. Fifth, in the spirit of Hayashi (1982), Section 1.2.3 develops a suitable micro-foundation setting of firms' investment choices and securities issuance as related to the returns on

[3]A very general formulation of a continuous-time growth model that admits closed-form solution for the class of HARA preferences is, for example, in Menoncin and Nembrini (2018).

households' risky assets holdings and equilibrium prices.

Finally, Section 1.2.4 discusses some important aspects concerning the relationship between the representative agent's problem and the pricing of risky assets in continuous-time economies.

1.2.1 A Toy Model: AK Growth

The social planner problem This section summarises the core features of a AK stochastic growth model with a unique source of systematic risk, where capital and output goods are exchanged at a 1-to-1 ratio. This first step formulation will be useful to: a) Review some basic tools of stochastic dynamic programming; b) Generalize the model to consider adjustment costs for converting perishable output good into physical capital (1.2.2.1), and idiosyncratic risk (1.2.2.2).

There exists a unique productive-consumption good, physical capital (henceforth, capital). The stock of capital evolves with dynamics

$$T_t: \quad dK_t = Y_t dt - K_t d\Delta_t - C_t dt, \tag{1.1}$$

where C_t represents the instantaneous flow of consumption, while $d\Delta_t$ is the (stochastic) depreciation with dynamics given by a Arithmetic Brownian Motion (ABM)

$$d\Delta_t = \delta dt - \sigma dW_t,$$

where dW_t is a measurable uni-variate Wiener process defined over a suitable probability space. The output Y_t is produced instantaneously by a linear technology with marginal productivity A, so that

$$Y_t = AK_t.$$

The economy is populated by a representative infinitely-lived household maximising the inter-temporal utility of her consumption. The household is endowed with an initial stock of capital K_0 at time $t = 0$. Formally, her problem reads as follows

$$H_0 := \max_{\{C_t\}_{t\in[0,\infty)} \in T_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t} U(C_t) dt, \tag{1.2}$$

subject to (1.1), where ρ denotes the constant instantaneous discount rate, and U is a standard separable utility function. In general, the differential representation of the productive technology may pass trough different approaches. In this setting, the dynamics of uncertainty is defined explicitly as a Wiener process and it affects the stock of capital directly rather than the aggregate TFP.[4]

[4]This approach correspond to what is defined in Wälde (2011) as *Standard II* or

Solution The optimal consumption path $\{C_t\}_{t\in[0,\infty)}$, that is solution to problem (1.2), satisfies the HJB equation

$$\rho H_t dt = U(C_t)dt + \mathbb{E}_t dH_t,$$

and, by Itô's Lemma,

$$\rho H_t = \max_{\{C_t\}_{t\in[0,\infty)}\in T_t} \left\{ U(C_t) + \frac{\partial H}{\partial t} + \frac{\partial H}{\partial K}\left[K_t\left(A-\delta\right)-C_t\right] + \frac{1}{2}\frac{\partial H^2}{\partial K^2}K_t^2\sigma^2 \right\},$$

and the FOC on consumption implies that

$$\frac{\partial U}{\partial C} = \frac{\partial H}{\partial K}.$$

Henceforth, we focus on the case of CRRA utility

$$U(C) := \begin{cases} \frac{C^{1-\gamma}}{1-\gamma} & \gamma \neq 1 \\ \ln C & \gamma = 1, \end{cases}$$

where γ represents the households' relative risk aversion. Under this assumption, the long-run value H ($\frac{\partial H}{\partial t} = 0$) solves the following ODE

$$\rho H = \left(\frac{\partial H}{\partial K}\right)^{\frac{\gamma-1}{\gamma}} \frac{\gamma}{1-\gamma} + \frac{\partial H}{\partial K}K\left(A-\delta\right) + \frac{1}{2}\frac{\partial H^2}{\partial K^2}K^2\sigma^2. \qquad (1.3)$$

By a proper ansatz of the form $H = \kappa^{-\gamma}\frac{K^{1-\gamma}}{1-\gamma}$, Equation (1.3) implies that the optimal consumption rate is fixed and equals $\frac{C}{K} = \kappa$, where $\kappa = \left[\rho + \frac{1-\gamma}{\gamma}\left(\frac{\gamma}{2}\sigma^2 - A + \delta\right)\right]^{\frac{1}{\gamma}}$. Accordingly, the dynamics of capital stock follows a Geometric Brownian Motion (GBM)

$$\frac{dK_t}{K_t} = \underbrace{\left(A-\delta-\kappa\right)}_{\mu} dt + \sigma dW_t, \qquad (1.4)$$

that, given the initial condition $K_0 > 0$, leads to the well know solution

$$K_T = K_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_T}.$$

1.2.2 A Toy Model in Competitive Equilibrium

The aim of this section is to highlight how the structure of Section 1.2.1 can be equivalently reformulated in a general equilibrium framework, so that it takes into account consumption, investments, and portfolio choices of the household's

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stochastic depreciation. Modelling capital accumulation as a risky process is an assumption that determines only indirectly the stochastic nature of aggregate output and, at the same time, it prevents Y_t from assuming negative values.

between a risk-free bond versus a risky asset. In particular, we assume that the return on risky claims is proportional to the return on capital stock (1.1).

As we shall see, although the equilibrium dynamics of capital holds the same, this new perspective will be useful to generalize the model to account for the case of adjustment costs when producing physical capital out of perishable output good. Moreover, it will suggest useful takeaways concerning the asset pricing implication associated to the equilibrium dynamics.

The household is endowed with initial wealth E_0. She allocates her wealth between risky claims, with stochastic return dR_t, and risk-free bonds, with instantaneous return $r_t dt$. The risky return is proportional to the dynamics of output and, at each instant of time t, the household optimally decides what fraction ι_t of her wealth to re-invest for generating new capital at $t + dt$. Therefore, the dynamics of household's risky investments depends on her own investment choice.[5] Accordingly, the (total) absolute return on capital stock K_t has dynamics

$$K_t dR_t = \underbrace{(Y_t - K_t \iota_t)}_{\text{Dividend yield}} dt + \underbrace{dK_t}_{\text{Capital gain}}, \tag{1.5}$$

where ι_t is the re-investment rate, and the dynamics of capital stock reads as

$$dK_t = \underbrace{I_t}_{\iota_t K_t} dt - K_t \delta dt + K_t \sigma dW_t, \tag{1.6}$$

where I_t is the stock in investments and δ the depreciation rate. What is relevant to highlight is that the former component of (1.5) corresponds to the amount of output good (consumption, in equilibrium) paid out from capital stock K_t. In this term, it can be interpreted as a dividend yield. Accordingly, the second component can be read as capital gain, as it accounts for the residual capital stock (value) available for new investment at $t + dt$.

In summary, the households' problem reads as follows

$$H_0^g \coloneqq \max_{\{C_t, \omega_t, \iota_t\}_{t \in [0,\infty)} \in T_t^g} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \frac{C_t^{1-\gamma}}{1-\gamma} dt \tag{1.7}$$

subject to

$$T_t^g : dE_t = E_t \omega_t dR_t + E_t (1 - \omega_t) r_t dt - C_t dt,$$

where ω is the wealth-portfolio share allocated to risky claims, r_t the endogenous return on risk-free bonds, and C_t the consumption flow. The optimal strategy

[5] As we discuss at length in Section 1.2.3 the risky claims can be interpreted as the inter-temporal equity issuance of a firm who collects physical capital from the household, act as a lessor, and rents it to another firm producing output good.

$\{C_t, \omega_t, \iota_t\}_{t\in[0,\infty)}$ for problem (1.7) satisfies the HJB Equation

$$\rho H_t^g = \frac{C_t^{1-\gamma}}{1-\gamma} + \frac{\partial H}{\partial t} + \frac{\partial H^g}{\partial E} E_t \left[r_t + \omega_t (A - \delta - r_t) - \frac{C_t}{E_t} \right] + \frac{1}{2} \frac{\partial H^{g,2}}{\partial E^2} (\omega_t E_t \sigma)^2 ,$$

and the FOCs satisfy

$$C_t = \left(\frac{\partial H^g}{\partial E} \right)^{-\frac{1}{\gamma}},$$

$$\omega_t = -\frac{\frac{\partial H^g}{\partial E}}{\frac{\partial H^{g,2}}{\partial E^2}} \frac{1}{E_t} \frac{A - \delta - r_t}{\sigma^2},$$

where ι_t and r_t will be determined by market clearing conditions in equilibrium. By considering a proper ansatz for the value function $H_t^g := \kappa_t^{-\gamma} \frac{E_t^{1-\gamma}}{1-\gamma}$ and setting $\frac{\partial H}{\partial t} = 0$ we have that

$$\frac{C_t}{E_t} = \kappa_t,$$

$$\omega_t = \frac{1}{\gamma} \frac{A - \delta - r_t}{\sigma^2},$$

where

$$\kappa_t = \frac{1}{\gamma} \left[\rho - (1-\gamma) r_t - \frac{1}{2} \frac{1-\gamma}{\gamma} \frac{(A - \delta - r_t)^2}{\sigma^2} \right]. \tag{1.8}$$

Competitive equilibrium Informally, the equilibrium is defined as a map from histories of systematic shocks to all the relevant aggregates so that the household maximises the inter-temporal utility of her consumption and all markets clear. Market clearing conditions are so that: the risk-free rate r_t adjusts according to the bonds zero net supply; the re-investment rate ι_t is so that aggregate consumption equals the stock of output not deployed to generate new capital. The market for physical capital (1-to-1 with risky claims) clears by Walras' Law. Formally,

1. Consumption/output good market

$$C_t = Y_t - I_t; \tag{1.9}$$

2. Bonds market (zero-net supply)

$$E_t (1 - \omega_t) = 0. \tag{1.10}$$

By matching (1.9) and (1.10) to (1.6) and (1.8), it is straightforward that the interest rate is constant (and so it is the function κ) and equals

$$r = A - \delta - \sigma^2 \gamma.$$

Accordingly, $\iota = A - \kappa$ and the dynamics of capital and household's wealth reduces to

$$\frac{dK_t}{K_t} = \frac{dE_t}{E_t} = \mu dt + \sigma dW_t,$$

that is equivalent to (1.4).

1.2.2.1 Output and Capital

In this section, the previous general equilibrium framework is generalized by taking into account the presence of adjustment costs when producing new physical capital from output. While the household's problem holds the same as in (1.7), we assume the investments at time t to generate new capital at time $t + dt$ at a rate $\Phi(\iota_t)$, where Φ is an increasing and concave function. Therefore, the stock of capital evolves with dynamics

$$dK_t = K_t \left[\Phi(\iota_t) - \delta\right] dt + K_t \sigma dW_t, \tag{1.11}$$

where dW_t is a Wiener process defined over a suitable probability space. From now on we assume that, due to its analytical tractability, $\Phi(\iota) = \frac{1}{\theta}\ln(1 + \theta\iota)$, where θ is a parameter that summarise the conversion cost of output into physical capital (see also Brunnermeier and Sannikov, 2016a).

Under this new assumption it is convenient to choose consumption as the numéraire of the economy. Accordingly, the aggregate capital stock is valued $K_t q_t$, where q_t is the price of capital stock in consumption good.[6] Another relevant aspect of introducing adjustment costs it that the dynamics of capital stock differs from the one of its value. To characterise the dynamics of the latter, we postulate the price q_t to follow a diffusion process

$$dq_t = q_t \mu_t^q dt + q_t \sigma_t^q dW_t,$$

where μ^q and σ^q are suitable stochastic processes adapted to the filtration spanned by the Wiener process driving (1.11). By Itô's Lemma, the value of capital stock evolves as

$$d\left(K_t q_t\right) = q_t K_t \left[\Phi(\iota_t) - \delta + \mu_t^q + \sigma_t^q \sigma\right] dt + q_t K_t \left(\sigma + \sigma_t^q\right) dW_t,$$

while the return on risky claims is so that

$$K_t q_t dR_t = \underbrace{\left(A - \iota_t\right) K_t}_{\text{Dividend yield}} dt + \underbrace{d\left(K_t q_t\right)}_{\text{Capital gain}}.$$

Given a proper ansatz of the form $H_t^{g,q} =: h(q_t)^{-\gamma} \frac{E_t^{1-\gamma}}{1-\gamma}$, the optimal strategy

[6] Note that in the case when $\Phi(\iota_t) = \iota_t$ the problem reduces to (1.2.2), and $q = 1$.

$\{C_t, \omega_t \iota_t\}_{t\in[0,\infty)}$ that is solution to problem (1.7) satisfies

$$C_t = h(q_t)E_t; \tag{1.12}$$

$$\omega_t = \frac{\mu_t - r_t}{\gamma \sigma_t^2} + \epsilon_{h,q} \frac{\sigma_t^q}{\sigma_t},$$

where $\epsilon_{h,q} = \partial_q h \frac{q_t}{h(q_t)}$ is the elasticity of consumption rate to capital prices (see also Equation 1.12), and

$$\partial_\iota \Phi = \frac{1}{q_t}. \tag{1.13}$$

Equation (1.13) can be read as a *Tobin's Q*, as it relates the price of capital to the equilibrium re-investment rate of the economy.

Competitive equilibrium As before, the market clearing conditions hold for physical capital (risky asset), risk-free bonds, and consumption. We solve the model by looking for that equilibrium where q is constant ($\mu^q = \sigma^q = 0$). In that equilibrium, h will also be constant, and the return on risky claims reads as

$$dR_t = \left[\frac{A - \iota}{q} + \Phi(\iota) - \delta\right] dt + \sigma dW_t,$$

and the risk-free rate (also constant) equals

$$r = \frac{A - \iota}{q} + \Phi(\iota) - \delta - \gamma\sigma^2.$$

Proof. A sketch of the proof is in Appendix A.1. □

By substituting the optimal strategies into the household's dynamic budget constraint, it is easy to see that it evolves as a GBM, and follows the same dynamics as the aggregate stock of capital

$$\frac{dK_t}{K_t} = \frac{dE_t}{E_t} = \underbrace{\left(\frac{1}{\theta}\ln q - \delta\right)}_{\mu^{g,q}} dt + \sigma dW_t.$$

1.2.2.2 Idiosyncratic Risk and Financial Frictions

This section is structured as follows: First, we briefly consider a further extension of model (1.2.2.1) featuring exogenous idiosyncratic shocks on top of the systematic one. Then, we compare the equilibrium outcomes to the baseline models discussed in the previous sections.

This new economy is populated by a continuum of households indexed $h \in \mathbb{H}$, where $\mathbb{H} := [0, 1)$, all endowed with an initial stock of wealth e_0^h at time $t = 0$. In the aggregate it holds that $\int_{\mathbb{H}} e_0^h dh = E_0 = K_0 q_0$. As before, households consume, invest, and allocate their wealth between risk-free and/or risky claims to maximise the inter-temporal utility of their consumption. We extend the model as follows: when making their portfolio choice, the agents may decide whether to invest in their own risky claim, affected by both idiosyncratic and systematic shocks, or to the aggregate portfolio, affect by systematic shocks only, after the payment of an exogenous pooling cost.

As we shall see, in equilibrium, although the aggregate volatility of capital stock dynamics will be not affected, the households' portfolio, investments, and consumption optimal strategies will fundamentally differ. Formally, the main equations in Section 1.2.2.1 modify as follows: the dynamics of capital stock k_t^h that belongs to household h evolves with dynamics

$$dk_t^h = k_t^h \left[\Phi(\iota_t^h) - \delta\right] dt + \underbrace{k_t^h \sigma dW_t}_{\text{Systematic}} + \underbrace{k_t^h \tilde{\sigma} d\tilde{W}_t}_{\text{Idiosyncratic}} , \tag{1.14}$$

where dW_t and $d\tilde{W}_t^h$ are measurable Wiener processes defined over a suitable probability space. We also assume that in the aggregate idiosyncratic shocks utterly cancel out $\int_{\mathbb{H}} d\tilde{W_t}^h = 0$. Under this new set of assumptions, the h household's problem reads as

$$H_0^{h,p} := \max_{\{c_t,\Omega_t,\iota_t\}_{t\in[0,\infty)}\in T_t^p} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \frac{\left(c_t^h\right)^{1-\gamma}}{1-\gamma} dt, \tag{1.15}$$

subject to

$$T_t^p : \quad de_t^h = e_t^h \omega_t^h dR_t^h + \omega_t^{p,h} dR_t^p + (1 - \omega_t^{p,h} - \omega_t^h) r_t dt - c_t^h dt,$$

where $\Omega_t := \left\{\omega_t^{p,h}, \omega_t^h \geq 0\right\}$ is the set of controls for the households' portfolio, while dR_t^h and dR_t^p are the return on un-pooled and pooled portfolios, respectively, with dynamics

$$dR_t^h = \frac{A - \iota_t^h}{q_t} dt + \frac{d\left(k_t^h q_t\right)}{k_t^h q_t};$$

$$dR_t^p = \int_{\mathbb{H}} dR_t^h - \underbrace{\eta_t dt}_{\text{Pooling cost}} .$$

The function η_t represents the instantaneous cost rate of investing in the pooled portfolio; it is a reduced form summarising the administrative and monitoring cost of holding the aggregate portfolio.[7]

[7] To study the relationship between risk pooling intermediation services, financial sector capitalization, and the business cycle, Chapter 2 extends this framework by developing a model with institutionally heterogeneous agents, intermediaries plus households, and restricted market participation. In particular, we will assume that intermediaries

Competitive equilibrium We solve the model for its competitive equilibrium (a sketch of the solution is in Appendix A.2). For sake of simplicity, we look for the equilibrium so that the price of capital q is constant, and consider the following market clearing conditions:

1. Physical capital (risky claims)
$$E_t\left(\omega_t^{p,h}+\omega_t^h\right)=K_t q_t; \tag{1.16}$$

2. Risk-free bond
$$E_t\left(1-\omega_t^{p,h}-\omega_t^h\right)=0 \Longrightarrow \omega_t^{p,h}+\omega_t^h=1; \tag{1.17}$$

3. Consumption
$$\left(A-\iota_t\right)K_t-\omega_t\eta_t K_t=C_t. \tag{1.18}$$

According to equilibrium conditions (1.16)-(1.18), the optimal strategies that are solution to problem (1.15) satisfy

$$\frac{c_t^h}{e_t^h}=h_t;$$

$$\omega_t^h=\frac{\eta_t}{\gamma\tilde{\sigma}^2},$$

where h and q solve the following non-linear system

$$\begin{cases}\frac{\rho}{1-\gamma}=\frac{\gamma}{1-\gamma}h_t+\left[\mu_t-\left(1-\omega_t\right)\eta_t\right]-\frac{1}{2}\gamma\left(\sigma^2+\omega_t^2\tilde{\sigma}^2\right)\\ q_t=\frac{1+A\theta-\frac{1}{\tilde{\sigma}}\eta_t\left(\frac{1}{\gamma\tilde{\sigma}}\eta_t\right)}{1+\theta h_t}\end{cases},$$

and the (shadow) risk-free rate equals

$$r=\frac{A-\iota-\eta}{q}+\Phi(\iota)-\delta-\sigma^2+\sigma^2\left(\frac{1}{\gamma\tilde{\sigma}^2}\eta-\beta\frac{\sigma}{\tilde{\sigma}}\right).$$

Also in this last case, the aggregate dynamics of capital evolves as a GBM

$$\frac{dK_t}{K_t}=\frac{dE_t}{E_t}=\underbrace{\left(\frac{1}{\theta}\ln q^p-\delta\right)}_{\mu^p}dt+\sigma dW_t. \tag{1.19}$$

To compare the equilibrium results of the models discussed thought sections (1.2.2)-(1.2.2.2), Table 1.1 reports the numerical values of the models main equi-

only are allowed to pool idiosyncratic risks. In equilibrium, they supply the economy with risk mitigation by leveraging their balance sheet. In turn, this will affect the real and financial macro-dynamics of the economy.

	Baseline	Adj. costs	Idiosy. risk
κ	0.105	0.078	0.0007
ω^h	1	1	0.1389
ι	0.095	0.0415	0.0605
q	1	1.083	1.121
r	0.120	0.1061	0.0109
$\frac{1}{dt}\mathbb{E}_t\frac{dK_t}{K_t}$	0.095	0.04	0.0380
$\frac{1}{dt}\mathbb{V}ar_t\frac{dK_t}{K_t}$	0.2	0.2	0.2

Table 1.1: Consumption rate, portfolio choices, risk-free rates, re-investment rate, prices, average growth rate and capital tock volatility of models in Sections 1.2.2 (left), 1.2.2.1 (centre), and 1.2.2.2 (right). Parameters: $\delta = 0$, $\sigma = 0.2$, $\rho = 0.05$, $\gamma = 2$, $A = 0.2$, $\eta = 0.1$, and $\theta = 2$.

librium variables. In order from the top: consumption rate, prices, risk-free rates, average growth rate and volatility of capital. For this purpose we assume that $\eta_t = \eta$, and set the following baseline values: $\delta = 0$, $\sigma = 0.2$, $\rho = 0.05$, $\gamma = 2$, $A = 0.2$, $\eta = 0.1$, and $\theta = 2$.

What stands out is that the presence of adjustment costs and idiosyncratic risk progressively reduce the aggregate consumption and as well as growth rate of capital. However, it does not change its volatility. At the same time, both frictions increase the equilibrium price of physical capital and, in turn, investments. Moreover, introducing frictions reduces the equilibrium risk-free interest rate due to the presence of uninsurable idiosyncratic risk.

1.2.3 Micro-foundation

The purpose of this section is to provide a micro-foundation that frames the return on risky claims that belong to households' portfolio within the maximization problems of heterogeneous capital and output producing firms. The overall structure is meant to be the continuous-time equivalent of Ljungqvist and Sargent (2012), Chapter 12. The section is organized as follows: first, we introduce the output and capital producing firms', respectively. Second, we discuss how the return on firms' issuance of risky claims relates to the households' risk-neutral measure (no-arbitrage condition) and, in turn, to their portfolio choices. We begin with a descriptive summary of the micro-structure.

There exist two types of firms: the former endowed with the technology to produce output good by taking physical capital as an input, the latter endowed with the technology to produce physical capital with perishable consumption as an input. Capital producing firms face adjustment costs when deploying input for generating new capital.

Capital producing firms live one period. At each instant t, they are constituted by transfers of resources executed by utility maximising agents, and liquidated at $s = t + dt$. Accordingly, those firms finance their constitution by issuing risky claims whose pay-off is written on their net instantaneous revenues. Capital producing firms earn revenues by instantaneously renting capital to output producing firms, and choose the re-investment rate of capital to maximise the expected return (pay-off) on their risky claims issuances. More formally, the firms' problem read as follows:

Output producing firms There exists a continuum of unitary mass of output producing firms indexed $i \in \mathbb{I}$, where $\mathbb{I} =: [0, 1)$. Those firms consist of an output producing technology f^i that inputs physical capital. At each instant of time t, the i^{th} productive firm chooses the physical capital k_t^i in order to solve a static problem

$$\max_{k_t^i \geq 0} \left\{ y_t^i - p_t^i k_t^i \right\}, \tag{1.20}$$

s.t.

$$y_t^i \leq f^i(k_t^i), \tag{1.21}$$

where p_t^i is the cost rate of physical capital.

Problem (1.20) has an interior solution only when the following zero-profit condition is satisfied:

$$p_t^i = f_k^i. \tag{1.22}$$

Thus, their profits equal $\Pi_t = f^i(k_t^*) - p_t^i k_t^*$, where k_t^* satisfies (B.2). To the aim of this work, we will always assume $f = Ak$, so that $p = A$ and the i firm profits Π equal zero. Accordingly, it always break even, it is willing to supply any market demand, and its size is indeterminate. In this framework, we assume the i^{th} output producing firm to rent physical capital from a j^{th} capital producing firm, that is newly constituted over each time interval dt by the capital transfers of utility maximising agents. The activity of capital producing firms reads as follows.

Capital producing firms At each instant t, capital producing firms are constituted by transfers of physical capital executed by utility maximising agents, and liquidated at $s = t + dt$. Let us consider a capital productive technology so that the dynamics of capital stock is affected by both idiosyncratic and systematic shocks as in Equation (1.14). We assume that there exists a continuum of unitary mass of capital producing firms, indexed $j \in \mathbb{J}$ where $\mathbb{J} =: [0, 1)$ who own that technology.

The j^{th} firm rents physical capital to the i^{th} output producing firms at the equilibrium rate p_t^i. Then, it transforms output into capital, stores it, and earns revenues out of its activity. At each instant of time t, firm j chooses how much value of capital $k_t^j q_t$ to store in order to earn stochastic returns dR_t^j per unitary capital, and how much numéraire $\iota_t^j k_t^j$ to purchase to generate new capital $\Phi(\iota_t^j)k_t^j$ at $t+dt$, where Φ is an increasing concave function of the re-investment rate ι_t. Note that this formulation is equivalent to having convex adjustment costs). Firm j finances itself by issuing state-contingent claims to the agent who supplies the capital stock with stochastic return dR_t^j on its net revenues.

Formally, between t and $s = t + dt$, the j^{th} firm solves the following problem

$$\max_{\{k_t^j, \iota_t^j\}} \left\{ \mathbb{E}_t^{\mathbb{Q}^j} \underbrace{\left[v_s e^{-\int_t^s r_s du} \right]}_{\text{Discounted net revenues}} - \underbrace{k_t^j q_t}_{\text{Cost of capital}} \right\},$$

subject to

$$T^j: \quad \frac{d\left(k_t^j q_t\right)}{k_t^j q_t} = \left(\Phi(\iota_t^j) - \delta + \mu_t^q - \sigma_t^q \sigma\right) dt + (\sigma - \sigma_t^q)\, dW_t + \tilde{\sigma} d\tilde{W}_t^j, \quad (1.23)$$

where $\mathbb{Q}^j$ is the risk neutral measure. The revenues v_s are "net" the cost of purchasing the input, which in returns reads as $e^{-\int_t^s \frac{\iota_u^j}{q_u} du}$ for unit of capital. By (B.3), it holds that

$$v_s = \underbrace{k_t^j q_t e^{\int_t^s \left(\Phi(\iota_u^j) - \delta + \mu_u^q - \sigma_u^q \sigma\right) du - \frac{1}{2}\|\Sigma_t^j\|^2 du + \int_t^s \Sigma_t^i d\mathbf{W}_t}}_{k_s q_s} e^{\int_t^s \frac{p_u - \iota_u^j}{q_u} du},$$

where $\Sigma_t^j = \begin{bmatrix} \sigma_t & [\mathbb{I}_{i=p}]\, \tilde{\sigma} \end{bmatrix}$ and $d\mathbf{W}_t = \begin{bmatrix} dW_t \\ d\tilde{W}_t \end{bmatrix}$.

The FOC on ι_t^j requires that

$$\partial_\iota \Phi = \frac{1}{q_u}, \quad \forall u \in (t, s)\,.$$

By (B.2), the FOC on k_t^j implies a zero-profit condition as

$$\mathbb{E}_t^{\mathbb{Q}^i} \left[e^{\int_t^s \left(\mu_u - \frac{1}{2}\|\Sigma_t^i\|^2 - r_u\right) du + \int_t^s \Sigma_u^i d\mathbf{W}_u} \right] = 1. \quad \forall j, \quad (1.24)$$

The zero profit condition (B.4) must be consistent with the equilibrium return on the j^{th} risky claim dR_t^j, so that

$$\mu_t := \frac{\mathbb{E}_t \left[dR_t^j\right]}{dt} = \frac{p_t^i - \iota_t^j}{q_t} + \Phi(\iota_t^j) + \mu_t^q - \delta - \sigma_t^q \sigma,$$

$$\| \Sigma_t \|^2 = (\sigma - \sigma_t^q)^2 + \tilde{\sigma}^2 = \frac{\mathbb{V}ar_t \left[dR_t^i \right]}{dt} \Longrightarrow \sigma_t := \sigma - \sigma_t^q .$$

Condition (B.4) can be also read as a non-arbitrage/asset pricing condition, i.e. the return on risky claims issued by output producing firms (equity), must be such that their present discounted value equals the current value of physical capital stock $k_t^j q_t$ supplied by the agents. If such, the j firm breaks even, its size is indeterminate, and it is willing to supply each market demand.

Note that p_t^i differs from q_t, the equilibrium price of physical capital, because while the former represents the rate at which output producing firms rent capital from capital producing firms, the latter stands for the price of the claims issued by the same firms for financing their activity. Stated differently, the price q_t is the price of each risky claim (unit of capital) that grants dividends payments $\frac{p_t^i - \iota_t}{q_t} dt$ over the interval $[t, s]$ plus the rebate of the residual value of capital stock $k_s q_s$ at the end of the period, when firms are liquidated.

Risk-neutral measure To grant the existence (and uniqueness) of the competitive equilibrium, one must find the risk-neutral measure $\mathbb{Q}^j$. Given the zero-profit condition in (B.4), by *Girsanov Theorem III* (see Øksendal, 2013), the correspondent Radon-Nykodym derivative equals

$$\frac{d\mathbb{Q}^j}{d\mathbb{P}} = \exp \left\{ - \int_t^s \xi_u dW_u - \int_t^s \tilde{\xi}_u d\tilde{W}_u - \frac{1}{2} \int_t^s \left(\xi_u^2 + \tilde{\xi}_u^2 \right) du \right\} .$$

where $\mathbb{P}$ is the real probability measure, while ξ_t and $\tilde{\xi}_t$ represent the market prices of systematic and idiosyncratic risk, respectively.

We choose the risk-neutral measure conditional on the existence of an aggregate portfolio so that idiosyncratic shocks are utterly pooled, and the systematic risk component shall be uniquely priced. Accordingly, the no-arbitrage condition for the aggregate portfolio with expected return μ^f reads as

$$\mathbb{E}_t^{\mathbb{Q}^f} \left[e^{\int_t^s \left(\mu_u^f - \frac{1}{2} \sigma_u^2 - r_u \right) du + \int_t^s \sigma_u^2 dW_u} \right] = 1,$$

and the measure $\mathbb{Q}^f$ satisfies the following differential

$$\frac{d\mathbb{Q}^f}{d\mathbb{P}} = \exp \left\{ - \int_t^s \xi_t du - \frac{1}{2} \int_t^s \xi_t^2 dW_u \right\} ,$$

where

$$\xi_t = \frac{\mu_t^f - r_t}{\sigma_t} . \tag{1.25}$$

Conditional on (1.25), the martingale measure $\mathbb{Q}^j$ is so that

$$d\mathbf{W}_t^{\mathbb{Q}^j} = \begin{bmatrix} \xi_t \\ \tilde{\xi}_t \end{bmatrix} dt + d\mathbf{W}_t. \tag{1.26}$$

where $\tilde{\xi}_t = \frac{\mu_t - \mu_t^f}{\tilde{\sigma}}$.

Under the measure (1.26), it hold that

$$k_s^j q_s e^{-\int_t^s \left(r_u - \frac{p_t^i - \iota_u}{q_u}\right) du} = k_t^j q_t e^{-\int_t^t \left(r_u - \frac{p_t^i - \iota_u}{q_u}\right) du} + \int_t^s \Sigma_t' d\mathbf{W}_t^{\mathbb{Q}^j},$$

$$\mathbb{E}_t^{\mathbb{Q}^j}\left[k_s^i q_s e^{-\int_t^s \left(r_u - \frac{p_t^i - \iota_u}{q_u}\right) du}\right] = k_t^i q_t + \underbrace{\mathbb{E}_t^{\mathbb{Q}^j}\left[\int_t^s \Sigma_t d\mathbf{W}_t^{\mathbb{Q}^j}\right]}_{0},$$

so that $k_t^j q_t$ is a martingale under $\mathbb{Q}^j$.

1.2.4 Asset Pricing

In this section we briefly introduce the relationship between a representative agents' problem and the pricing of a risky claim in continuous-time economies. For a more general discussion, we refer to Cochrane (2009). The results summarised in this section will be useful to better comprehend the connection between intermediaries capitalization and asset pricing proposed in the seminal paper by He and Krishnamurthy (2013) *Intermediary asset pricing*, discussed at length in Section 1.4.2.

Let us consider an endowment economy and a representative agent that maximises the inter-temporal utility of its consumption. Let us assume that the agent may invest into a risky asset by paying a price q_t, and each asset pays out dividends D_t at each instant in time t.

The agent's problem can be then written as

$$\max_{\omega} \mathbb{E}_t \int_t^{\infty} e^{-\rho(s-t)} u(c_s) ds; \quad de_s = \omega\left(D_s ds + dq_s\right) - c_s ds,$$

where the consumption between t and $t+s$ is bounded by the value of asset in the portfolio plus the stream of dividends for quantity of the risky asset ω.

The FOC on the control ω implies that

$$\mathbb{E}_t \int_t^{\infty} e^{-\rho(s-t)} \partial_c u(c_s) \left(D_s + \frac{dq_s}{ds}\right) ds = 0,$$

and so

$$\partial_c u(c_t) q_t = \mathbb{E}_t \int_t^{\infty} e^{-\rho(s-t)} \partial_c u(c_s) D_s ds \tag{1.27}$$

which is the equivalent of the classical *Euler equation* in discrete-time models. Let us define the consumption-based *Stochastic Discount Factor* (SDF) as $\Lambda_t := e^{-\rho t}\partial_c u(c_t)$ from which we write equation (1.27) as

$$\Lambda_t q_t = \mathbb{E}_t \int_t^\infty \Lambda_{t+s} D_{t+s} ds. \tag{1.28}$$

Let us now define the equilibrium equation (1.28) over a generic sub-interval $[t+\Delta, \infty]$, where $\Delta > 0$, and take the conditional expected value, so that

$$\mathbb{E}_t \left[\Lambda_{t+\Delta} q_{t+\Delta}\right] = \mathbb{E}_t \int_{t+\Delta}^\infty \Lambda_s D_s ds. \tag{1.29}$$

By taking the difference between (1.28) and (1.29), we have that

$$\Lambda_t q_t - \mathbb{E}_t \left[\Lambda_{t+\Delta} q_{t+\Delta}\right] = \mathbb{E}_t \int_t^\Delta \Lambda_s D_s ds. \tag{1.30}$$

For Δ small enough, the first term of the right hand side in (1.30) can be approximated by

$$\Lambda_t q_t \approx \Lambda_{t+\Delta} D_{t+\Delta} \Delta + \mathbb{E}_t \left[(\Lambda_{t+\Delta} q_{t+\Delta} - \Lambda_t q_t) + \Lambda_t q_t\right].$$

By taking the limit $\Delta \to 0$, it follows that

$$0 = \Lambda_t D_t dt + \mathbb{E}_t \left[d\left(\Lambda_t q_t\right)\right], \tag{1.31}$$

where, by Itô's Lemma,

$$d\left(\Lambda_t q_t\right) = q_t d\Lambda_t + \Lambda_t dq_t + dq_t d\Lambda_t,$$

that leads, once substituted in (1.31), to the dynamic pricing condition

$$0 = \frac{D_t}{q_t} dt + \mathbb{E}_t \left[\frac{d\Lambda_t}{\Lambda_t} + \frac{dq_t}{q_t} + \frac{dq_t d\Lambda_t}{q_t \Lambda_t}\right]. \tag{1.32}$$

Equation (1.32) must be satisfied by all assets traded in the economy. Thus, if we consider a risk-free bond with unitary price $q = 1$ and interest payments $D_t = r_t$, we obtain the well known condition on the dynamics of the SDF

$$r_t = -\frac{1}{dt} \mathbb{E}_t \left[\frac{d\Lambda_t}{\Lambda_t}\right], \tag{1.33}$$

that implies, for a generic risky asset that satisfies (1.32), that the risk premium is such that

$$\underbrace{\mathbb{E}_t \left[\frac{dq_t}{q_t} + \frac{D_t}{q_t} dt\right]}_{dR_t} - r_t dt = -\mathbb{E}_t \left[\frac{dq_t d\Lambda_t}{q_t \Lambda_t}\right]. \tag{1.34}$$

To further characterise the pricing equation (1.34), we may develop the dynamics of the SDF Λ_t as a function of consumption. By Itô's Lemma

$$d\Lambda_t = \partial_t \left(e^{-\rho t}\partial_c u(c_t)\right) dt + \partial_c \left(e^{-\rho t}\partial_c u(c_t)\right) dc_t + \frac{1}{2}\partial^2_{cc} u(c_t) \left(e^{-\rho t}\partial_c u(c_t)\right) dc_t^2,$$

and so

$$\frac{d\Lambda_t}{\Lambda_t} = -\rho dt + \frac{\partial^2_{cc} u(c_t)}{\partial_c u(c_t)} dc_t + \frac{1}{2}\frac{\partial^3_{ccc} u(c_t)}{\partial_c u(c_t)} dc_t^2. \tag{1.35}$$

We now characterise the result in (1.35) by considering the case of power utility as well as the limit case when the risk aversion goes to one, i.e. the log utility.

An example: power utility We consider the case of Constant Relative Risk Aversion (CRRA) utility, so that $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, where γ represents the agent relative risk aversion. In this case, the dynamics of the SDF in (1.35) reduces to

$$\frac{d\Lambda_t}{\Lambda_t} = -\rho dt - \delta \frac{dc_t}{c_t} + \frac{\delta}{2}(\gamma + 1)\left(\frac{dc_t}{c_t}\right)^2,$$

so that the asset pricing condition reads as

$$\mathbb{E}_t dR_t - r_t = \gamma \mathbb{C}ov_t \left[\frac{dq_t}{q_t}, \frac{dc_t}{c_t}\right]. \tag{1.36}$$

It the special case when $\gamma \to 1$ (log-utility) $C_t \propto E_t$, and

$$\omega = \frac{\frac{1}{dt}\mathbb{E}_t dR_t - r_t}{\frac{1}{dt}\mathbb{V}ar_t \left[\frac{dq_t}{q_t}\right]},$$

where the risk-free rate equals

$$r_t = \rho dt + \frac{1}{dt}\mathbb{E}_t \frac{dE_t}{E_t} - \frac{1}{dt}\mathbb{V}ar_t \frac{dE_t}{E_t}. \tag{1.37}$$

1.3 Baseline Models

In this section, we discuss and replicate a few relevant models in the recent macro-finance literature that further extend the toy models introduced in Section 1.2. In particular, we introduce both institutional heterogeneity between different classes of agents and (extreme) financial frictions. In the framework of this paper, the models of this section are classified as "baseline" due to their substantial analytical tractability.

The remaining of this section is structured as follows. First, 1.3.1 discusses a *"A Macroeconomic Model with Extreme Financial Frictions"* by Klimenko et al. (2017), as it embodies most methods that characterise continuous-time macro-finance models with financial frictions, such as the characterization of equilibrium dynamics by the joint solution of backward (HJB) and forward equations (Fokker-Plank). Second, in the spirit of Basak and Cuoco (1998), Section 1.3.2 introduces the most basic model with restricted market participation, where heterogeneity is not persistent and risk-free assets are introduced in the form of short term bonds. Section 1.3.2.2 steps a little forward, and discusses a simple extension that features persistent heterogeneity.

Finally, Section 1.3.3 studies the economic implications of a model featuring restricted market participation and different sources of risk (idiosyncratic plus systematic risks) where money act as a an instrument of risk mitigation (see also Brunnermeier and Sannikov, 2016b).

1.3.1 A Model with Extreme Financial Frictions

This section reviews and replicates the main results of Klimenko et al. (2017). In particular, the focus is on: a) The features of the stationary distribution; b) The relationship that exists between long-run average and the (deterministic) steady-state of the economy in equilibrium.

Outline The paper proposes a treatable framework of a macroeconomic model with extreme financial frictions. The aim of the paper is to study the endogenous fluctuation of exogenous systematic shocks due to the interaction of heterogeneous classes of agents (a more advanced framework with similar research questions is Brunnermeier and Sannikov, 2014, whose features are discussed at lenth in Section 1.4.3). Another important contribution of the paper is to discuss the "gap" between the long-run average and the correspondent steady-state of the equilibrium dynamics. Among its core results, the paper shows how approximating economic variables of interests by their steady-state values can be utterly misleading.

1.3.1.1 The Model

Frictions and productive technologies Time is continuous and the economy is populated by risk-neutral *Landlords* (L) and risk-averse *Farmers* (F) (henceforth, indexed as $l \in \mathbb{L}$ and $f \in \mathbb{F}$, respectively). There exist two different goods: consumption y_t (apples, the numéraire) and physical capital stock K_t (land). The aggregate stock of land K_t is in fixed supply and equals one.

The stock of land belongs to the landlords, and it can be used for growing apples at a rate A^i, $i \in \{L, F\}$, depending on the agent who disposes of the land. The farmers rent the stock of land at an instantaneous rate q_t, to be determined in equilibrium.

The farmers are more proficient than landlords at managing land, so that one unit of land yields apple as

$$dy_t^F = A^F dt + \sigma dW_t,$$

where dW_t is a Brownian motion defined on a probability space $(\Omega, \mathbb{P}, \mathcal{F})$ where $\mathcal{F}$ is the natural filtration. The parameters A^F and σ represent the average productivity of the farmers and the sensitivity of their "crop" returns to aggregate shocks, respectively.

The landlords are less proficient at producing apples, so that when they manage land, the produce at a rate

$$dy_t^L = A^L dt + \sigma dW_t,$$

where A^L stochastic and continuously distributed over $(0, A^F)$.

The financial frictions in the economy are *extreme*, i.e. the farmers have no endowment by themselves (no collateral), and they cannot borrow land from the landlords. Moreover, there do not exist financial markets where the agents can negotiate claims to hedge against aggregate shocks.

Since the farmers are more productive at producing apples, the equilibrium rental rate of land will follow a stochastic process over the support $(0, A^F)$ so that q_t evolves as

$$\frac{dq_t}{q_t} = \mu_t^q dt + \sigma_t^q dW_t, \tag{1.38}$$

where μ_t^q and σ_t^q are $\mathcal{F}$-adapted stochastic process and will be determined in equilibrium.

Landlords and farmers Landlords are impatient and have infinite elasticity of inter-temporal substitution. In summary each landlord will: rend land as long as $q_t \geq A^L$, crop it herself otherwise. What follows is that, given that the aggregate productivity of landlords' is defined over the continuum $(0, A^H)$, their supply $K_S(q_t)$ would also be continuous and increasing in the equilibrium rate q_t.

For the model purpose, landlords' supply is not modelled explicitly, but is assumed is a reduced form as a power function of the rental rate q_t as

$$K_t^S(q_t) = \left(\frac{q_t}{A^F}\right)^\beta, \tag{1.39}$$

where β represents the elasticity of land supply.

The farmers are instead risk-averse, and solve the following problem

$$V_0 =: \max_{\{C_t, \omega_t\}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \ln C_t dt, \tag{1.40}$$

subject to

$$\frac{dE_t}{E_t} = \omega_t \left(dy_t^F - q_t dt \right) - \frac{C_t}{E_t} dt, \tag{1.41}$$

where $\omega = \frac{K_t^D}{E_t}$ is the fraction of capital demanded by the farmers over the value of their savings. The farmers' problem (1.40) leads to the following HJB

$$\rho V_t = \max_{\{C_t, \omega_t\}} \left\{ \ln C_t + \frac{1}{dt} \mathbb{E}_t \left(\frac{\partial V}{\partial t} dt \frac{\partial V}{\partial E} dE + \frac{\partial V}{\partial q} dq + \frac{1}{2} \frac{\partial^2 V}{\partial E^2} dE^2 + \frac{1}{2} \frac{\partial^2 V}{\partial q^2} dq^2 + \frac{\partial^2 V}{\partial E \partial q} dE dq \right) \right\},$$

that, given an ansatz of the form $V_t \coloneqq v(q_t) + \frac{1}{\rho} \ln E_t$, implies the optimal policies to be

$$\frac{C_t}{E_t} = \rho; \quad \omega_t = \frac{A^F - q_t}{\sigma^2},$$

and the stationary HJB satisfies

$$\rho v_t = \ln \rho - 1 + \frac{1}{2\rho} \frac{\left(A^F - q_t\right)^2}{\sigma^2} + v_q q_t \mu_t^q + \frac{1}{2} v_{qq} \left(q_t \sigma_t^q\right)^2,$$

with Neumann boundary conditions

$$v(0) = \frac{1}{\rho} \left[\ln \rho - 1 + \frac{1}{2\rho} \frac{\left(A^F\right)^2}{\sigma^2} \right]; \quad v_q(0) = 0.$$

Competitive equilibrium and long-run dynamics The model is solved for its competitive equilibrium so that

1. The agents maximize their utility (1.40);

2. All markets (consumption and land) clear

$$\omega_t E_t = K_t^S. \tag{1.42}$$

The overall equilibrium can be characterized by finding drift and diffusion of the equilibrium process (1.38). By Itó's Lemma,

$$dE_t = \left[E_q q_t \mu_t^q + E_{qq} \left(q_t \sigma_t^q\right)^2 \right] dt + E_q q_t \sigma_t^q dW_t,$$

and matching the drift and diffusion to the farmers' dynamic budget constraint

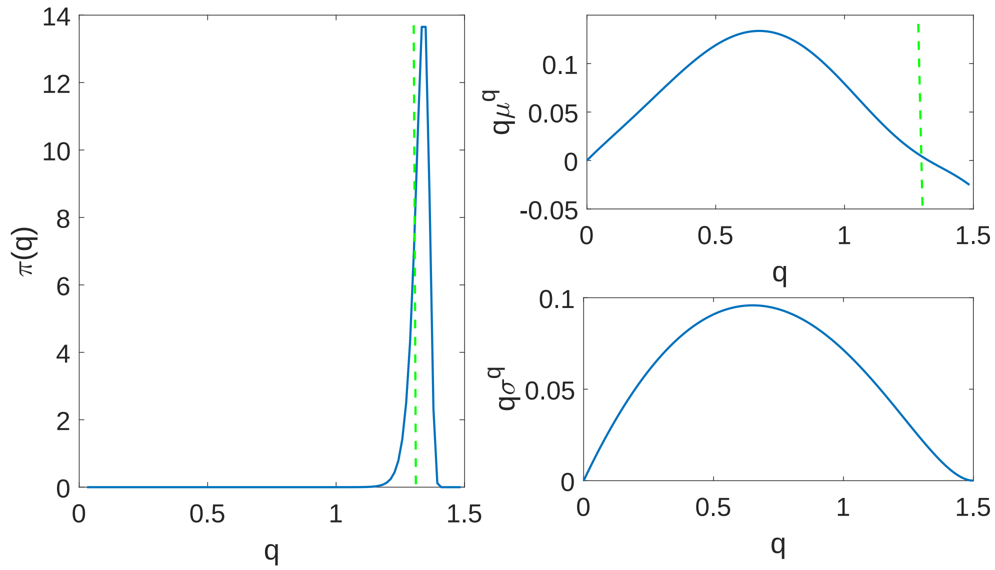

Figure 1.1: Left: Stationary distribution of the state variable q_t as a function of $q_t \in (0, A^F)$. Right: Drift (top) and diffusion (bottom) of the stochastic process of the rental rate dynamics dq_t as a function of $q_t \in (0, A^F)$. The green dashed line depicts the associated deterministic steady-state of the equilibrium.

process (1.41) under the optimal strategy $\{C_t, \omega_t\}$,

$$\mu_t^q = \frac{\frac{A^F - q_t}{\sigma^2} - \rho - \frac{E_{qq}}{E_t}\left(q_t \sigma_t^q\right)^2}{\frac{E_q}{E_t} q_t}; \quad \sigma_t^q = \frac{1}{\frac{E_q}{E} q_t}\left[\frac{A^F - q_t}{\sigma}\right].$$

By considering the reduced form for the landlords' land supply (1.39) and the market clearing (1.42), it follows that $E_t = \left(\frac{q_t}{A^F}\right)^\beta \frac{\sigma^2}{A^F - q_t}$, and so

$$\mu_t^q = \frac{\frac{A^F - q_t}{\sigma^2} - \rho - \left[4\left(\beta - 1\right)\beta + \frac{4\beta}{A^F - q_t} q_t\right]\left(\sigma_t^q\right)^2}{\beta + 2q_t}; \quad \sigma_t^q = \frac{1}{\beta + 2q_t}\left(\frac{A^F - q_t}{\sigma}\right).$$

In Figure 1.1, we plot the stationary density of the state π (left) jointly with the drift (right, top) and diffusion (right, bottom) of the process dq_t as a function of $q_t \in (0, A^F)$. Here and henceforth, we consider the following parametric specification $A^F = 1.5$, $\beta = 2$, $\rho = 0.05$, $\sigma = 0.4$. The green dashed line depicts the associated deterministic steady-state of the equilibrium.

An important contribution of the paper is the argument against the usual method for analysing the long-term behaviour of macro-variables in a DSGE. Most of times, when a global solution is not feasible, conclusions are drawn by looking at the response function of the log-linearisation around a deterministic steady-state to unanticipated ("MIT") exogenous shocks. The equivalent in this model is to consider the special trajectory of Equation (1.38) where the realization of exogenous systematic shocks is $\{dW_t\}_t = \mathbf{0}$. In such a case, the dynamics of the system would be explained by the first order ODE

$$\frac{dq_t}{dt} = \mu_t^q,$$

whose deterministic steady state would be the root $\bar{q}$ that solves the following equation

$$A^F - \bar{q} = \sigma^2 \rho \left[4\left(\beta - 1\right)\beta + \frac{4\beta\bar{q}}{A^F - \bar{q}}\right] \left[\frac{1}{\beta + 2q_t}\left(\frac{A^F - \bar{q}}{\sigma}\right)\right]^2; \quad \in (0, A^F).$$

Note that the "gap" between the long-run average of the full equilibrium dynamics and the associated deterministic steady-state $\bar{q}$ may be arbitrarily far from each other. In general, the more "spread" and asymmetric the stationary density of the state, the farther they are. In our specific case, the level of q_t with the greatest density $\pi(q)$ equals 1.346, whereas the deterministic steady state $\bar{q} = 1.303$.

1.3.2 Leverage and Restricted Market Participation

The aim of this section is to explore the mechanism beneath the easiest continuous-time model with financial frictions similar to the one proposed by Basak and Cuoco (1998) (see also Brunnermeier and Sannikov, 2016b). The model features *restricted market participation*, and two institutionally heterogeneous agents, as each of them is restricted with respect to the quality of assets she holds in her portfolio.

Outline The model fits two agents: expert (producer) and lender. At time zero, each agent has an initial capital endowment. The producer agent borrows capital from the lender at an endogenous risk-free interest rate. The lender is constraint in its holdings, and cannot dispose of physical capital by itself if not by lending it. Figure 1.2 shows a schematic representation of the interaction between producer and lender in equilibrium. The top panel depicts the agents' balance sheets at time t, while the bottom panel represents the exchange between capital endowment and risk-free bonds over dt (grey boxes).

As we shall see, the aggregate (disposable) output and consumption do not change as a response to borrower's leverage.[8] Nonetheless, leverage influences interest rates, the dynamics of wealth distribution among agents and, in turn,

[8] This aspect we address in Chapter 2, where we propose a generalization of this model by introducing idiosyncratic risk, in the spirit of Brunnermeier and Sannikov (2016a), and considering the case of restricted market participation. We aim at exploring the relationship between financial sector leverage, its risk pooling capacity, and business cycle fluctuations.

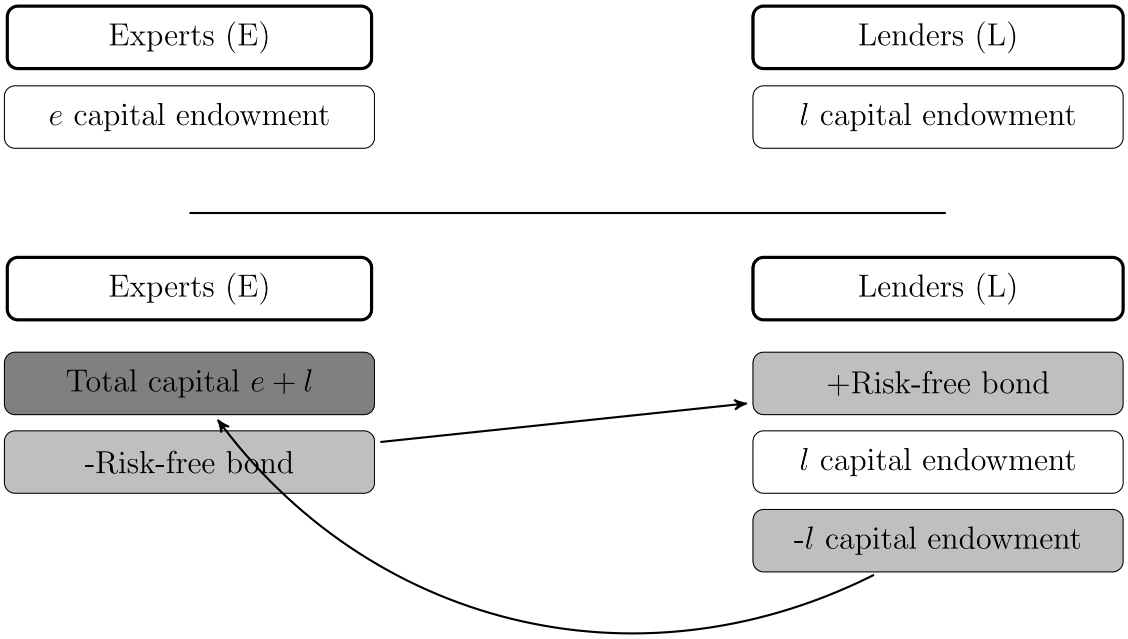

Figure 1.2: Structure of agents' interaction. Synthetic balance sheets at time t (upper panel) and $t + dt$ (lower panel).

the speed of convergence to the unique absorbing state. In the long-run, only experts survive and the model degenerate to the AK growth discussed in Section 1.2.2.[9]

1.3.2.1 The Model

Agents, frictions, and productive technologies There exist two goods, physical capital (such as a tree) and consumption (such as apples). The two goods do not exchange at a 1-to-1 ratio, as each is produced by a particular type of firms (a possible micro-foundation is discussed at length in Section 1.2.3). Time is continuous and there exist two different utility maximising agents: *expert* (E) and *lender* (L).[10]

The economy is characterised by *restricted market participation*, so that the expert only is allowed to manage physical capital over time.

The productive technologies within the economy are as introduced in Section

[9] In Section 1.3.2.2, we provide a simple variant of this model where instead heterogeneity is persistent.

[10] The model can be generalized to consider a continuum of unitary mass for each class of agents that populates the economy so that the set of experts is defined over the interval $\mathbb{E} \in [0, 1]$ and indexed by $e \in \mathbb{E}$. Similarly, lenders belong to $\mathbb{L} \in (1, 2]$ and are indexed as $l \in \mathbb{L}$. This generalization will be meaningful once idiosyncratic shocks affecting the capital stocks dynamics are introduced.

1.2.2.1. Thus, the aggregate capital stock evolves according the following SDE

$$dK_t = K_t \left[\Phi(\iota_t) - \delta + \mu_t^q + \sigma_t^q \sigma\right] dt + \sigma K_t dW_t, \tag{1.43}$$

where dW_t is a Wiener process defined over a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that represents the aggregate shocks affecting the economy, ι and δ are the re-investment and depreciation rates, respectively, and $\Phi(\iota_t)$ an increasing and concave investment function. Accordingly, the value of aggregate capital evolves with dynamics

$$\frac{d\left(K_t q_t\right)}{K_t q_t} = \left[\Phi(\iota_t) - \delta + \mu_t^q + \sigma_t^q \sigma\right] dt + \underbrace{\left(\sigma_t^q + \sigma\right)}_{\sigma_t} dW_t,$$

where μ_t^q and σ_t^q are $\mathcal{F}$-adapted stochastic processes and will be endogenously determined in equilibrium. It follows that the return on risky claims dR_t evolves as

$$dR_t = \frac{A - \iota_t}{q_t} dt + \frac{d\left(K_t q_t\right)}{K_t q_t}. \tag{1.44}$$

At time $t = 0$, each agent is endowed with initial wealth stock E_0^i, $i \in \{l, e\}$, and the value of their wealth stock $E_0^l + E_0^e = K_0 q_0$ sums up to the total stock of physical capital within the economy.

Both lender and expert discount the future at a common fixed rate ρ. They choose consumption and allocates their wealth to risk-free bonds and risky claims with returns (1.44) to maximize their inter-temporal log-utility. As we shall see, due to restricted market participation, the lender will allocate its whole wealth stock in risk-free bonds and, accordingly, the expert will leverage out all lender's capital endowment. Formally, the agents' problems read as follows:

$$V_0 \coloneqq \max_{\left\{C_t^i, \omega_t^i\right\} \in B_i} \mathbb{E}_0 \int_0^\infty \ln C_t^i dt; \quad i \in \{l, e\}, \tag{1.45}$$

subject to

$$\frac{dE_t^i}{E_t^i} = \omega_t^i dR_t + \left(1 - \omega_t^i\right) r_t dt - \frac{C_t^i}{E_t^i} dt, \tag{1.46}$$

where ω_t^i and C_t^i represent the portfolio and consumption choices, respectively. The optimal strategy of the agents' are such that (see the computations in Appendix A.1, when $\gamma \to 1$)

$$\frac{C_t^i}{E_t^i} = \rho; \quad \omega_t^i = \frac{\mu_t - r_t}{\sigma_t^2}. \tag{1.47}$$

As we shall see, in equilibrium, due to the restricted market participation, $\omega_t^l = 0$, while $\omega_t^e \geq 1$.

Competitive equilibrium Informally, the equilibrium of this economy is defined as a map from histories of exogenous systematic shocks to all relevant aggregates so that:

1. Agents maximize their utility;
2. Firms' maximise their profits;
3. All markets clear.

The equilibrium is characterized by postulating a stochastic process for the price of physical capital q_t first (see also Section 1.2.2.1), by defining proper state variables then. In this case, the state is defined as the relative share of wealth owned by experts ψ_t:

$$\psi_t := \frac{E_t^e}{E_t^e + E_t^l} \in [0, 1] .$$

Within the framework of this model, it is possible to show (see Appendix A.3) that both the equilibrium price q and the re-investment rate ι are constant and equal

$$q = \frac{1 + \theta A}{1 + \theta \rho}, \tag{1.48}$$

$$\iota = \frac{1}{\theta} \left(\frac{1 + \theta A}{1 + \theta \rho} - 1 \right), \tag{1.49}$$

where (1.49) holds positive as long as $A > \rho$. Equation (1.48) implies that μ_t^q and σ_t^q equal zero.

Due to the restricted market participation, the expert leverage the whole capital stock from the lender's balance sheet. Thus, $\omega_t^e = \frac{1}{\psi_t}$ and, by (1.47), the risk-free interest rate equals

$$r_t = \frac{A - \iota}{q} + \Phi(\iota) - \delta - \frac{\sigma^2}{\psi_t}.$$

Finally, it is possible to show (see Appendix A.3) that the state variable ψ_t evolves with dynamics

$$\frac{d\psi_t}{\psi_t} = \frac{(1 - \psi_t)^2}{\psi_t^2} \sigma^2 dt + \frac{(1 - \psi_t)}{\psi_t} \sigma dW_t, \tag{1.50}$$

where both drift and diffusion components are positive and proportional to the expert's short position. Accordingly, the higher her leverage, the more sensitive is the dynamics of the state variable to exogenous systematic shocks. It is also relevant to highlight that $\lim_{\psi_t \to 0} r_t = -\infty$, i.e. an infinitely big risk premium is required to hold risky asset.

To complete the picture, before we discuss the equilibrium dynamics in the long-run, in Figure 1.3 we plot the drift (top, left) and diffusion (top, right)

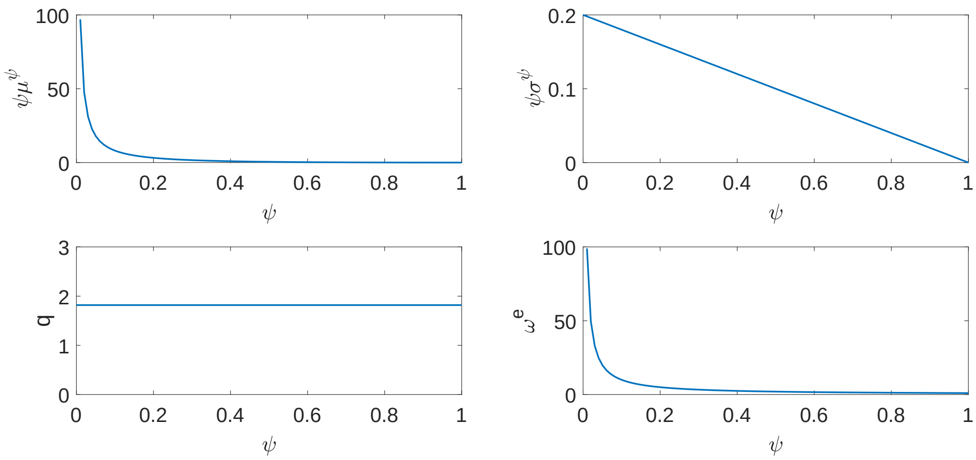

Figure 1.3: State drift (top, left), diffusion (top, right), capital price (bottom, left), and expert's leverage (bottom, right) as a function of the state. Baseline parameters: $\sigma = 0.2$, $\rho = 0.05$ $\theta = 2$

of the state process, as well as the equilibrium price (bottom, left) and expert leverage (bottom, right).

Equilibrium in the long-run In the setting described so far, the equilibrium is not characterised as a point in the state space only (its deterministic steady state). On the contrary, the state space exists jointly with its distribution that, under proper condition, does not trivially decay to a unique absorbing state.

In such a framework, studying the equilibrium dynamics as a response after unexpected exogenous perturbation (impulse-response analysis) may be misleading. In this sense, the traditional log-linearised equilibrium mechanism does not properly represent a dynamic system subject to highly non-linear fluctuations, where the agents' choice endogenously affect the sensitivity of the economy to systematic shocks.

The strength of continuous-time method also leans on the possibility of characterizing a probability distribution of the equilibrium over the whole state space by solving a PDE, the so-called *Fokker-Plank Equation* (FP or *Kolmogorov Forward*).[11]

Let us define $\pi(t, \psi_t)$ as the density function that associates to of the (Markov) stochastic differential equation

$$\frac{d\psi_t}{\psi_t} = \mu_t^{\psi} dt + \sigma_t^{\psi} dW_t.$$

[11]While the solution to the HJB equation gives the optimal strategy of the agents at each t conditional on their expectations, the FP gives the distribution of the state given the agents' optimal choices. For a theoretical reference see Bjork (2009) and Øksendal (2013).

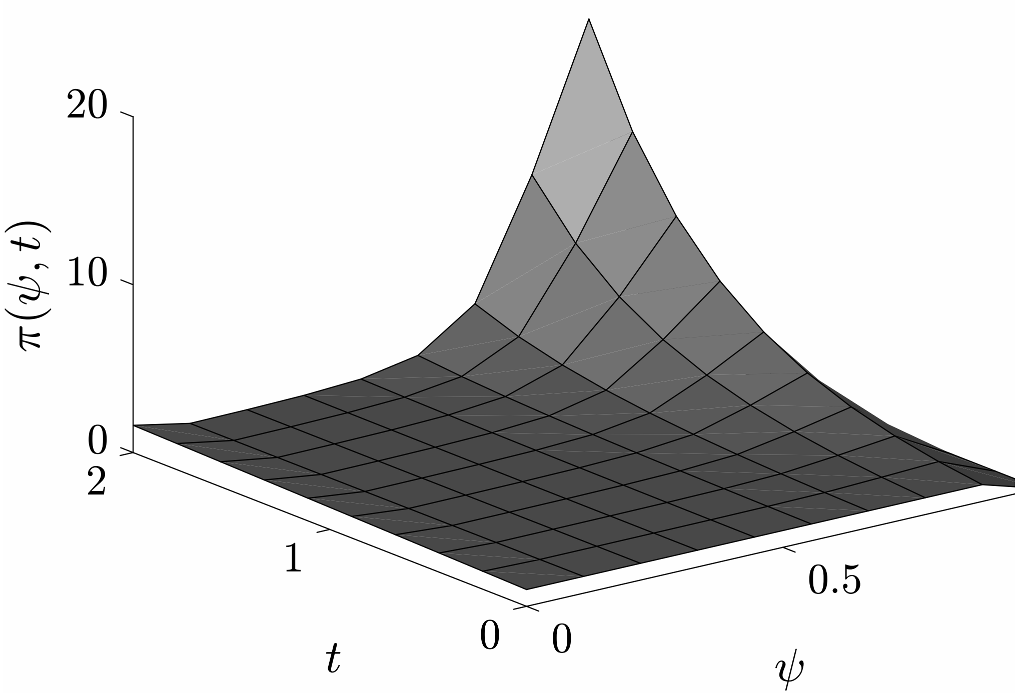

Figure 1.4: Numerical solution of the FP equation over the time interval $t \in [0; 2]$. Parameters: $\sigma = 0.2$.

Then, π is solution to the PDE:

$$\frac{\partial \pi}{\partial t}(t, \psi_t) = -\frac{\partial}{\partial \psi}\left\{\psi \mu_t^{\psi} \pi(t, \psi_t) - \frac{\partial}{\partial \psi}\left[\frac{1}{2}\psi^2\left(\sigma_t^{\psi}\right)^2 \pi(t, \psi_t)\right]\right\}. \tag{1.51}$$

Whereas the solution of (1.51) gives the *transitional* distribution conditional on time, its solution by setting $\frac{\partial \pi}{\partial t}(t, \psi_t) = 0$ corresponds to the stationary density of ψ.

In Figure 1.4, we plot the numerical solution (via Matlab PDEPE solver) of (1.51) over a sub-interval of time $t \in [0; 1,000]$. It is clear that, as the drift of (1.50) holds positive for each $\psi_t \in (0, 1)$, the unique absorbing state is $\psi = 1$. This feature is due to the advantage of expert versus the lender.

To conclude, we briefly comment on the transitional dynamics of some important variables of model (1.50), as for $t \to \infty$ there is no need to solve (1.51), and $\pi(1)$ is a Dirac centred in one. To this purpose we consider the following parametric values: $\sigma = 0.2$, $\rho = 0.05$, $\delta = 0.05$, $\theta = 2$, and $\psi_0 = 0.5$. Figure 1.5 (top) plots two simulated paths of ψ in blue and green, respectively. Each line maps a history of exogenous systemic shocks to the share of wealth ψ_t over $T = 1,000$ periods. In the same figure, we plot the corresponding risk-free rate r_t (middle) and the Sharpe ratio ξ_t (bottom).

What stands out is that, when the expert is relatively under-capitalized, the wealth share process becomes much more volatile. In those states, the risk-free

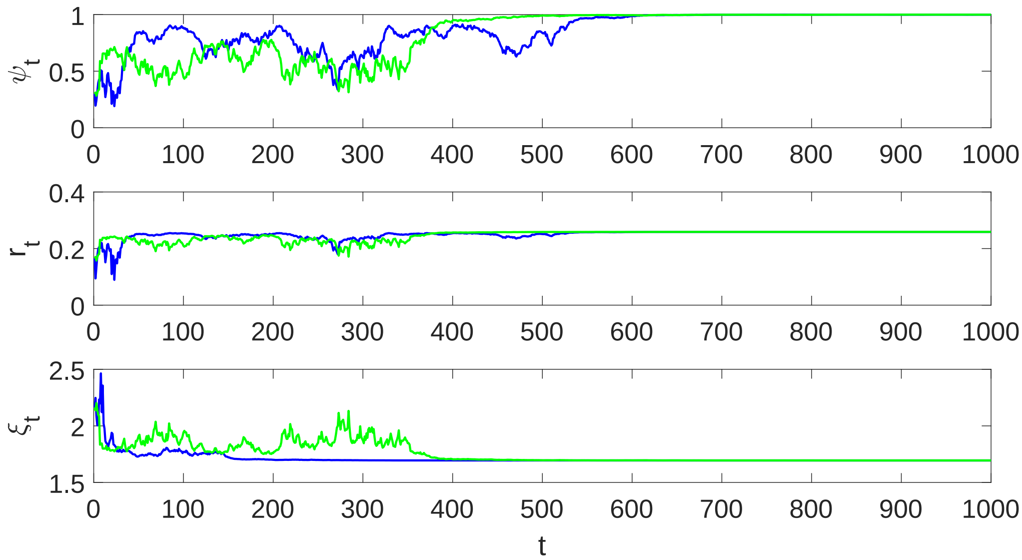

Figure 1.5: Two simulated paths of the relative wealth share dynamics (top) and the corresponding risk-free rate (middle) and Sharpe ratio (bottom). Baseline parameters: $\sigma = 0.2$, $\rho = 0.05$, $\delta = 0.05$, $\theta = 2$, and $\psi_0 = 0.5$.

rate shrinks considerably, due to the expert's high leverage. Accordingly, the states where risk-free rate is lower are also those with a higher Sharpe ratio.

Conversely, when ψ_t approaches the absorbing state, r_t converges to its long run level. What is also relevant to highlight is that the transition before the economy reaches $\psi = 1$ may be arbitrarily long, and relates to the volatility parameter σ.

1.3.2.2 Leverage and Persistent Heterogeneity

In this Section we develop a simple framework that extends model 1.3.2.1 in a way that *heterogeneity is persistent*, i.e. both agents survive in the long-run. In particular, we do so by re-interpreting both the expert and lender as follows.

The model key assumptions hold as before, but the expert can be read as a *financial intermediary* (f) who collects resources from the *household* (h), its unique shareholder. While the latter cannot invest in physical capital by herself, it supplies its wealth stock to the intermediary who pays out dividend flows to maximise its market value for its shareholder. Moreover, we assume that the financial institution and the household discount the future at different rates, $\rho^f < \rho^h$, i.e. the household is less patient than the intermediary.

Therefore, the households' problem holds as in (1.45), but its dynamic balance sheet changes as

$$B_t^h : \frac{dE_t^h}{E_t^h} = \omega_t^h dR_t + (1 - \omega_t^h) r_t dt - \frac{C_t^h}{E_t^h} dt + \frac{\Delta_t}{E_t^h} dt,$$

where Δ_t is the instantaneous flow of dividends that she receives from the financial intermediary. Note that the household does not internalize the optimal dividends payout policy of the intermediary, and she takes it as given.

In a similar fashion, the intermediary problem changes af follows:

$$V_0 =: \max_{\{\delta_t,\omega_t\}\in B_t^f} \mathbb{E}_0 \int_0^\infty e^{-\rho^e t} \ln \delta_t dt,$$

subjected to the dynamic budget constraint

$$B_t^f : \frac{dE_t^f}{E_t^f} = \omega_t^e dR_t + (1-\omega_t^f) r_t dt - \Delta_t dt,$$

where $\Delta_t = \frac{\delta_t}{E_t^f}$ represents the dividend pay out rate with respect to the intermediary book value. It is relevant to highlight that the value function V_0 represents the current expected value of the future dividend stream paid out by the intermediary. Thus, it equals its market value. In this framework, the log utility can be read as a risk adjusted capital requirement that prevents the intermediary from defaulting. As we shall see, in equilibrium, the flow of dividends paid out by the banking sector prevents the household (lender) from disappearing, i.e. both agents survive in the long-run.[12]

Equilibrium and state As already discussed in Section 1.3.2.1, the competitive equilibrium is defined as a map from histories of systematic shocks $\{W_t\}_{t\in(0,\infty)}$ to all relevant economic aggregates so that intermediaries and households maximise their utility, firms maximise their profits, and all markets clear. Once again, the state variable will be the intermediary relative share of wealth, or $\psi_t =: \frac{E_t^f}{E_t^f + E_t^h}$. Given the optimal strategies of the agents' jointly with the market clearing conditions, intermediary's and household's wealth stocks evolve as

$$\frac{dE_t^f}{E_t^f} = \left(r_t + \frac{\mu_t - r_t}{\psi_t} - \rho_t^f \right) dt + \frac{\sigma_t}{\psi_t} dW_t,$$

and

$$\frac{dE_t^h}{E_t^h} = \left[r_t - \rho^h + \rho^f \left(\frac{\psi_t}{1-\psi_t} \right) \right] dt.$$

By Itô's Lemma, ψ_t has dynamics

$$\frac{d\psi_t}{\psi_t} = \left(\bar{\rho} - \psi_t \rho^h \right) dt + \sigma \frac{1-\psi_t}{\psi_t} \left[1 + \theta \rho^h (1-\psi_t) \right] dW_t,$$

where $\bar{\rho}$ is the gap between household's and intermediary's inter-temporal dis-

[12] A similar approach in a dynamic model with a financial intermediary is for instance in He and Krishnamurthy (2019), where the households receive an exogenous wage.

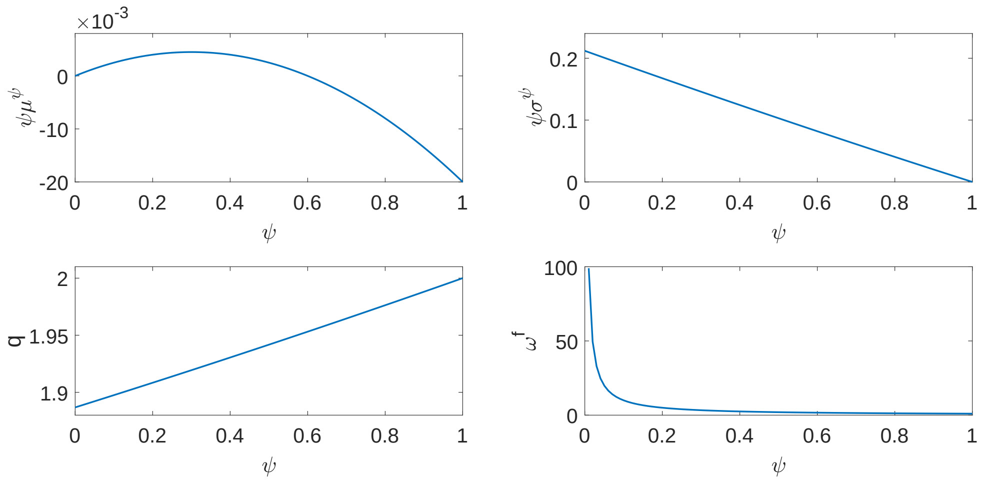

Figure 1.6: Top: Drift (left) and diffusion (right) of the state process $d\psi$. Bottom: Equilibrium price of capital (left) and intermediaries' financial leverage (right). Baseline parameters: $\sigma = 0.2$, $\rho = 0.05$, $\delta = 0.05$, $\theta = 2$, and $\psi_0 = 0.5$.

count rates.[13] To conclude, Figure 1.6 depicts drift (left) and diffusion (right) of the state prices (top panel). In the same Figure (bottom panel) we report the equilibrium price q (left) and financial leverage ω^f (right).

What stands out is that, in general, states of high financial capitalization associate to low volatility, leverage, and higher prices. Accordingly, those are also the states where ψ decreases sharply, since it distributes higher absolute dividend flows. On the contrary, those states where financial relative capitalization is lower are also those where the financial sector grows the most, due to its high leverage and low (absolute) dividend payouts.

The features of the equilibrium can be better understood by looking at the stationary density of ψ. To do so, Figure 1.7 plots the volatility of the returns on risky claims (left) jointly with the stationary density $\pi(\psi)$ (right). What is relevant to highlight is that, since all states where the financial sector is poorly capitalised are also those featuring higher financial leverage, in those states the volatility is the highest. Thus, they may be highly persistent. Overall, the financial sector's wealth always grows faster than the households', due to leverage.

[13]The endogenous volatility component σ_t^q, that determines the volatility σ_t, can be obtained by solving the first order ODE

$$\sigma_t^q q_t = \frac{\partial q}{\psi_t} \psi_t \sigma_t^\psi,$$

and equals

$$\sigma_t^q = \sigma \frac{\frac{\theta \rho^h (1-\psi_t)}{1+\theta \rho^h (1-\psi_t)}}{1 - \frac{\theta \rho^h (1-\psi_t)}{1+\theta \rho^h (1-\psi_t)}}.$$

A sketch of the proof is in Appendix A.4.

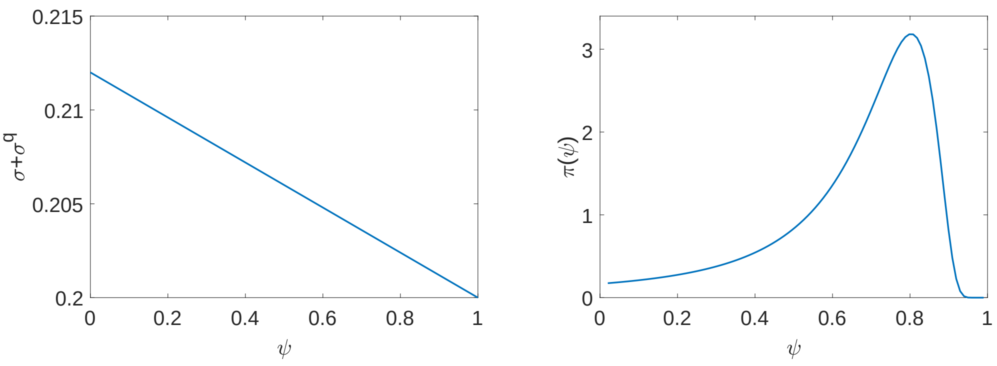

Figure 1.7: Left: total volatility of risky claims. Right: stationary state density. Baseline parameters: $\sigma = 0.2$, $\rho = 0.05$, $\delta = 0.05$, $\theta = 2$, and $\psi_0 = 0.5$.

Therefore, the density mass is concentrated to the states with relatively high financial capitalization.

1.3.3 Money as a Risk-free Asset

In this section we introduce a simplified model featuring money as a risk-free asset. This setting will serve as a baseline to disclose the features and mechanism of the seminal paper *"The I Theory of Money"* (Brunnermeier and Sannikov, 2016a)[14]

The modelling structure is similar to the baseline discussed in Section 1.3.2.1. Nevertheless, it fundamentally differs under several aspects. First, experts who manage physical capital are exposed to exogenous idiosyncratic risks on top of a common systematic one. Moreover, as the financial markets are incomplete, experts cannot diversify the idiosyncratic risk of their capital holding by themselves.

Second, lenders, whose market participation is still fundamentally restricted, are endowed with a money-like asset that does not pay any dividends flows. Differently from previous models, there does not exist a risk-free bond. Money is a special (nominal) asset whose role is to store value, as the dynamics of its price is affected by systematic shocks only. As we shall see, in equilibrium, money is valuable, and it allows experts to deal with idiosyncratic risks (money is a bubble, in the sense of Tobin, 1965). In this term, the lender act as a financial sector, and supplies the experts a tool through which they hedge against idiosyncratic shocks.

We proceed as follows: first we reproduce and discuss both structure and main features of the model. Then, we focus on the role and meaning of the

[14]We postpone a full-length discussion of "The I Theory" to Section 1.4.4. For a general introduction to money as a risky-free asset see also Brunnermeier and Sannikov (2016b).

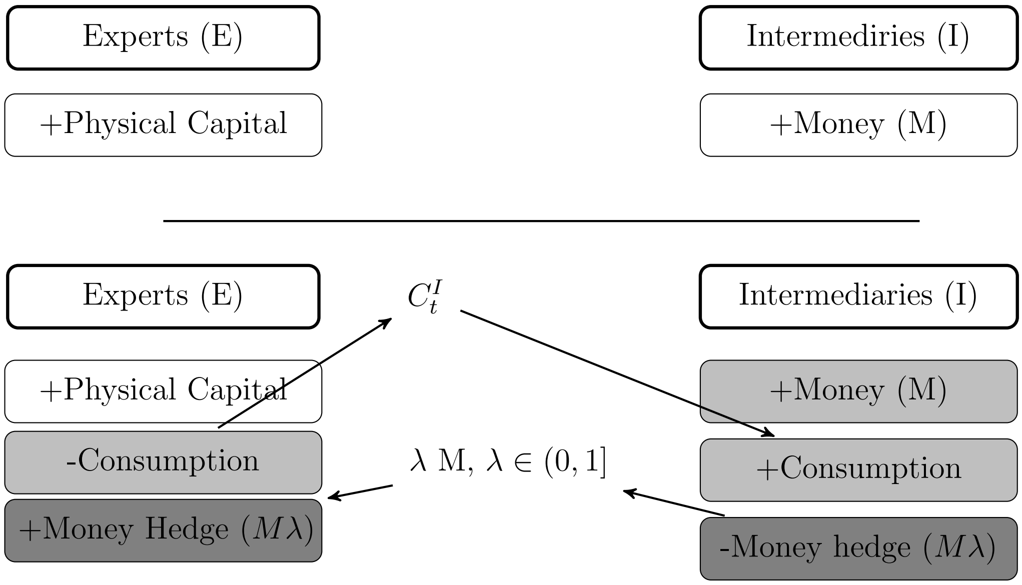

Figure 1.8: Structure of agents' interaction in the M model. Synthetic balance sheets at time t and $t + dt$ in upper and lower panels, respectively.

money asset in equilibrium.

Overview The model fits two classes of agents, lenders and experts (henceforth, intermediaries). Intermediaries are endowed with a fixed amount of money asset and are restricted in their assets market participation, so that they cannot hold physical capital. On the contrary, experts are endowed with an initial stock of physical capital access the risky claim issued by their own capital producing firm.

The capital dynamics is affected by both systematic and idiosyncratic shocks, whereas the dynamics of money directly evolves after systematic shocks only. In this setting, intermediaries have a role as insurer since, by issuing money asset for buying consumption, they allow experts to partially hedge their exposure to idiosyncratic risk.

In Figure 1.8, we show a snapshot representation of the equilibrium interaction among agents in the M model. At each instant of time t, experts are willing to purchase a fraction λ of intermediaries' money in exchange for consumption C_t^I. In the long-run, since experts are the only ones allowed to deal with capital producing firms, they pump all money out the intermediaries' balance sheet.

In the top panel, we depict the balance sheet of agents' at time t. In the bottom panel we represent by (light grey boxes) the consumption versus money flows among experts and intermediaries over dt. The dark grey boxes depict the increasing stock of money accumulated by experts over time.

1.3.3.1 The Model

Agents, frictions, and productive technologies There exist three goods: physical capital (such as a tree), consumption (such as apples), and money. As in model 1.2.2.2, consumption and capital do not exchange at a 1-to-1 ratio, and each good is produced by a particular type for firms. Moreover, capital producing firms are affected by both idiosyncratic and systematic shocks. On the contrary, money is "bubble" asset that does not pay any cash flows and it exists in a fixed aggregate stock M. Time is continuous and there exist two different classes of utility maximising agents: *experts* $\mathbb{E} \in [0,1]$, indexed as $e \in \mathbb{E}$, and *intermediaries* $\mathbb{I} \in (1,2]$, indexed $i \in \mathbb{I}$. As in model (1.3.2.1), both classes are infinitely lived, discount the future as a common fixed rate ρ, and maximise the inter-temporal log-utility of their consumption.

At time $t = 0$, each expert is endowed with an initial wealth stock in physical capital $e_0^e = k_0^e q_0$, where q is the price of capital. In the aggregate, capital endowments across experts sums up to the whole stock within the economy $\int_{\mathbb{E}} e_0^e de = E_0^e = K_0 q_0$. Similarly, each intermediary is endowed with an initial stock of wealth in money $e_0^i = m^i \tilde{p}_0$, where $\tilde{p}_0$ is the price of money. Thus, the money endowment across intermediaries sums up to the aggregate money stock within the economy $\int_{\mathbb{E}} e_0^i di = E_0^i = M\tilde{p}_0$.

As the aggregate money endowment is fixed and exogenous, it is useful to normalize the price of money $\tilde{p}$ with respect to the $M\tilde{p}_0 = K_0 p_0$, so that the aggregate wealth within the economy can be expressed in terms of capital stock as $K_0 (p_0 + q_0)$, where $p_0 = \frac{M\tilde{p}_0}{K_0}$.

Further relevant assumptions characterizing the economy are extreme financial frictions, so that intermediaries' market participation is restricted to hold money only. As such, experts are the only ones allowed to hold physical capital. Moreover, experts' financial markets are utterly incomplete, and they cannot diversify the idiosyncratic exposure of their risky claims by trading them among each other.

In summary, experts' and intermediaries' wealth stock evolve with dynamics

$$\frac{de_t^e}{e_t^e} = dR_t^m + \omega_t^e \left(dR_t^k - dR_t^m \right) - \frac{c_t^e}{e_t^e} dt, \tag{1.52}$$

$$\frac{de_t^i}{e_t^i} = dR_t^m - \frac{c_t^i}{e_t^i} dt, \tag{1.53}$$

where c_t is the agents consumption, while dR_t^m and dR_t^k depict the return on money and capital, respectively.

As experts are heterogeneous (and each of them accesses her own firm only), there also exists a continuum of capital producing firms indexed $j \in \mathbb{J}$. Thus, the stock of capital under each firm's management has dynamics

$$\frac{dk_t^j}{k_t^j} = \left[\Phi(\iota_t^j) - \delta + \mu_t^q + \sigma_t^q \sigma\right] dt + \underbrace{\sigma dW_t}_{\text{Systematic risk}} + \underbrace{\tilde{\sigma} d\tilde{W_t}^j}_{\text{Idiosyncratic risk}} \tag{1.54}$$

where ι_t is the re-investment rate, δ the depreciation rate, and $dW_t \perp d\tilde{\tilde{W}}_t^j \perp d\tilde{W}_t^z$, $j \neq z$, are Wiener processes defined over a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and represent systematic and idiosyncratic risk sources affecting capital stock dynamics. As we shall see, the presence of idiosyncratic risks is the core feature that makes money valuable, as it may be used as instrument of partial risk insurance by experts.[15]

Competitive equilibrium The competitive equilibrium is defined as a map from histories of systematic shocks to the all relevant aggregates so that:

1. Agents maximize their utility;
2. The firms' maximise their profits;
3. All markets clear (capital, money, and consumption).

A first interesting relationship that connects the price of money to the price of capital can be obtained by firms optimal investments $\partial_\iota \Phi = \frac{1}{q_t}$ (for details, see Section 1.2.2.1) jointly with the market clearing condition on consumption

$$\rho\left(E_t^e + E_t^i\right) = \rho K_t\left(p_t + q_t\right) = \left(A - \iota_t\right) K_t, \tag{1.55}$$

leading to

$$q_t = \frac{1 + \theta A + \theta \rho p_t}{1 + \theta \rho}.$$

Of course, the price of capital in an equilibrium with valueless money ($p_t = 0$) coincides with the equilibrium outcome of a model with no money asset at all. Not surprisingly, an increase in the price of physical good is associated with a decrease in price of money

$$\frac{\partial q_t}{\partial p_t} = -\frac{\theta \rho}{1 + \theta \rho}.$$

Equilibrium characterization Similarly to all models with adjustment costs described in the previous sections, to characterise the equilibrium one shall: first, guess a proper stochastic process for the price of both money and physical capital. Second, define a proper state variable. As before, the unique state will

[15] In the aggregate, idiosyncratic shocks are pooled away, and the capital stock within the economy K_t has dynamics as in (1.43). However, due to market incompleteness, idiosyncratic shocks still affect experts' welfare through their portfolio choices.

be experts' (aggregate) relative wealth share. Thus, we look for an equilibrium where prices dynamics dq_t and dp_t evolve as diffusion processes

$$dq_t = q_t\mu_t^q dt + q_t\sigma_t^q dW_t, \tag{1.56}$$

$$dp_t = p_t\mu_t^p dt + p_t\sigma_t^p dW_t, \tag{1.57}$$

where μ_t^q, μ_t^p, σ_t^q, and σ_t^q are $\mathcal{F}$-adapted stochastic processes whose values are jointly determined in equilibrium. In this context, it is meaningful to assume that idiosyncratic shocks do not to affect the dynamics of equilibrium prices, as markets are perfectly competitive and they are determined by (aggregate) market clearing.

By Itô's Lemma, the returns on investments in risky money and capital assets in equilibrium are described by the following SDEs[16]

$$dR_t^k = \underbrace{\frac{A-\iota_t}{q_t}dt + \left[\Phi(\iota_t) - \delta + \mu_t^q + \sigma_t^q\sigma\right]}_{\mu_t^k} dt + \left(\sigma + \sigma_t^q\right) dW_t + \tilde{\sigma} d\tilde{W}_t, \tag{1.58}$$

$$dR_t^m = \underbrace{\left[\Phi(\iota_t) - \delta + \mu_t^p + \sigma_t^p\sigma\right]}_{\mu_t^m} dt + \left(\sigma + \sigma_t^p\right) dW_t. \tag{1.59}$$

By substituting (1.58) and (1.59) into (1.52) and (1.53), respectively, jointly with the agents' optimal consumption strategy, the agents' dynamic budget constraints evolve as

$$\frac{dE_t^e}{E_t^e} = \left(\mu_t^m - \rho\right) dt + \omega_t^e\left(\mu_t^k - \mu_t^m\right) dt + \left[\left(\sigma + \sigma_t^p\right) + \omega_t^e\left(\sigma_t^q - \sigma_t^q\right)\right] dW_t + \underbrace{\omega_t^e\tilde{\sigma}\int_{\mathbb{E}} d\tilde{W}_t}_{=0},$$

$$\frac{dE_t^i}{E_t^i} = \left(\mu_t^m - \rho\right) dt + \left(\sigma + \sigma_t^p\right) dW_t.$$

What follows is that the agents' SDFs (see also Section 1.2.3) evolve with dynamics

$$\frac{d\Lambda_t^e}{\Lambda_t^e} = -r_t^e dt - \xi_t^e dW_t - \tilde{\xi}_t d\tilde{W}_t,$$

$$\frac{d\Lambda_t^i}{\Lambda_t^i} = -r_t^i dt - \xi_t^i dW_t,$$

[16] As both the value of money and capital stock are linear functions of prices and capital, the square terms of the expansion equal zero, and

$$dR_t^k = \frac{A-\iota_t}{q_t}dt + \frac{d\left(k_t p_t\right)}{j_t p_t} = \frac{A-\iota_t}{q_t}dt + \frac{1}{k_t q_t}\left[\frac{\partial\left(k_t q_t\right)}{\partial k_t}dk_t + \frac{\partial\left(k_t q_t\right)}{\partial q_t}dq_t\right],$$

$$dR_t^m = \frac{d\left(k_t p_t\right)}{k_t p_t} = \frac{1}{k_t p_t}\left[\frac{\partial\left(k_t p_t\right)}{\partial k_t}dK_t + \frac{\partial\left(k_t p_t\right)}{\partial p_t}dp_t\right].$$

where ξ_t and $\tilde{\xi}_t$ are the prices of systematic and idiosyncratic risks, respectively, whereas r_t^e and r_t^i are the shadows risk-free rates. Note that, in principle, the two rates may be different, as there does not happen any trading of risk-free assets neither between nor within each class of agents.

State dynamics and numerical solution The model can be solved by looking for those equilibria where the dynamics of prices (1.56) and (1.57) evolve deterministically, i.e. $\sigma_t^q = \sigma_t^p = 0$. From this assumption, it follows that both experts' and intermediaries' wealth dynamics have constant exposition to exogenous systematic shocks (and equal to σ). This implies that the kernel they use to price systematic risk is common, i.e. $\xi_t^e = \xi_t^i = \xi_t$. As the portfolio choice of the intermediaries' is constrained by restricted market participation, ξ_t is pinned down by their optimal portfolio choice

$$\frac{\mu_t^m - r_t}{\sigma^2} = 1 \Longrightarrow \xi_t = \sigma. \tag{1.60}$$

Equation (1.60) implies that, even though agents do not directly trade on risk-free assets, they agree on the equilibrium risk-free rate, as it is indirectly defined by their trading in money assets, and the risky claims written on capital stock can be priced accordingly. In general, we know that (see for example Bjork, 2009) the risk premium of a risky asset n must hold as

$$\mu_t^n - r_t = \sum_N \sigma_t^n \xi_t^n,$$

where N is the number of risk sources to which asset n is exposed. In our specific case,

$$\mu_t^k - r_t = \xi_t \sigma + \tilde{\xi}_t \tilde{\sigma},$$

and, by experts' optimal strategy,

$$\tilde{\sigma}^2 \omega_t^e = \mu_t^k - \mu_t^m \Longrightarrow \tilde{\xi}_t = \tilde{\sigma} \omega_t^e. \tag{1.61}$$

We now have all the elements to characterize the dynamics of experts' relative share of wealth ψ_t, defined as

$$\psi_t := \frac{E_t^e}{E_t^e + E_t^i} \in [0, 1], \tag{1.62}$$

that evolves with dynamics given by the following ODE (a sketch of the proof is in Appendix A.5)

$$\frac{d\psi_t}{dt} = \tilde{\sigma} \left(\omega_t^e\right)^2 \psi_t \left(1 - \psi_t\right), \tag{1.63}$$

where

$$\omega_t^e = \frac{1}{\psi_t}\left(\frac{q_t}{q_t + p_t}\right). \tag{1.64}$$

What stands out is that, in general, equilibria where money are more valuable are also those where experts allocate a lower share of their wealth to capital in order to hedge their exposure to idiosyncratic shocks. Accordingly, the expected growth rate of ψ is lower.

Equilibrium in the long-run and "financial deepening" Since the drift of (1.63) holds positive for every ψ, when $t \to \infty$ then q and p converge to their long-run (constant) values, and $\mu_t^q = \mu_t^p = \mu^\psi = 0$, while $\omega = \frac{q}{p+q}$. By market clearing equations for physical capital

$$\omega_t^e E_t^e = K_t q_t, \tag{1.65}$$

money,

$$\left(1 - \omega_t^i\right) E_t^i + (1 - \omega_t^e) E_t^e = K_t p_t,$$

and consumption (1.55), equilibrium prices and investments are jointly defined and equal

$$\iota = \frac{q-1}{\theta}, \tag{1.66}$$

$$q = \frac{1 + \theta(A - \rho p)}{1 + \theta\rho},$$

$$p = \frac{1 + \theta A - (1 + \theta\rho)}{\theta\rho}.$$

What is relevant to stress is that the equilibrium where money is valuable exists for values of $p > 0$ so that

$$q < \frac{1 + \theta A}{1 + \theta\rho},$$

where q is lower than it would be if money was not valuable. This implies that in a model with idiosyncratic risks where money is valuable experts are less willing to bear risk and, in turn, investments are lower (see Equation 1.66). This reflects on the growth rate of the economy that is proportional to the growth rate of physical capital stock (1.54) and, in expectation,

$$\mathbb{E}_t \frac{dK_t q}{K_t q} = \frac{1}{\theta} \ln(q) - \delta.$$

To better understand the relationship that holds between equilibrium prices and idiosyncratic risk, it may be useful to write q and p as functions of $\tilde{\sigma}$ at the absorbing state. By capital market clearing and optimal portfolio Equations

(1.64) and (1.65) we know that

$$\frac{\mu^k - \mu^m}{\tilde{\sigma^2}} = \omega = \frac{q}{q+p}, \tag{1.67}$$

while the risk premium on risky assets returns over money equals

$$\mu^k - \mu^m = \frac{A-\iota}{q}. \tag{1.68}$$

By matching (1.67) and (1.68), the following equality must hold

$$\frac{A-\iota}{q}\frac{1}{\tilde{\sigma}^2} = \frac{q}{p+q},$$

and, rearranging,

$$p = \frac{\tilde{\sigma} - \sqrt{\rho}}{\sqrt{\rho}} q. \tag{1.69}$$

From (1.69) it is clear that, for a positive q, a money equilibrium exists as long as idiosyncratic risk is big enough $\tilde{\sigma} > \sqrt{\rho}$. In this sense, the bigger the volatility of idiosyncratic shocks, the more valuable is the money asset. Finally, by substituting (1.69) into (1.69) and rearranging,

$$q = \frac{1+\theta A}{1+\theta\tilde{\sigma}\sqrt{\rho}}.$$

By the last few equations, it is clear what in (Brunnermeier and Sannikov, 2016b,a) is defined as *"financial deepening"* effect: as long as financial intermediaries manage to reduce the households' exposure to idiosyncratic risks, i.e. the financial sector deepens, the equilibrium price of physical capital rises, while the value of money reduces. In turn, this fosters economic grows as it stems into a higher investment rate ι (see Equation 1.66). Milder idiosyncratic fluctuations also associate to higher risk-free interest rates.[17]

The nature of the relationship between idiosyncratic risks, prices, and economic growth strictly relates to the nature of financial frictions considered in the model and, in turn, to the degree of institutional heterogeneity among different classes of agents. In this term, the effect of reducing idiosyncratic risks over the dynamics of (relative and) aggregate wealth remains unclear as it strictly depends on the functions of the financial sector that are considered within the

[17] The correspondent shadow risk-free interest rate of this economy equals

$$r = \rho\left(\frac{p+q}{q}\right) + \frac{1}{\theta}\log\left[\frac{1+\theta A}{1+\theta\sqrt{\rho}\tilde{\sigma}}\right] - \delta - \sigma^2 - \left(\frac{q}{p+q}\right)\tilde{\sigma^2},$$

which reduces, under the assumption af complete financial markets -no idiosyncratic risk-, to one implicit in the absorbing state of model 1.3.2.1.

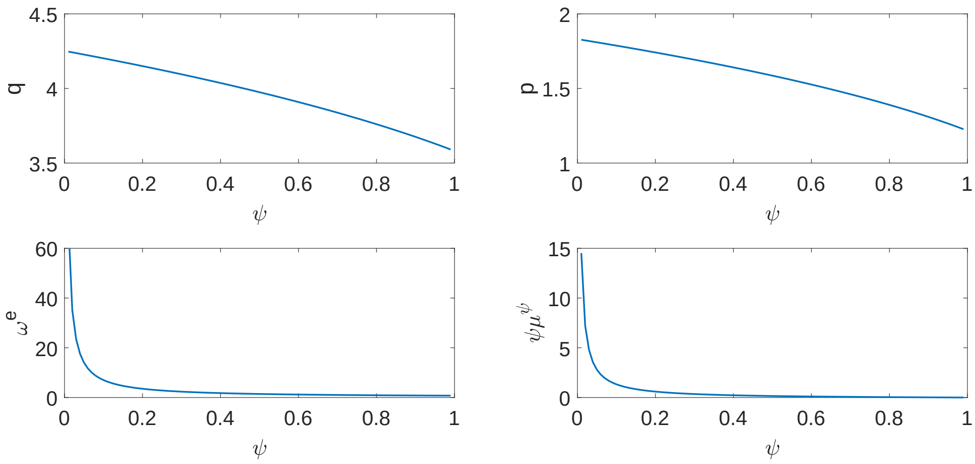

Figure 1.9: Top: Price of physical capital (left) and money (right). Bottom: experts' leverage (left), and drift of the state process (right). Baseline parameters: $\sigma = 0.2$, $\rho = 0.05$, $\delta = 0.05$, $\theta = 2$, and $\tilde{\sigma} = 0.6$.
.

model. As each agent does not internalize the effect of the distribution of aggregate capital stock over the dynamics of their own wealth (pecuniary externality), their interaction associate to endogenous feedbacks, and the wealth allocation may be not optimal in terms of welfare. To study this aspect, in Chapter 2 we develop a model where the financial sector offers risk mitigation services by issuing risk-free bonds and, at the same time, it pools idiosyncratic risky claims after then payment of intermediation costs. In such a framework we show that financial risk pooling activities fundamentally relate to real and financial macro-dynamics. Accordingly, households' welfare also depends on the size of the financial sector.[18] We conclude this section by solving the model numerically and briefly commenting on the results.

Numerical solution Figure 1.9 shows the numerical solution (by Matlab *ode45* solver) of the equilibrium: prices of capital (top, left), money (top, right), experts' portfolio share in capital (bottom, left), and drift of the state process (bottom, right). What stands out is that experts' higher exposure to the risky assets ω_t^e (the lower its hedging), the higher the growth rate of its relative share of wealth. Accordingly, a high price for physical capital also associates to a high price of money. In particular, both are decreasing as the stock of wealth within the economy gradually flows towards the experts'.

[18]On the contrary, in a model without financial frictions and complete markets, the distribution of wealth does not count since the flow to most productive agents is unconstrained and naturally stems from their optimal choices.

1.4 Advanced Models

In this section, we discuss and replicate a few important theoretical contributions to the macro-finance literature in continuous-time. In the framework of this paper, those models are classified as "advanced" because due to their structural complexity and the resulting non-trivial equilibrium state dynamics. In order of presentation: first, Section 1.4.1 discusses Klimenko et al. (2016), that studies the relationship between aggregate bank capital and credit dynamics. Second, Section 1.4.2 introduces the seminal work by He and Krishnamurthy (2013), concerned to explain the relationship between financial intermediation, asset pricing, and financial crisis. Third, Section 1.4.3 explores the amplification mechanism of exogenous systematic proposed in Brunnermeier and Sannikov (2014). Forth, Section 1.4.4 explores the relationship between money, intended as an instrument of risk mitigation, and the so-called liquidity and dis-inflationary spirals proposed in Brunnermeier and Sannikov (2016a).

1.4.1 Aggregate Bank Capital and Credit Dynamics

This section aims at reproducing and understanding mechanisms, assumptions, and methodologies of a recent working paper by Klimenko et al. (2016). In Chapter 3, we will adopt some of these assumptions, as for example the structure of the banking sector, to develop a general equilibrium model aimed at studying the relationship linking banks' recapitalization policies and long-run households' welfare.

Outline The paper proposes a model of a dynamic endowment economy with an aggregate banking sector. In particular, it studies the relationship between banks' lending activities and the real economy and, in turn, it shows that aggregate bank capital is a fundamental driver of the bank lending itself. A relevant feature of the model is that, because of financial frictions, the banks' issuance of new equity is costly. Therefore, in equilibrium, banks build equity buffers aimed at absorbing negative exogenous systematic shocks. In such a framework, aggregate bank capital also determines the overall dynamics of lending and, interestingly, the equilibrium loan rate is a decreasing function of aggregate banks' capitalization.

Moreover, the paper shows the competitive equilibrium between banks, households, and firms to be utterly inefficient. This is because the banks do not internalize the consequences of individual lending decisions for the future loss-absorbing capacity of their own capital. It follows that the banks lend too much, and so that proper capital ratios may help at improving the stability of the banking system.

1.4.1.1 The Model

Firms, investment projects, and households Time is continuous and the economy is populated by households, firms, and commercial banks, indexed $h \in \mathbb{H}$, $f \in \mathbb{F}$, and $b \in \mathbb{B}$, respectively. Both the aggregate of banks and firms are owned by households.

At each instant of time t, firm f, that is endowed with an investment project, may invest one unit of capital good go generate A units of the same good over the time interval dt. The distribution of A is exogenous and bounded over the interval $(0, \bar{A})$. To collect the capital required to finance its project, the f firm subscribe a loan with the b bank and, upfront, it agrees to pay out a stochastic return q_t. The rental rate q_t is endogenous and determined in equilibrium; moreover, the project will be finance as long as $A > q_t$. Another relevant assumption is that the firms' projects may fail with a certain intensity p_t that evolves as a GBM

$$\frac{dp_t}{p_t} = pdt - \sigma dW_t,$$

where dW_t is a Wiener process defined over a suitable probability space $(\Omega, \mathbb{P}, \mathcal{F})$ and $\mathcal{F}$ its the natural filtration. In summary, the stochastic return on banks' loan reads

$$dR_t = (q_t - p)\, dt - \sigma dW_t. \tag{1.70}$$

In the model, firms' demand for loans is represented in reduced form as

$$L(q_t) := \left(\frac{\bar{A} - q_t}{\bar{A} - p - \rho}\right)^{\beta}, \tag{1.71}$$

where ρ is the discount rate of the economy and β the elasticity parameter.

The households are infinitely lived, risk-neutral, and discount the future at a fixed rate ρ. They maximise the current value of their consumption flows (linear utility) plus the liquidity value of their deposits. In summary, they allocate their wealth between consumption and/or bank deposits. Formally, the households' lifetime utility reads as follows

$$\mathbb{E}_0 \int_0^{\infty} e^{-\rho t} \left[dC_t + \kappa(D_t)dt\right], \tag{1.72}$$

subject to

$$dC_t = dT_t + r_t D_t dt,$$

where κ is a concave increasing function of the deposits, and dT_t summarises the transfers from/to the banking sector plus firms' profits. For practical purposes, here I consider $\kappa(D_t) = \kappa D_t$ so that, at the optimum, the return on deposits is constant and satisfies $r = \rho - \kappa$.

Note that the value r can be read as a *liquidity premium* since it implies that households invest in deposits up to the point where the marginal utility from liquidity services equals the gap between their discount rate minus the risk-free rate on deposits.

Banks The banking sector consists of a continuum of commercial banks owned by the households. They aim at maximizing their *market value* (franchise value) for their shareholders. The banks are risk-neutral and finance risky investment to productive firms by means of households' deposits plus their own equity endowment e_t^b. In the aggregate, the banking sector's equity sums up to $\int_{\mathbb{B}} e_t^b db = E_t$. Formally, the banks' problem reads as follows:

$$B_0 := \max_{\left\{d\delta_t^b, d\pi_t^b, \omega_t\right\}_{t\in[0,\infty)} \in T_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \left[d\delta_t^b - (1+\lambda)d\pi_t^b\right], \tag{1.73}$$

subject to

$$T_t : \quad \frac{de_t^b}{e_t^b} = dR_t\omega + (1-\omega_t)r_t dt - \frac{dT_t^b}{e_t^b}, \tag{1.74}$$

where the transfer function holds as $dT_t^b = d\delta_t^b - d\pi_t^b$, λ is a reduced form summarizing the (costly) re-capitalization friction, and dR_t is the return on firms' risky projects (1.70).

Under the optimal strategy $\left\{d\delta_t^b, d\pi_t^b, \omega_t\right\}$, the banks' HJB equation holds as

$$\rho B_t dt = d\delta_t^b - (1+\lambda)d\pi_t^b + \mathbb{E}_t\left[dB_t\right].$$

The solution of the problem is found by a proper ansatz of the form $B_t := b(E_t)e_t^b$, meaning that the banks' value is linear in their own capitalization and non-linear in the aggregate capital E_t of the banking sector itself (for a sketch of the solution, see Appendix A.6). In this term, the function b can be interpreted as the banking sector *market-to-book value*. It follows that the optimal dividend pay-out and re-capitalization strategies $\left\{d\delta_t^b, d\pi_t^b\right\}$ are so that

$$\begin{cases} d\delta_t^b \geq 0, & b(E_t) = 1; \\ d\pi_t^b \geq 0, & b(E_t) = 1+\lambda. \end{cases}$$

zero otherwise. This means that the banks pay out dividends when the marginal value of their equity is below or equal to one. Conversely, the issue new (costly) equity when it is above the marginal cost or re-capitalization.

The optimal leverage policy $\{\omega_t\}$ is so that the rental rate q_t satisfies

$$q_t = p + r - \sigma^2 \epsilon_{b,E}\Omega_t, \tag{1.75}$$

where Ω_t is the aggregate banking sector leverage, and $\epsilon_{b,E} = \frac{\partial b}{\partial E}\frac{E_t}{b}$ is the elasticity of banks' market-to-book value with respect to the aggregate banking capitalization.[19] Accordingly, the function b satisfies the following ODE

$$(\rho - r)\, bE_t = b_E\left[E_t r_t + E_t\Omega_t\left(q_t - p - r\right)\right] + \frac{1}{2} b_{EE}\left(E\Omega_t\right)^2 \sigma^2,$$

with boundary conditions $b(E^{min}) = 1 + \lambda$ and $b(E^{max}) = 1$. Note that Equation (1.75) summarises the implicit risk-aversion of banks' stemming from the sensitivity of the marginal value of their equity with respect to aggregate bank's equity. Therefore, banks' are risk-neutral but they act "as if" they where risk-averse.

Equilibrium and state For each value of aggregate banks' capital $E_t \in (E^{min}, E^{max})$, the competitive equilibrium of the economy is an adapted stochastic process that maps histories of exogenous systematic shocks $\{dW_t\}$ to all relevant economic aggregates so that

1. Banks' lending supply satisfies firms' demand (exogenously given by Equation 1.71)
$$E_t\Omega_t = L(q_t(E_t)).$$
2. Households' maximise their utility (1.72);
3. Banks' maximize their franchise value (1.73);
4. All markets clear.

By market clearing conditions, equation (1.75) can be equivalently written as

$$b(E_t) = \exp\left\{\int_E^{E^{max}} \frac{q_t(E_t) - r - p}{L(q(E_t))\sigma^2} dE\right\},$$

by considering the proper boundary condition $b(E^{max}) = 1$. Accordingly, $q(E_t)$ satisfies the ODE

$$q_E = -\frac{2\left\{\lambda - \frac{(q_t - r - p)}{\sigma^2}\left[q_t - r - p + \frac{E_t r}{L_t}\right]\right\} - b_E L_t\left(q_t - r - p\right)}{L_t + L_q\left(q_t - r - p\right)},$$

with boundary condition $q(E^{max}) = r + p$. Therefore, the equilibrium rental rate q will be a decreasing function of aggregate banking capital E.

1.4.2 Intermediary Asset Pricing

This section aims at reproducing and understanding the mechanisms and methodologies of the seminal paper *"Intermediary Asset Pricing"* by He and

[19] Of course, due to banks' homogeneity, in equilibrium it holds that $\omega_t = \Omega_t$.

Krishnamurthy (2013).[20]

Outline The main contribution of the paper is to relate the risk premium to the size of a financial sector in a general equilibrium model of an endowment economy with financial frictions and heterogeneous agents. The economy is populated by households and intermediaries. Financial frictions consist of households' *restricted market participation* plus a maximal allowed leverage (constraint) that bounds intermediaries' portfolio choices.

The paper proposes a theoretical framework where, when the economy falls into the constrained region, i.e. enters into a crisis, the risk premium on risky assets is increasing in a highly non-linear fashion, as the risk-free rates drastically decrease (flight to quality). Another contribution of the paper is to study the dynamics of the recovery path after a crisis by tracking the (sticky) half-life of the risk premium captured by the model.

Finally, the paper explores three types of policies: a) Lowering borrow rate; b) Direct purchase of risky asset; c) Capital Infusions. All in all, they highlight that, within the model framework, the c) policy is the most effective at countering the effect of crises.

1.4.2.1 The Model

Intermediary and asset pricing Time is continuous and there exist two goods: perishable consumption, (acting as a numéraire) and physical capital K_t (fixed and equal to 1), valued q_t. In the aggregate capital stock it is written a risky claim (asset) that pays off a risky stream of dividends. The risky asset is modelled as a Lucas tree (Lucas, 1978), so that the dynamics of dividends D_t is a GBM

$$\frac{dD_t}{D_t} = \mu dt + \sigma dW_t \tag{1.76}$$

where the drift μ and diffusion σ are constant, and dW_t is a Wiener process defined on the probability space $(\Omega, \mathbb{P}, \mathcal{F})$ with natural filtration $\mathcal{F}$. Therefore, the total return on the risky asset has the following dynamics:

$$dR_t = \underbrace{\frac{D_t}{q_t}dt}_{\text{Dividend yield}} + \underbrace{\frac{dq_t}{q_t}}_{\text{Capital Gain}} .$$

The economy is populated by two agents: a *financial intermediary* (I) and a *household* (H). Financial markets are utterly restricted in their participation, i.e. the investment opportunity consists of the risky asset in which only the intermediary can invest. In this setting, the channel by which households may

[20] A recent related spin-off contribution is He and Krishnamurthy (2019).

invests in risky assets is through the intermediary. Otherwise, the household may invest in risk-free (short-term) bonds with return r_t.

At each instant of time t, the household allocates a certain fraction of its own endowment $H_t = \omega_t^H E_t^H$ in order to buy a share of the intermediary equity. Conversely, intermediary endowment invested into risky equity is denoted as E_t^I. Thus, the intermediary allocates the wealth stock $E_t^I + H_t$ among assets, and she is not constrained at holding either short or long positions in risky claims and risk-free bonds.

By ω_t^I we denote the fraction hold in risky asset over its whole endowment E_t^I. The value of ω_t^I, if greater then 1, captures leverage. Accordingly, the total return on the intermediary capital endowment is stochastic and has dynamics

$$dR_t^I = r_t dt + \omega_t^I \left(dR_t - r_t dt\right).$$

The intermediary consumes and allocates its own resources plus what is collected though household's deposits aiming at maximizing the inter-temporal utility of its consumption. Formally

$$V_0 := \max_{\left\{C_t^I, \omega_t^I\right\}_{t\in[0,\infty)} \in T_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u\left(C_t^I\right) dt \tag{1.77}$$

subject to

$$T_t : \quad \frac{dE_t^I}{E_t^I} = dR_t^I - \frac{C_t}{E_t} dt, \tag{1.78}$$

and to the static constraint

$$H_t \leq m E_t^I,$$

meaning that financial leverage is bounded from above as a fraction of the intermediaries equity value m. Conversely, the household side of the economy is modelled as a continuous-time OLG, where each single agent has a lifespan defined over the instantaneous time interval dt, i.e. each generation lives between t and $t+dt$. Thus, each generation consumes, invests, and leaves a certain bequest to the following generation. Formally, the household has endowment E_t^H and maximises its utility according to

$$U_t := \max_{C_t^H, \omega_t^H} \rho \ln C_t^H dt + (1 - \rho dt)\, \mathbb{E}_t \ln E_{t+dt}^H, \tag{1.79}$$

where E_{t+dt}^H represents the bequest of generation t for the succeeding generation at $t + dt$. In addition to the bequest from its predecessor, each generation receives labour income of $LD_t dt$, where $L \in (0, 1)$ is a fraction of total output (dividend).[21]

Household's choices and liquidity frictions The model assumes that a minimum amount of the household wealth λE_t^H must be kept invested into short-term debt (locally risk-free deposits) of the intermediary sector. This constraint can be read as demand for liquidity.[22] Given the log utility in (1.79), the household consumes a fixed fraction of its wealth $C_t^H = \rho E_t^H$, and allocates λE_t^H in the intermediary debt issuance at a price r_t. The residual share of her wealth, $(1-\lambda-\rho)E_t^H$ is either allocated in risky equity or in risk-free debt.

In this setting, the household optimal portfolio allocation ω_t^H is the solution of a linear quadratic problem, so that

$$\omega_t^H := \arg\max \left\{ \mathbb{E}_t \left[dR_t^I \right] - \frac{1}{2} \mathbb{V}ar_t \left[dR_t^I \right] \right\},$$

under the static budget constraint

$$H_t := \omega_t^H E_t^H (1-\lambda) \leq m E_t^I, \tag{1.80}$$

where ω_t^H denotes the fraction of wealth (portfolio share) invested in the risky equity, and m is the maximum amount of resources the intermediary may collect through bonds issuance. Thus, households' wealth has dynamics

$$\frac{dE_t^H}{E_t^H} = \underbrace{\left(\frac{D_t L}{E_t^H} - \rho \right) dt}_{\text{Wage minus consumption rate}} + \underbrace{r_t dt + \omega_t^H (1-\lambda) \left(dR_t^I - r_t dt \right)}_{\text{Portfolio component}}$$

Equilibrium and state The competitive equilibrium of this economy is defined as a set of adapted stochastic processes mapping the histories of systematic shocks to price processes $\{P_t, r_t\}$ so that

1. Intermediary (1.77) and household (1.79) maximise their utility;

2. The market for risky assets clears,

$$P_t = \omega_t^I \left(E_t^I + H_t \right) \Longrightarrow P_t = \omega_t^I \left[E_t^I + \omega_t^H E_t^H (1-\lambda) \right]; \tag{1.81}$$

3. The consumption good market clears,

$$C_t^I + C_t^H = D_t (1+L). \tag{1.82}$$

[21]This additional source of income prevents equilibria in which the household side of the economy get progressively marginal and disappears.

[22]This assumption is fundamental since it allows for intermediary leverage even when the capital constraint is slack.

The competitive equilibrium can be characterized by means of two state variables, i.e. the dividends stream D_t jointly with ψ_t, where $\psi_t = \frac{E_t^I}{E_t^I + E_t^H} \in (0,1)$ is the relative wealth share of the intermediary. We now discuss the two alternative equilibrium scenarios, when the static constraint (1.80) is either binding or not.

Case 1: binding constraints In this first case, (1.80) holds with equality, so that $H_t = mE_t^I$, and the market clearing condition for the risky asset (1.81) implies that

$$q_t = \omega_t^I \left(E_t^I + mE_t^I \right)$$

and thus, given $q_t = E_t^I + E_t^H$,

$$\omega_t^I = \frac{1}{\psi_t(1+m)}. \tag{1.83}$$

By equation (1.83), it is straightforward that the risk premium is decreasing in the constraint m. Conversely, the lower the wealth share of the intermediary ψ_t, the higher equilibrium leverage, the higher the risk premium.[23] Therefore, the risk premium on risky claims holds as and

$$\mathbb{E}_t dR_t^I - r_t = \frac{\mathbb{V}ar_t \left[\frac{dq_t}{q_t} \right]}{\psi_t(1+m)}.$$

Case 2: slack constraints In the case when the constraint is slack ($H_t < mE_t^I$), the share of capital that is not "required" to be saved by the constraint, $(1-\lambda)$, is utterly invested in risky capital, i.e. $\omega_t^H = 1$. It follows that

$$q_t = \omega_t^I \left(E_t^I + E_t^H (1-\lambda) \right) \Rightarrow \omega_t^I = \frac{1}{\psi_t + (1-\psi_t)(1-\lambda)}.$$

Of course, as long as $\lambda = 0$, there is no liquidity constraint for the household, and so the share of capital invested in the risky asset is constant and equal to one. This implies that the risk premium in the unconstrained region, where we do not have any financial friction, would be constant.

In case $\lambda > 0$, the household are constrained at holding a fraction of their wealth in risk-free bonds, and the leverage of the intermediary is positive and greater than one. In general, the grater the intermediary wealth share, the lower the level of leverage induced by the liquidity constraint.

How much higher is the leverage in the constrained case with respect to its unconstrained counterpart? Here we consider the ratio of the constraint over

[23] See also the results of Section 1.2.4.

the unconstrained intermediary portfolio shares ω_t^I:

$$\frac{\omega_t^{I,C}}{\omega_t^{I,U}} = \frac{1}{(1+m)}\left[\frac{1}{\psi_t} - \frac{\lambda\left(1-\psi_t\right)}{\psi_t}\right]. \tag{1.84}$$

What stands out it that, the higher the capital that the households is allowed to transfer (m), the lower the excess of leverage. Nevertheless, the distribution of wealth is determinant: the higher the concentration of wealth to the household $(1-\psi_t)$, the greater the equilibrium leverage.

Sketch of the solution Given the market clearing condition (1.82), it follows that

$$C_t^I = D_t\left[(1+L) - \rho\frac{q_t}{D_t}(1-\psi_t)\right],$$

and the state ψ_t is re-defined as

$$\psi_t = 1 - \underbrace{(1-\psi_t)\frac{q_t}{D_t}}_{y}\frac{1}{F(y)},$$

where y is the household relative share of wealth normalized by the dividend D_t, and $F(y) = \frac{q_t}{D_t}$. Accordingly,

$$C_t^I = D_t\left[(1+L) - \rho y_t\right]. \tag{1.85}$$

At this point, we look for an equilibrium so that y follows a diffusion process of the type

$$\frac{dy_t}{y_t} = \mu_t^y dt + \sigma_t^y dW_t, \tag{1.86}$$

where μ_t^y and σ_t^y are endogenously determined in equilibrium. By Itô's Lemma, the dynamics of intermediaries consumption evolves as

$$\frac{dC_t^I}{C_t^I} = \frac{d\left\{D_t\left[(1+L)-\rho y_t\right]\right\}}{D_t\left[(1+L)-\rho y_t\right]} = \frac{dD_t}{D_t} - \frac{\rho}{(1+L)-\rho y_t}dy_t - \frac{dD_t}{D_t}\frac{\rho dy_t}{\left[(1+L)-\rho y_t\right]},$$

and, given (1.76) and (1.86),

$$\frac{dC_t^I}{C_t^I} = \left(\mu - y_t\frac{\rho\sigma\sigma_t^y + \mu_t^y}{(1+L)-\rho y_t}\right)dt + \left(\sigma - y_t\sigma_t^y\frac{\rho}{(1+L)-\rho y_t}\right)dW_t. \tag{1.87}$$

In the same fashion, one can obtain the return on risky claims

$$dR_t = \frac{D_t}{q_t}dt + \frac{dq_t}{q_t}, \tag{1.88}$$

where, by Itô's Lemma, the capital gain process evolves as

$$\frac{dq_t}{q_t} := \frac{d\left(F_t D_t\right)}{F_t D_t} = \frac{dF_t}{F_t} + \frac{dD_t}{D_t} + \frac{dF_t}{F_t}\frac{dD_t}{D_t}, \tag{1.89}$$

and F has dynamics

$$dF_t = F_y\left(\mu_t^y y_t dt + \sigma_t^y y_t dW_t\right) + \frac{1}{2}F_{yy}\left(\sigma_t^y y_t\right)^2 dt. \tag{1.90}$$

By matching (1.88) and (1.89) to (1.90), we obtain the dynamics of the return on risky assets as

$$dR_t = \left[\mu + \frac{1}{F_t} + \frac{\partial_y F_t}{F_t}\mu_t^y y_t + \frac{1}{2}\frac{\partial_{yy}^2 F_t}{F_t}\left(\sigma_t^y y_t\right)^2 + \sigma_t^y \sigma y_t\right] dt + \left(\sigma + \frac{\partial_y F_t}{F_t}\sigma_t^y y_t\right) dW_t. \tag{1.91}$$

Finally, by considering the asset pricing conditions (1.36) and (1.37) in Section 1.2.4, jointly with (1.87) and (1.91), we have that F must satisfy the following ODE

$$\begin{aligned}\left[\mu + \frac{1}{F_t} + \frac{\partial_y F_t}{F_t}\mu_t^y y_t + \frac{1}{2}\frac{\partial_{yy}^2 F_t}{F_t}\left(\sigma_t^y y_t\right)^2 + \sigma_t^y \sigma y_t - \rho\right] + \\ - \left(\mu - y_t \frac{\rho\sigma\sigma_t^y + \mu_t^y}{(1+L) - \rho y_t}\right) + \left(\sigma - y_t \sigma_t^y \frac{\rho}{(1+L) - \rho y_t}\right)^2 + \\ - \left(\sigma + \frac{\partial_y F_t}{F_t}\sigma_t^y y_t\right)\left(\sigma - y_t \sigma_t^y \frac{\rho}{(1+L) - \rho y_t}\right) = 0.\end{aligned} \tag{1.92}$$

Equation (1.92) is solved numerically, must be satisfied by $y(\psi_t)$, and evaluated for both the constrained and unconstrained region. The threshold of relative wealth share where intermediary leverage constraint starts binding is given by equation (1.84)

$$\psi^* = \frac{1-\lambda}{1+m-\lambda}.$$

1.4.3 A Macroeconomic Model with a Financial Sector

This section introduces and discusses the main features and mechanisms of the seminal paper *"A Macroeconomic Model with a Financial Sector"* by Brunnermeier and Sannikov (2014).

Outline The paper studies the general equilibrium dynamics of a productive economy with financial frictions, heterogeneous classes of agents, and complete financial markets. The model shows that, due to the highly non-linear amplification effects that stem from the interaction of agents, the economy is prone to instability and may occasionally enter volatile crisis episodes. In particular, they dissect the mechanism through which endogenous risk is generated by the presence of financial frictions and argue that such a risk may be persistently increasing as systematic volatility decreases, as it relates to equilibrium financial leverage (*volatility paradox*). Interestingly, the existence of contracts improving the risk sharing of the agents, may lead to higher leverage, and thus more frequent crises.

1.4.3.1 The Model

The agents and their preferences Time is continuous, and the economy is populated by two representative infinitely lived classes of agents: *experts* and *households*, henceforth, indexed as $x \in \mathbb{X}$ and $h \in \mathbb{H}$, respectively. Each class of agents is defined over a continuum of unitary mass, so that $\mathbb{X} := [0, 1)$ and $\mathbb{H} := [1, 2)$. Both actors are risk-neutral and maximise the inter-temporal utility of their consumption flows $d\zeta_t^i$ by allocating their wealth to either risky claims on capital or risk-free bonds. A first layer of heterogeneity comes from the discount factor r^i of experts' and households'. In particular, it is assumed that $r^e > e^h$ so that experts are less patient than households. Moreover, experts are not allowed to have negative consumption flows. Formally, agents' problems read as

$$V_i := \max_{\left\{d\zeta_t^i, \omega_t^i\right\}_{t \in [0,\infty)} \in T_t^i} \mathbb{E}_0 \int_0^\infty e^{-r_i t} d\zeta_t^i; \quad i \in \{x, h\}\,, \tag{1.93}$$

subject to

$$T_t^i : \frac{de_t^i}{e_t^i} = \omega_t^i dR_t^i + \left(1 - \omega_t^i\right) r_t dt - \frac{d\zeta_t^i}{e_t^i},$$

and

$$d\zeta_t^x \geq 0, \tag{1.94}$$

where ω_t^i represents the wealth share invested in risky claims with return dR_t, r_t is the return on risk-free bonds, and $d\zeta_t^i$ is the consumption rate. It is relevant to stress that equation (1.94) summarises the core financial friction of the model and, as we shall see, it grants the equilibrium risk-free to be constant and equal to the discount rate of the households' r^h.

Productive technologies There exist two types of non-fungible goods: physical capital (such as trees) and output good (such as apples). Henceforth, we denote agent-wise variables with a lower case letter. Conversely, we denote aggregate variables by capital letters. For instance, k_t^h denotes the physical capital that belongs to the household, whereas K_t denotes the aggregate capital stock within the economy.

Each type of good, either capital or output, is produced by a particular technology. However, even though both classes are able to manage capital, the experts are more productive at producing output and, in equilibrium, they will finance their activity by issuing risk-free bonds.[24]

Let dW_t be a Wiener process defined over a filtered probability space $(\Omega, \mathcal{H}, \mathbb{P})$, the unique source of uncertainty of the economy, and $\{\mathcal{H}_t, t \geq 0\}$ represents the natural filtration over the measurable space $\{\Omega, \mathcal{H}\}$. Then, the capital stock managed by agent i evolves as an Itô's diffusion:

$$\frac{dk_t^i}{k_t^i} = \left[\Phi(\iota_t^i) - \delta^i\right] dt + \sigma dW_t, \tag{1.95}$$

whereas the output good is produced at a rate

$$y_t^i = A^i k_t^i. \tag{1.96}$$

In Brunnermeier and Sannikov (2014), the capital stock managed by the households produces output at a lower rate, and depreciates at a higher rate, so that $A^e \geq A^h$ and $\delta^e \leq \delta^h$.

Return on risky claims and capital markets All agents are price takers and exchange physical capital in a perfectly competitive market at the endogenous (equilibrium) price q_t whose dynamics is given by

$$\frac{dq_t}{q_t} = \mu_t^q dt + \sigma_t^q dW_t, \tag{1.97}$$

whose drift, diffusion, and level are endogenous and will be determined in equilibrium. Therefore, output acts as numéraire and aggregate capital stock is valued $K_t q_t$. Given the productive technologies (1.95) and (1.96) and the price process (B.24), the returns on agent i risky claims have dynamics

$$dR_t^i = \frac{A^i - \iota_t^i}{q_t} dt + \left[\Phi(\iota_t^i) - \delta^i + \mu_t^q + \sigma\sigma_t^q\right] dt + (\sigma + \sigma_t^q)\, dW_t. \tag{1.98}$$

Competitive equilibrium Given an initial endowment of wealth at time zero $E_0^h + E_0^x = K_0 q_0$, the competitive equilibrium of the economy is defined as a map

[24]For a micro-foundation of the production process we refer to the framework described in Section 1.2.3. An alternative way to think the production process is that the households have their investment opportunity set restricted to those firms that are relatively inefficient, so that they invest in capital producing firms whose depreciation rate is higher, and output producing firms whose productivity is lower.

from histories of systematic shocks to prices, asset allocation, and consumption choices so that:

1. The agents maximise their expected utilities;

2. All markets clear (capital stock, consumption, risky, and risk-free assets).

Since the households are not financially constrained, i.e. their consumption may hold negative, the FOCs of problem (3.48), given (1.98) and $i = h$, lead to the following condition

$$\frac{A^h - \iota_t}{q_t} + \Phi(\iota_t) - \delta^i + \mu_t^q + \sigma\sigma_t^q \leq r^h, \tag{1.99}$$

that holds with equality as long as they own a positive stock of wealth. Equation (1.99) implies that the households, in equilibrium, are willing to give up their capital stock (in fact, they are indifferent between holding risky and risk-free assets) as long as its expected return equals their discount rate r^h. This result stems from the agents' risk neutrality.

The choice of experts, being financially constrained, is harder. They access firms with higher productivity and lower depreciation rate while facing a dynamic problem, since the marginal value of their wealth stock relates to the externality through which they (indirectly) affect equilibrium prices.[25]

To this regard, the risk premium of the experts' relates to the covariance between their risky assets returns and the dynamics of their wealth stock marginal value. It is possible to show (see Brunnermeier and Sannikov, 2014, pp 391-392) that, given the ansatz for their value $V_t^x := \theta_t e_t^x$, the process $\{\theta_t\}$ under the optimal strategy $\{d\zeta_t^x \geq 0, \omega_t^x\}$ satisfies the HJB equation:

$$\rho\theta_t dt = d\zeta_t^x + \mathbb{E}_t\left[\frac{d\left(\theta_t e_t^x\right)}{e_t^x}\right],$$

with transversality condition $\lim_{t\to\infty}\mathbb{E}_0\left[e^{-r_x t}\theta_t e_t^x\right] = 0$. The solution of the problem is characterized by guessing a stochastic process describing the evolution of θ over time:

$$\frac{d\theta_t}{\theta_t} = \mu_t^\theta dt + \sigma_t^\theta dW_t, \tag{1.100}$$

so that μ_t^θ and σ_t^θ are endogenous, and determined in equilibrium. According to the ansatz in (1.100), it follows that the optimal strategy $\{d\zeta_t^x \geq 0, \omega_t^x\}$ is such that:

[25] As we shall see, the higher their leverage ($\omega^x \geq 1$), the more exogenous systematic shocks are amplified, the higher the marginal value of their wealth.

1. Consumption $d\zeta_t^x$ holds as

$$d\zeta_t^x \geq 0; \quad \theta = 1.$$

Else,

$$d\zeta_t^x = 0; \quad \theta > 1.$$

2. Asset allocation ω^x is so that

$$\frac{A^x - \iota_t}{q_t} + \Phi(\iota_t) - \delta^x + \mu_t^q + \sigma\sigma_t^q - r^h \leq -\mathbb{C}ov_t\left(dR_t^x, \frac{d\theta_t}{\theta_t}\right), \tag{1.101}$$

that holds with equality if $\omega_t^x > 0$.

Before we discuss the features of experts' equilibrium strategies, it is useful to introduce the state variable through which it is possible to describe the whole equilibrium dynamics: experts' *relative wealth share* ψ defined as

$$\psi_t =: \frac{E_t^x}{E_t^h + E_t^x} \in [0, 1], \tag{1.102}$$

where E_t^i is the aggregate stock of wealth of the agents within class i. In the same fashion, it is useful to define as $\vartheta_t =: \omega_t^x \psi_t$ the fraction of capital stock allocated to the experts. Note that when $\vartheta_t = 1 \Rightarrow \omega_t^x = \frac{1}{\psi_t}$ so that, in equilibrium, the experts leverage the whole capital stock out of the householders' balance sheet.

The right-hand side of equation (1.101) is fundamental, and represents the risk-premium of the experts'. In particular, it summarises their *precautionary motif*, that relates to the covariance between risky assets returns and the marginal value of their own wealth stock e^x. A higher correlation makes the premium higher with respect to the excess return on capital, and so it reduces the experts' leverage such that $\omega^x < \frac{1}{\psi}$. This happens because the risk-free rate is constant and equal to the household discount rate. Conversely, by market clearing conditions, $1 - \vartheta_t =: \omega_t^h (1 - \psi_t) \in [0, 1]$.

An important takeaway from Equation (1.101) is that the *precautionary motive* of experts' increases in their aggregate leverage. This means that, the lower their relative share of total wealth, the higher the share of their wealth allocated in risky claims. Conversely, it disappears when experts invest in capital without using leverage $\omega^x = \frac{1}{\psi}$. Therefore, the incentives of individual experts to take on risk are decreasing in the risks taken by aggregate of experts.

State variable By Itô's Lemma, if we set $\delta^x = \delta^h$, it is possible to show that

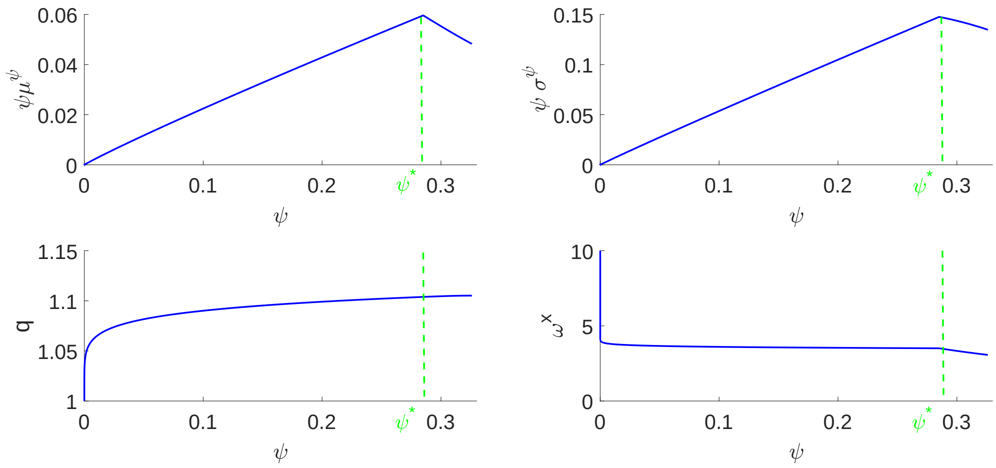

Figure 1.10: Top: Drift $\psi\mu^{\psi}$ (left) and diffusion $\psi\sigma^{\psi}$ (right) of the state process $d\psi_t$. Bottom: equilibrium price q and experts' leverage ω^x. In dashed green, the threshold ψ^* so that $\vartheta = 1$.

the unique state variable ψ evolves as a regulated diffusion

$$\frac{d\psi_t}{\psi_t} = \mu_t^{\psi} dt + \sigma_t^{\psi} dW_t - d\zeta_t, \tag{1.103}$$

where

1. The term $d\zeta_t =: \frac{1-\psi_t}{\psi_t}\left[d\zeta_t^x - d\zeta_t^h\right]$ is an impulse variable that adjusts the dynamics of (1.103) by creating a "regulated" diffusion and

$$d\zeta_t = \begin{cases} \frac{1-\psi_t}{\psi_t}\left[d\zeta_t^x - d\zeta_t^h\right] \geq 0; & \psi_t \geq \psi^*; \theta_t \geq 1, \\ 0; & else; \end{cases}$$

2. The drift μ^{ψ} and diffusion σ^{ψ} terms equal, respectively:

$$\mu_t^{\psi} = \underbrace{-\left[\frac{A^x - \iota_t}{q_t} + \Phi(\iota_t) - \delta^x + \mu_t^q + \sigma\sigma_t^q - r^h\right]}_{\sigma_t^{\theta}(\sigma+\sigma_t^q)} \frac{\vartheta_t - \psi_t}{\psi_t} + \\ - \frac{\vartheta_t - \psi_t}{\psi_t}(\sigma + \sigma_t^q)^2 + \frac{A^x - \iota_t}{q_t}; \tag{1.104}$$

$$\sigma_t^{\psi} = \frac{1-\psi_t}{\psi_t}\left(\omega_t^h + \omega_t^x\right)(\sigma + \sigma_t^q) = \frac{\vartheta_t - \psi_t}{\psi_t}\left(\sigma + \sigma_t^q\right). \tag{1.105}$$

In Figure 1.10, we show the numerical solution of the model. In particular, we plot the drift (top left) and diffusion (top right) of the state process as well as the equilibrium price level (bottom left) and experts' leverage (bottom right) as a function of the state ψ (details are in Brunnermeier and Sannikov, 2014).[26] In general, both drift and diffusion of the state are non-monotonic, and reach their maximum at the inner threshold ψ^*, where $\vartheta = 1$ so that the experts leverage the whole capital stock within the economy. Accordingly, the experts' leverage ω^x decreases in their relative wealth share and, since experts are more productive than households, i.e. $A^x > A^h$, equilibrium prices are an increasing function of the state.

Amplification and stationary density Perhaps the most compelling result of the model is the amplified sensitivity of the state dynamics to exogenous systematic shocks jointly with the consequent "switch" between regimes that is triggered by streams of positive and negative systematic shocks. To analyse this aspect, Figure 1.11 shows the diffusion term of the risky assets return (left) jointly with the stationary distribution of the state $f(\psi)$.

Because of endogenous non-linearities, the price volatility component σ_t^q is higher for intermediate states, and amplifies the magnitude of those exogenous aggregate shocks effecting the economy. Conversely, the endogenous amplification is lower in the neighbourhood of upper and lower boundaries, respectively. Therefore, the economy spends most of time either to high (more capital allocated to more productive experts) or lower (more capital allocated to less productive households) states.[27]

Due to those non-linearities, once the economy drifts trough one of the stable low/high states, a positive/negative stream of aggregate shocks may get progressively amplified, rapidly leading the economy towards the opposite state, where the allocation of capital is skewed towards households (left-hand side) or experts (right-hand side). Of course, due to the higher productivity of experts', those states where their leverage is low, most of capital is under their management, and prices are higher, are utterly preferable (see also Figure 1.10) The duration of "shifts" between regimes, below and above the steady state ψ^*, may be arbitrarily persistent, depending on the parametric values such as the productivity gap between experts and households $A^x - A^h$, and the aggregate volatility σ.

1.4.4 The I Theory of Money

The purpose of this section is to reproduce and discuss the main features and mechanism underneath the cornerstone paper *"The I Theory of Money"* by Brunnermeier and Sannikov (2016a).

[26] Henceforth, we adopt the following parametric specification $A^x = 0.15$, $A^h = 0.05$, $r = 0.02$, $\rho = 0.05$, $\theta = 4$, $\delta = 0.03$, and $\sigma = 0.2$, while $\Phi(\iota) = \frac{1}{\theta}\ln(1 + \theta\iota)$.

[27] Although the shape the long-run state density is robust, the relative mass below each tail is strongly effected by the parametric values.

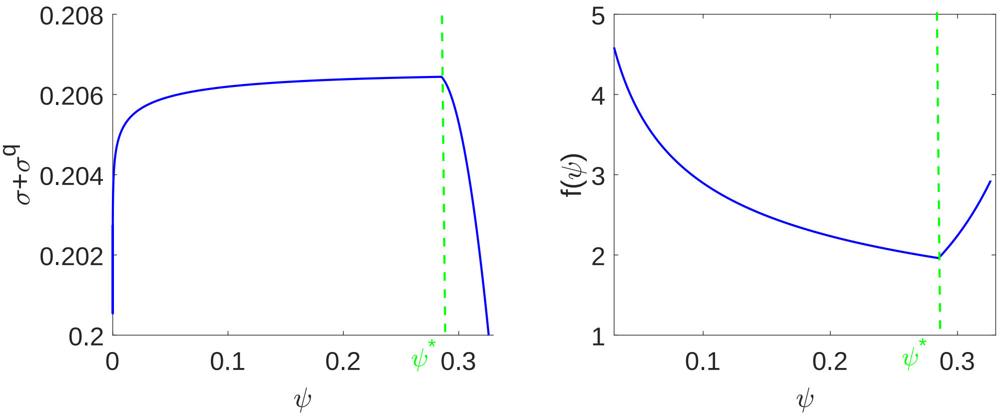

Figure 1.11: Left: Risky assets return diffusion. Right: Stationary density of the state. In dashed green, the threshold ψ^* so that $\vartheta = 1$.

Overview The paper develops a *theory of money value* as it relates to the role of financial intermediaries at diversifying risks and, in turn, to their supply of inside money to the economy. From the policy point of view, the paper argues upon the redistributive role of Monetary Policy (MP) and shows that an expansionary MP (indirectly) recapitalizes balance impaired financial intermediaries. Thus, it mitigates adverse liquidity and disinflationary spirals. Notwithstanding, MP is limited since it cannot control risk-taking and premiums separately. To this regards, macro-prudential policies aimed at limiting leverage may improve welfare.

Similar to Brunnermeier and Sannikov (2014), one of the most peculiar aspects of the model is that the interaction among agents generates endogenous amplification of systematic shocks, jointly channelled by two different dynamics: a) Financial intermediaries are able to diversify idiosyncratic risks and issue (inside) money to finance their position in outside equity; b) Intermediaries are not allowed to purchase outside capital coming from some sector of the economy, i.e. capital flows are constrained.

Perhaps the most important contribution of the paper is that it features the so-called *liquidity* and *deflationary* spirals. In a nutshell, these effects work as follows: whenever an averse stream of shocks affect physical capital, agents react by rebalancing their portfolios - overweighting money - (safe asset) with respect to physical capital. This pattern associates to a reduction in the value of intermediaries' equity, that stems into a shrinkage of their inside money supply. This triggers deflationary effects on money. The panic selling of physical capital puts pressure to further capital depreciation and, in turn, addition deflation.

1.4.4.1 The Model

Agents and preferences The economy is populated by a continuum of (heterogeneous) households, indexed $a \in \mathbb{A} \vee b \in \mathbb{B}$, where $\mathbb{A} \cup \mathbb{B} := [0,1)$, and a financial sector, indexed I. All actors have identical log preferences and discount the future at a fixed common rate ρ. All actors consume and optimally allocate their wealth between risky claims issued by capital producing firms (inside equity) and money asset. Households' risky allocation of may be alternatively to projects of type a or b, with stochastic return dR_t^a and dR_t^b, respectively. Both types of projects are exposed to both systematic and project specific idiosyncratic risks. As we shall see, in equilibrium, the return on risky claims from different projects must be so that households are indifferent between a and b. Moreover, those households who choose to allocate their wealth to projects of type b may also decide to issue equity χ_t that is purchased by the financial intermediary (outside equity), who pools risky claims issued by the continuum of households. Formally, the agents' problem read as follows:

$$V_0 := \max_{\left\{c_t^j, \omega_t^j, \chi_t^{I} \leq \bar{\chi}\right\}_{t \in [0,\infty)} \in T_t^j} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \ln c_t^j dt;$$

subject to

$$\frac{de_t^j}{e_t^j} = \omega_t^j dR_t^j + \left(1 - \omega_t^j\right) dR_t^m - \frac{c_t^j}{e_t^j} dt, \tag{1.106}$$

where dR_t^j, $j \in \{a, b, I\}$, denotes the return on agents' risky assets, and dR_t^m.

Productive technologies Similar to the framework introduced in Section 1.2.2.1, there exist two not-fungible goods: output (numéraire) and physical capital, valued $K_t q_t$. Output Y_t is produced by allocating physical capital to either projects of type a or projects of type b by the following CES production function

$$Y(\phi) = K_t A \left[\phi^r + (1-\phi)^r\right]^{\frac{1}{r}}, \tag{1.107}$$

where $\frac{1}{r-1}$ is the elasticity of substitution, while ϕ_t and $1-\phi_t$ represent the shares of total capital K_t devoted to projects a and b, respectively.[28] Accordingly, the marginal contribution of technologies a and b to aggregate output (see also Appendix A.7.1) equals

$$Y^a(\phi) = Y(\phi) + (1-\phi) Y'(\phi),$$

[28]Note that, when $\phi = 1$, the production function reduces to the baseline AK case.

and

$$Y^b(\phi) = Y(\phi) - \phi Y'(\phi).$$

Similar to the model in Section 1.2.2.2, the capital allocation follow a stochastic process whose dynamics is affected by idiosyncratic as well as systematic shocks. Households may decide to *alternatively* undertake projects of type a or b whose dynamics are given by

$$\frac{dk_t^a}{k_t^a} = [\Phi(\iota_t^a) - \delta]\, dt + \sigma^a dW_t + \tilde{\sigma^a} d\tilde{W_t}^a, \tag{1.108}$$

$$\frac{dk_t^b}{k_t^b} = \left[\Phi(\iota_t^b) - \delta\right] dt + \sigma^b dW_t + \tilde{\sigma^b} d\tilde{W_t}^b, \tag{1.109}$$

where $dW_t^{a,b} \perp d\tilde{W_t}^{a,b}$ are pairwise independent Wiener processes defined over a suitable probability space that represent systematic and (project specific) idiosyncratic shocks. According to (1.107), (1.108), and (1.109), by setting $dW_t^a = dW_t^b$, the stock of aggregate capital evolves as

$$\frac{dK_t}{K_t} = \left\{\Phi(\iota_t^b) + \phi_t \left[\Phi(\iota_t^a) - \Phi(\iota_t^b)\right] - \delta\right\} dt + \underbrace{\left[\sigma^a \phi_t + \sigma^b (1 - \phi_t)\right]}_{\sigma_t^K} dW_t.$$

Return on assets, financial markets, and money Similarly to what we discussed in previous sections, the model core dynamics are characterizing by guessing stochastic processes describing the dynamics of capital and money value (see also Section 1.3.3). By Itô's Lemma, the return on risky claims written on project k evolves as

$$dR_t^j = \underbrace{\frac{Y^j(\phi_t) - \iota_t^j}{q_t} dt + \left[\Phi(\iota_t^j) + \mu_t^q + \sigma_t^q \sigma - \delta\right] dt}_{\mu_t^j} + \underbrace{\left(\sigma^j + \sigma_t^q\right)}_{\sigma_t^j} dW_t + \tilde{\sigma}^j d\tilde{W_t}^j, \tag{1.110}$$

while the return on money has dynamics

$$dR_t^m = \underbrace{\left\{\Phi(\iota_t^b) + \phi \left[\Phi(\iota_t^a) - \Phi(\iota_t^b)\right] - \delta + \mu_t^p + \sigma_t^p \sigma_t^K\right\}}_{\mu_t^m} dt + \underbrace{\left(\sigma_t^K + \sigma_t^p\right)}_{\sigma_t^m} dW_t. \tag{1.111}$$

Finally, as the financial intermediary pools risky claims from the continuum of households' investing the sector b, the return on its risky assets satisfies

$$dR_t^I \leq \int_{\mathbb{B}} dR_t^b,$$

and $\int_{\mathbb{B}} d\tilde{W_t}^b$. An important aspect of the model that is implicit in Equation

(1.111) relates to the role of money as an instrument of risk mitigation, i.e. households hedge against idiosyncratic risks through money. To this regard, it is relevant to highlight that money can either be *outside*, given as an exogenous endowment, or *inside*, i.e. generated by the intermediary through its balance sheet. Inside and outside money are indistinguishable and fungible. In summary, money is a (less) risky asset whose dynamics is affected by systematic shocks only and earns no "dividend" interest and whose returns can be interpreted as deflation (money is a bubble).

Equilibrium The competitive equilibrium of this economy is defined as a set of adapted stochastic processes mapping histories of exogenous systematic shocks to all equilibrium aggregate variables so that

1. Intermediary and households maximise their utility;
2. Firms maximize their profits:

$$\partial_{\iota^a}\Phi = \partial_{\iota^b}\Phi = \frac{1}{q_t} \Longleftrightarrow \iota_t^a = \iota_t^b;$$

3. All markets clear:

   (a) Capital/risky claims

$$\omega_t^a E_t^a + \omega_t^b E_t^b + \omega_t^I E_t^I = K_t q_t.$$

   (b) Money;

$$\left(1-\omega_t^a\right) E_t^a + \left(1-\omega_t^b\right) E_t^b + \left(1-\omega_t^I\right) E_t^I = K_t p_t.$$

   (c) Consumption;

$$\underbrace{C_t^I + C_t^a + C_t^b}_{\rho K_t (p_t + q_t)} = Y(\phi_t) - K_t\left[\phi \iota_t^a + (1-\phi_t)\,\iota_t^b\right].$$

A very important feature characterizing the competitive equilibrium is that, since households are allowed to invest into one type of project exclusively, they must be indifferent between a and b. Accordingly, those who choose b decide how much outside χ_t equity to issue on the market. What is relevant to highlight is that, in equilibrium, when the outside equity issuance constraint is binding, households are willing to issue more equity than how they are allowed to. Accordingly, it must hold that intermediary's outside equity earns less than the return on inside equity of households investing in projects b. Thus,

$$\chi_t = \bar{\chi} \Longrightarrow \mathbb{E}_0 dR_t^b > \mathbb{E}_0 dR_t^I. \tag{1.112}$$

On the contrary, when the equity issuance constraint is slack, households are free to issue as much outside equity as they want. Therefore,

$$\chi_t < \bar{\chi} \Longrightarrow \frac{1}{dt}\mathbb{E}_0 dR_t^b = \frac{1}{dt}\mathbb{E}_0 dR_t^I. \tag{1.113}$$

Before we move forward to describe the features of state dynamics and amplification mechanism, it is interesting to outline some relationship between the assets risk premiums and equity issuance in equilibrium. In general, we know that each asset carries a risk premium that must be proportional to the risk coming from the exposure to that specific asset (see Bjork, 2009; Duffie, 2010). Thus, it must hold that

$$\mu_t^a - r_t = \xi_t^a \sigma_t^a + \tilde{\xi}_t{}^a \tilde{\sigma}^a, \tag{1.114}$$

$$\mu_t^b - r_t = \xi_t^b \chi_t \sigma_t^b + \xi_t^I (1 - \chi_t) \sigma_t^b + \chi_t \tilde{\xi}_t{}^b \tilde{\sigma}^b, \tag{1.115}$$

and

$$\mu_t^m - r_t = \xi_t \sigma_t^m, \tag{1.116}$$

where r_t is the shadow risk-free rate, $1 - \chi_t$ is the fraction of b allocated capital issued as outside equity, and ξ_t^j tracks down the price of risk coming from source j. By matching (1.114)-(1.116), the agents' dynamics budget constraint (1.106), and the returns dynamics (1.110)-(1.110), it is possible to show (see Appendix A.7.2) that the premium of inside over outside equity equals

$$\frac{1}{dt}\mathbb{E}_0 \left[dR_t^b - dR_t^I\right] = \chi_t \left\{ \left[\sigma_t^q - \sigma_t^p + (1-\phi_t)(\sigma^a - \sigma^b)\right]^2 \left(\omega_t^I - \omega_t^b\right) + \omega_t^b \left(\tilde{\sigma}^b\right)^2 \right\}. \tag{1.117}$$

The main implication of Equation (1.117) is that, in equilibrium, either households "b" issue equity to the upper bound $\bar{\chi}$, or they do not issue any at all (see conditions 1.112 and 1.113). What is also relevant is that the gap between inside and outside equity returns is increasing in both idiosyncratic risk $\tilde{\sigma}^b$ and sensitivity gap to systematic shocks between different technologies $\sigma^a - \sigma^b$.

Amplification, liquidity, and disinflationary spirals We conclude the section by outlining the mechanism behind one of the main contributions of the paper, i.e. the amplification pattern of equilibrium state dynamics and its relationship to the so-called liquidity and inflationary spirals.

Let the unique state variable be the relative share of wealth that belongs to the financial sector be defined as

$$\psi_t =: \frac{E_t^I}{E_t^I + \int_{\mathbb{H}} e_t^h dh} \in (0, 1). \tag{1.118}$$

Then, it is possible to show (see Appendix) that (1.118) evolves as a diffusion process with dynamics

$$\frac{d\psi_t}{\psi_t} = \mu_t^{\psi} dt + \sigma_t^{\psi} dW_t,$$

with drift μ_t^{ψ}

$$\mu_t^{\psi} = (1 - \psi_t) \left\{ \left[\left(\omega_t^I\right)^2 - \left(\omega_t^b\right)^2 \right] \left(\sigma_t^q + \sigma^b - \sigma_t^m\right)^2 - \left(\omega_t^b \tilde{\sigma}^b\right)^2 \right\} + \\ - \left[\omega_t^I \left(\sigma_t^q + \sigma^b - \sigma_t^m\right) + \sigma_t^{\pi}\right] \sigma_t^{\pi},$$

where $\sigma_t^{\pi} = \frac{1}{dt}\sqrt{\mathbb{V}ar_t\left(\frac{q_t}{q_t+p_t}\right)}$, and diffusion

$$\sigma_t^{\psi} = \left[\omega_t^I \left(\sigma_t^q + \sigma^b - \sigma_t^m\right) + \sigma_t^{\pi}\right]. \tag{1.119}$$

To better understand it dynamics, the diffusion term in (1.119) can be also written as a sum of a fundamental plus an amplification term as follows:

$$\sigma_t^{\psi} = \underbrace{\omega_t^I \left(\sigma^b - \sigma_t^K\right)}_{\text{Fundamental}} - \underbrace{\sigma_t^{\pi} \left(\frac{\omega_t^I}{1 - \pi_t} - 1\right)}_{\text{Amplification}}. \tag{1.120}$$

A first element that is relevant to highlight is that the magnitude of *amplification* is scaled by the term σ_t^{π}, that is, the volatility of relative size between capital versus money prices. This term is what characterise the *liquidity* and *disinflationary* spiral, respectively.

A former spiral (liquidity) is the consequence of a negative stream of averse shocks that affect the stock of capital. In such a case, agents react by (panic) selling the risky assets in their portfolio (q_t decreases). As a consequence, households rebalance their portfolio share towards money while, at the same time, the intermediary reduces its supply of inside money. This triggers the second spiral (disinflationary), as households' increase their demand for risk mitigation, the intermediary reduces (p_t increases).

What follows is that, since her liabilities consist of money, the intermediary is willing to de-leverage even further. Therefore, money supply reduces, leading to even higher deflation and depressing the price of physical capital.

An alternative way to look at the relationship between the volatility of relative prices versus intermediary's wealth share reads as follows. Let π_t be defined as

$$\pi_t = \frac{p_t}{p_t + q_t},$$

i.e. the value of money over the value of the economy. Then, by Itô's Lemma (see Appendix A.7.4).

$$d\pi_t = \pi_\psi d\psi_t^2 + \frac{1}{2}\pi_{\psi\psi} d\psi_t^2.$$

Therefore,

$$\sigma_t^\pi = \epsilon_{\pi,\psi}\sigma_t^\psi,$$

where $\epsilon_{\pi,\psi}$ is the elasticity of relative money value with respect to intermediary's relative wealth share and, by (1.119),

$$\sigma_t^\psi = \frac{\omega_t^I\left(\sigma^b - \sigma_t^K\right)}{1 - \epsilon_{\pi,\psi}\left(1 - \frac{\omega_t}{1-\pi_t}\right)}. \tag{1.121}$$

The numerator of (1.121) represents fundamental risk and the second addend of the denominator acts as an amplification term. Instead, the former element depends on portfolio choices of individuals, i.e. on risk taking behaviour. In this term, amplification exists under two conditions:

1. $\pi_\psi < 0$; The relative value of money π reduces when the size of intermediary's ψ increases. Thus $\psi, q \uparrow \rightarrow$ Inside Money$\uparrow\rightarrow(1-\pi_t)\downarrow$ (Inflation);
2. $\omega_t^I > 1 - \pi_t$; Intermediary's position in risky assets must be higher than the relative value of capital over money (always satisfies as long as the intermediary hold leverage/issues inside money).

Chapter 2

Risk Pooling, Leverage, and the Business Cycle[1]

"Not every business cycle has a financial crisis. Frequently they do."

- Kenneth Arrow, *2009*

Abstract

This paper studies the impact of financial sector size and leverage on the business cycle and risk-free rates dynamics. We develop a general equilibrium model of a productive economy where financial intermediaries provide risk mitigation to households by pooling the idiosyncratic risks of their investment activities. In contrast to previous studies, we show that intermediaries' risk pooling capacity not only amplifies the fluctuations of the relative wealth between sectors, but may also mitigate business cycle fluctuations. Accordingly, households benefit the most when the financial sector is neither too small, thus avoiding high consumption fluctuations, nor too big, so that fewer resources are lost after intermediation costs.

Keywords Amplification, Business Cycle, Financial Frictions, Leverage, Risk Pooling.

JEL Classification E13, E32, E69, G12.

[1]Based on a joint work with Pietro Dindo and Loriana Pelizzon.

2.1 Introduction

There is widespread agreement that financial intermediation is not only a veil between savers and borrowers. On the contrary, it plays a fundamental role to properly characterize business and financial cycles altogether (Adrian and Shin, 2010; Borio, 2014; Brunnermeier and Sannikov, 2014, 2016a; He et al., 2017; He and Krishnamurthy, 2011, 2013, 2019).

The focus of many recent studies that embody a financial sector in a general equilibrium setting is on the negative externalities that come after intermediaries' activity. Endogenous risk - amplification of exogenous shocks - is generated by the interaction of heterogeneous agents in presence of financial frictions. Much less has been done to develop theoretical models showing that also positive externalities associate to financial sector's deeds.

Broadly speaking, one could think of various channels through which the financial sector may affect the real economy: its supply of payment services, its fundamental role at pricing and allocating risks, its capacity of converting illiquid assets into cash without undue loss of value, and many others. In this paper, we exclusively focus on two specific and related functions of the financial sector: risk pooling and mitigation.

In a nutshell, the main contribution of our paper is to work out a dynamic general equilibrium model of a productive economy where financial intermediaries, on the asset side, pool idiosyncratic risky stakes in firms, pay the associated intermediation cost, and bear only their systematic risk component. On the liability side, by issuing short-term risk-free debt (i.e. via leverage), they provide households with risk-mitigation instruments. In equilibrium, this mechanism stems into a structural mismatch between the risk of intermediaries' assets and liabilities, whose magnitude fundamentally determines the width of business cycle fluctuations jointly with the dynamics of risk-free interest rates. In particular, we show that the extent of intermediaries' risk mismatch associates to their risk-pooling capacity and generates: i) State-dependent mitigation of aggregate output and consumption fluctuations; ii) Pro-cyclical (possibly negative) risk-free interest rates; iii) Households benefit the most when the financial sector is neither too small, thus avoiding high consumption fluctuations, nor too big, so that fewer resources are lost after intermediation costs.

We model our economy in continuous-time, and solve for the equilibrium joint dynamics of capital prices and size of the financial sector.[2]

We assume that capital is either held by equally risk averse heterogeneous households or by an aggregate financial sector. Financial frictions are introduced

[2]From the methodological standpoint, we follow the approach proposed by He and Krishnamurthy (2011, 2013) and Brunnermeier and Sannikov (2014, 2016a). A more detailed discussion can be found in the literature review section.

by assuming *restricted market participation* and, in the spirit of Diamond (1984), the financial sector has a cost advantage at pooling idiosyncratic risks.

Due to restricted market participation, each household has free access to its own specific firm only, sustaining both systematic and idiosyncratic risks. Therefore, households are not able to diversify idiosyncratic risks among themselves, and purchase a risk-free bond issued by the financial sector; the bond acts as an instrument of risk mitigation and allows households to smooth consumption.[3]

The choice of risk mitigation through risk-free bonds implies that the financial sector leverage is counter-cyclical. After a negative shock, the financial sector relative capitalization decreases, while its leverage increases further, to keep up with the households' higher demand for risk-free bonds. The opposite holds as a response to positive shocks: the financial sector increases its relative capitalization, its leverage reduces, and so does its supply of risk-free bonds. This mechanism is consistent with recent empirical findings suggesting counter-cyclical financial leverage (see He et al., 2017).[4]

From the macroeconomic perspective, financial sector capitalization channels exogenous systematic shocks because the width of aggregate output and consumption fluctuations largely depends on its leverage (size). Most importantly, the dynamics of output is affected by intermediation costs both in its growth rate and volatility; due to those costs, the output per unit of capital depends by a factor that negatively relates to size of the financial sector.[5] Therefore, the output drift is decreasing in the size of the financial sector due to a *pecuniary*

[3]Note that, in equilibrium, the households' demand for risk-free bonds is supplied by the financial sector, that uses it to leverage its balance sheet. As a result, firms are financed by both households and intermediaries that provide venture capital. This means that intermediaries bear a fraction of all firms' equity (=assets), rather than financing them via debt securities, and firms neither do leverage nor default. We silence both the channels of firms' and households' leverage on purpose because our focus is on the effect that financial sector risk pooling has on the business cycle, without the indirect effects non-financial firms' and householders' leverage (differently from Brunnermeier and Sannikov (2014) and Korinek and Simsek (2016), among others).

[4]This stylized fact stays in stark contrast with previous evidence in Adrian et al. (2014) where leverage is pro-cyclical. This is due to our choice of considering financial intermediaries focusing on their activity as *central dealers* of idiosyncratic risky claims, and relates to the marginal value of the financial sector's aggregate wealth. Pro-cyclical leverage empirical evidence also features in Adrian and Shin (2010, 2013) and has a theoretical foundation in Adrian and Boyarchenko (2012). In this stream of the literature, pro-cyclical leverage is a consequence of pro-cyclical VaR leverage constraints. The problem of leverage cyclicality is also discussed in Adrian et al. (2016), where they consider the difference between market and book leverage.

[5]This result squares nicely with the empirical evidence in Philippon and Reshef (2012), claiming that the size of financial intermediaries relate to the remuneration of their executive; in fact, they show that the size distribution of financial firms explains about one fifth of the premium for their executives. This is relevant because financial services account for up to 25% of the overall increase in wage inequality since 1980. In particular, they argue that financiers may be overpaid from a social point of view.

externality, the larger the financial sector capitalization: the lower the aggregate productivity of capital (due to high intermediation costs per unit of capital), the lower its price, the lower the investments in new capital (growth rate).

On the other hand, the same externality positively affects the business cycle by mitigating the fluctuations of aggregate output.

This happens because the fraction of aggregate capital under intermediaries management is increasing in financial sector capitalization. Accordingly, when a positive systematic shocks increase intermediaries relative wealth (and so their managed capital stock), it also reduces equilibrium prices, dampening the fluctuations of capital real value.

In a similar fashion, aggregate and households' consumption dynamics benefit from the pecuniary externality offered by the financial sector risk pooling (mitigation) but experience low growth rates when leverage is too high because risk mitigation becomes increasingly costly.

In this regard, another theoretical result of our model concerns the equilibrium risk-free interest rates, that may turn negative when the financial sector is too small.[6] Note that this effect mirrors the households' demand of risk-free bonds, and does not require a crisis situation to take place.

In the last part of the paper we study agents' welfare as related to the size of the financial sector. Overall, we find that households benefit the most when the financial sector is neither too small (offering too little - and costly - risk mitigation) nor too big (so that households have a lower lower capitalization).

Motivated by this finding, we investigate whether static leverage constraints and redistributive taxation policies could improve households' welfare. According to our model: i) A tax that redistributes wealth from the financial sector to the households prevents the former from growing too large, and so to waste too many resources after intermediation costs; ii) Leverage constraints acts by preventing the financial sector from collecting too much capital, and so from paying the associated intermediation cost. In turn, this fosters additional growth as capital is allocated to the more productive households. However, it may negatively affect the mitigation of output and consumption fluctuations.[7]

All in all, out theoretical result suggest that there exist leverage constraints and redistributive taxation policies such that the size of the financial sector

[6]According to Gourinchas and Rey (2017), a weakened financial sector may lead to persistently low, or even negative, short term interest rates for an extended period of time.

[7]The welfare effect of imposing leverage constraints is thus two-sided: from the perspective of the financial sector, the result is a net welfare loss. This is because, when binding, the constraints keep the financial sector relative capitalization at a low(er) level with a high(er) probability. Conversely, leverage constraints may be welfare-improving for the households, as there exists a trade-off between the gain from the higher growth rate of their consumption versus the loss due to a weakened financial sector (less mitigation).

remains within an "optimal" range in order to improve households' welfare.

The paper proceeds as follows: Section 2.1.1 frames our results as related to the incumbent literature. Then, Section 2.2 outlines the model micro-foundation (2.2.1) and the agents' problems (2.2.2).

Section 2.3 derives the competitive equilibrium (2.3.1) and discusses its characterization (2.3.2). Section 2.4 focuses on the intermediate case where both classes of agents co-exist and characterise the link between financial sector leverage, risk-free interest rates (2.4.1), and the macroeconomic dynamics (2.4.2). Finally, Section 2.5 investigates the role of leverage constraints and redistributive taxation policies at increasing the households' welfare. Section 2.6 concludes.

2.1.1 Related Literature

This paper belongs to the body of literature describing the relationship between financial intermediation, the macroeconomic dynamics, and its welfare implications.

Methodologically, we are close to the seminal work of He and Krishnamurthy (2011, 2013) and of Brunnermeier and Sannikov (2014, 2016a). However, we substantially diverge in several dimensions: He and Krishnamurthy (2011, 2013) consider general equilibrium endowment economies, and study the dynamics of asset pricing as it relates to financial intermediaries capitalization and financial crises. On the contrary, we develop a productive economy to investigate financial intermediaries risk pooling activities as connected to real business cycle fluctuations and risk-free rates.

Brunnermeier and Sannikov (2014) build a model where more productive agents (experts) leverage their balance sheet. On the contrary, the households in our model do not leverage, even if they are the most productive agents; conversely, it is the financial sector that leverages up, and sells to households risk-free bonds in exchange of a fraction of their firms' risky capital (equity). What follows is that, in our model, more productive agents have extra risk exposure, and so demand for mitigation instruments. Accordingly, less productive financial intermediaries provide risk-pooling, by buying households' risky claims, and risk mitigation, by issuing risk-free bonds that they sell to households.

Another important difference concerns the financial friction: whereas in Brunnermeier and Sannikov (2014) is that the experts' consumption must hold positive (and households' may be negative), in our case the friction comes after the assumption of *restricted market participation*. These differences lead to substantially opposite equilibrium dynamics and stationary wealth share distribution. In these terms, our model is complementary to theirs.

We also connect to Brunnermeier and Sannikov (2016a), whose core contribution is to study the value of money when offered by financial intermediaries as a risk-mitigation tool to insure idiosyncratic risks faced entrepreneurs. Despite the fact that in both papers intermediaries are short in risk mitigation instruments and long in firms' risky stakes, the consequences of the associated risk-mismatch mechanism are rather different.[8] As in Brunnermeier and Sannikov (2016a) mitigation is provided via money, a negative aggregate shock that affects intermediaries' balance sheet decreases the provision of risk mitigation - generates higher demand for money - and increases its price. The resulting deflation further depreciates intermediaries' liability, and fosters additional decrease in the provision of risk mitigation (deflationary spiral). Monetary policy is the instrument to break the spiral. In our model instead, risk mitigation is provided via short-term risk free bonds. When a negative shock hits, the financial sector becomes smaller, the demand for deposit increases, and interest rates decline. The latter positively affect the provision of idiosyncratic risk-mitigation, leading to an increase of leverage. No spiral occurs and the natural policy instrument in our set-up is a leverage constraint, whose effect is primarily to decrease both the growth rate drift and volatility of financial intermediaries' wealth share (leading to a higher welfare for the households).

More specifically, we can structure our contribution along the following dimensions: the role of exogenous (systematic and) idiosyncratic risks in a dynamic model with frictions (IR); the role of financial sector leverage (LV) and size in amplifying (but also mitigating) the propagation of exogenous shocks (AM), as well as their effect over the business cycle, consumption, and their fluctuations (BC); how the allocation of risk and restricted market participation relates to asset pricing (AP); the welfare implications of leverage and size of the financial sector (W).

An early approach connecting the allocation of risk to portfolio choices *(IR)* in a general equilibrium set-up can be found in Heaton and Lucas (2004). Their analysis builds on the observation that idiosyncratic risk is priced by the market, since agents are risk averse and unable to diversify idiosyncratic shocks by themselves. Nevertheless, they do not consider any financial sector.

By introducing *restricted market participation*, our model also relates to the body of literature that studies *incomplete* markets and the role of aggregate uninsurable shocks in equilibrium dynamics. Seminal papers in this field are Aiyagari and Gertler (1991), Huggett (1993), as well as Aiyagari and Rao (1994).

In this paper, *endogenous risk* takes place as an amplification/mitigation (AM) of exogenous systematic shocks. As in Brunnermeier and Sannikov (2014),

[8]The source of the risk-mismatch in Brunnermeier and Sannikov (2016a) is as follows: it is assumed that there are two types of capital (both necessary to produce the consumption good) and while the equilibrium value of money captures risks associated to the production of both types, the financial intermediary can invest only in one type of risky firms.

our model features the so called *volatility paradox* (see also Brunnermeier and Adrian, 2016), i.e. lower exogenous risk may lead to higher endogenous volatility, especially when financial capitalisation is arbitrary low. However, our model differs in several substantial ways: first, we account for both systematic and idiosyncratic risks as determinants of aggregate fluctuations. This feature squares with empirical evidence suggesting a relationship between macroeconomic dynamics and the state of the financial system (Adrian et al., 2016). Second, in our model the effect of increasing idiosyncratic risk leads to further leverage. This is because, after *restricted market participation*, the households increase their demand for risk-free bonds. Another relevant element of our model is that equilibrium risk-free interest rates fluctuate over time (and may take negative values) instead of being constant.[9]

As for LV, our paper moves along the seminal stream accounting for financial frictions in general equilibrium (for a general discussion see Brunnermeier et al., 2012; Moritz and Taylor, 2012) and, more specifically, to those known as *post-crisis* macro models (see Haven et al., 2016). An important feature we share with the post-crisis literature is the connection between financial leverage and the magnitude of economic fluctuations.[10]

Still concerning LV, the core difference between the aforementioned stream of literature and our paper consists of both the source of frictions, *restricted market participation* in place of an agency problem as well as of their externalities: in our model intermediaries leverage endogenously generates a mitigation effect that may overtake the amplification of business cycle fluctuations.[11]

It is relevant to highlight that the core effect of introducing financial frictions through occasionally binding constraint is that central theorems of welfare do not hold, and the equilibrium risk allocation is inefficient (as for example in Mendoza and Bianchi, 2010; Bianchi, 2011).[12]

From the asset pricing perspective (AP), our contribution has common characteristics with the literature of general equilibrium models where financial cycles and constraints determine asset prices, as for example in He and Krishnamurthy

[9] In Brunnermeier and Sannikov (2014) and Phelan (2016) instead, agents are risk neutral, and thus the equilibrium risk-free rate equals the discount rate of the most conservative class of agent.

[10] The idea of the financial cycle being determinant of the business cycle is introduced in Carlstrom and Fuerst (1997). A similar setting with adjustment costs on capital investment can be found in Kiyotaki and Moore (1997), and it is developed in a New Keynesian setting by Bernanke et al. (1999).

[11] The idea that uninsurable risk associates to structural financial leverage is introduced in a theoretical setting by Krishnamurthy (2003).

[12] We obtain a similar effect by *restricted market participation*, yet suboptimal allocation is not contingent and happens also when the constraint is slack. An interesting exercise discussing how to counter pecuniary externality in a financial accelerator framework is in Korinek (2011). The paper proposes a tax based disincentive to extreme leverage.

(2011, 2013).

The core difference is that our model does not need the constraints to be binding in order to generate those effects. Moreover, we explicitly micro-found the demand for risk-free assets and show that in certain states of the world real risk-free interest rates could be negative.

In light of the role of financial leverage and size as related to the business cycle (BC) and risk-free interest rates during crises (see He et al., 2010), our results relate to those papers at the intersection between finance and macroeconomics treating systemic risk, as for example Nuno and Rey (2017).

With Nuno and Rey (2017) we share the trade-off between economic growth and stability, although our mechanism of amplification is deeply different, and so it is our definition of systemic crisis.[13]

Finally, we investigate the relationship between the size of the financial sector, leverage, and welfare (W). On this side, we are related to the work of Philippon (2010) that studies the interaction between the financial and non-financial sectors, and investigates whether it is optimal to subsidize or tax the former. However, our model is largely different, and so it is the role played by the financial sector. We also partially relate to the literature that investigates optimal financial leverage constraints, in particular to Phelan (2016) and Pancost and Robatto (2018).

A common element between this work and Phelan (2016) is the relationship linking financial leverage constraints and welfare. His paper suggests that a policy of recapitalizing banks, that mechanically decreases leverage to the optimal level, is welfare-improving. This relates to the concept of *welfare maximizing size* of the financial sector suggested by our model. Nevertheless, we strongly differ with respect to several aspects: first, we introduce *restricted market participation* as a friction that allows us to model the demand for risk-free assets. Second, our model displays a smooth dynamics rather then a step-wise process of aggregate consumption. This allows us to relate financial leverage to the economic macro-dynamics.

Similar to our setting, Pancost and Robatto (2018) consider the role of banks in providing risk pooling services as well as their role of supplying risk mitigation instruments through deposits (for a similar argument, see also DeAngelo and Stulz, 2015). Although we reach similar conclusion concerning the welfare improvement that may come to households after imposing leverage constraints, both mechanism and focus of our papers differ substantially. Pancost and Ro-

[13]To Nuno and Rey (2017) systemic risk is the probability of intermediaries default whereas in our model it can be interpreted as the probability of being below a certain capitalisation threshold. In these terms, our model is similar to He and Krishnamurthy (2019) where systemic risk is the conditional probability of reaching binding constraint states. However, this is not the focus of our paper.

batto (2018) argue the optimal capital requirement to be imposed on a risk neutral financial intermediary as dependent on the trade-off between good and bad-risk taking.[14] Conversely, this paper focuses on the relationship between financial sector risk pooling and the macro-financial dynamics in a model where, due to restricted market participation, the relative size of households and intermediaries matters.

In summary, the strength of our model (and of its theoretical predictions) is its ability to jointly consider very different dimensions: the role of systematic and idiosyncratic risks; how their allocation channels mitigation of exogenous systematic shocks; the role of leverage constraints as related to the dynamics of the financial sector size; their effect over the macroeconomic dynamics and, in turn, households' welfare.

2.2 The Model

In this section, we first introduce the overall economic environment. Then, we discuss the agents' problem and describe the features of the return on risky assets. We start with a narrative description of the model.

We consider a continuous-time infinite-horizon production economy with two goods: physical capital (such as a tree) and output (perishable good, such as apples). Each good is produced by a specific type of firms, the perishable good acts as numéraire.

There are two types of assets: risky claims and risk-free bonds. Risky claims are written on the profits of capital-producing firms. The risk to which they are exposed is both systematic (economy-wide) and idiosyncratic (firm-specific). Risk-free bonds have value as risk-mitigation instruments and are in zero-net supply.

The economy is populated by two classes of agents: *households* and *financial intermediaries*. Intermediaries are allowed to invest in all firms; accordingly, they pool idiosyncratic risks and are exposed to systematic shocks only. The expected return on their (risky) assets is reduced by a cost of intermediation paid for each unit of capital.[15] Conversely, due to restricted market participation, each household is allowed to invest in one capital producing firm only. Since households do not pay the intermediation cost, they earn higher expected returns. However, their over-exposure to idiosyncratic risk generates positive demand for risk-mitigation instruments. As we shall see, in equilibrium, this

[14] Capital requirements may improve welfare when the supply of inputs of firms facing idiosyncratic shocks is particularly rigid, due to a decrease in the price of inputs.

[15] The intermediation cost can be thought as a reduced form that represents the administrative costs that the intermediaries bear for screening and monitoring the firm activity, that instead the entrepreneur observes, as well as for operational purposes.

demand will be satisfied by the financial sector through its short position in risk-free bonds.[16,17]

The share of risky claims that is left un-pooled, i.e. that remains in the hands of households', determines the idiosyncratic risk allocation in the economy and with it consumption, output, risky assets, and equilibrium prices.

2.2.1 Technologies and Risky Claims

There exist two types of firms: Type I has the inter-temporal role of generating new physical capital (trees) through a concave technology $\Phi(\cdot)$ that uses the perishable good (apples) as input. Let $[0,1]$ be a continuum of type I firms and let $dW_t \perp d\tilde{W}_t^i \perp d\tilde{W}_t^j \ \forall i \neq j, \{i,j\} \in [0,1]$ be independent standard Brownian motions defined on the filtered probability space $(\Omega, \mathcal{H}, \mathbb{P})$, where $\{\mathcal{H}_t, t > 0\}$ is the natural filtration over the measurable space $(\Omega, \mathcal{H})$. The capital stock $k_t^i \in \mathbb{R}$ managed by firm $i \in [0,1]$ follows a bi-variate Itô diffusion

$$T_t^i: \quad \frac{dk_t^i}{k_t^i} = \left[\Phi(\iota_t^i) - \delta\right] dt + \sigma dW_t + \tilde{\sigma} d\tilde{W}_t{}^i, \text{ with } \Phi(\iota) = \frac{1}{\theta} \log(1 + \theta\iota), \quad (2.1)$$

where δ is the depreciation rate, ι_t^i is the reinvestment rate as dependent on the concavity parameter θ, σ and $\tilde{\sigma}$ are constant systematic and idiosyncratic diffusion terms, respectively.

Capital producing firms live one period. At each instant t, they are constituted by transfers of physical capital executed by either households or intermediaries, and liquidated at $s = t + dt$. Accordingly, type I firms finance their own constitution by issuing risky claims with stochastic pay-off dR_t^i over $[t, s]$ written on their net revenues. The total return on firms' risky claims is endogenous and determined in equilibrium. Firms of type I earn revenues by renting capital to firms of type II at the instantaneous rate p_t, and choose the re-investment rate of capital ι to maximise the expected return on their risky claims issuances. Note that the zero profit condition of capital producing firms' must be consistent with the equilibrium return on the risky claim dR_t^i. This is equivalent to a non-arbitrage condition, i.e. the return on firms' risky issuance (their equity) is such that the present discounted value of their revenues under the risk-neutral measure equals the current value of physical capital stock supplied by the agents. If

[16] From the households' perspective, the restricted access to financial markets is an exceedingly relevant topic. For instance, Davydiuka et al. (2018) provide a theoretical model that motivates the substantial decline of small firms going public in the last 20 years (as documented in Gao et al., 2013) by the presence of increasing financial frictions, such as IPO and regulatory-disclosure related costs.

[17] In Appendix B.1.1 we show that, even if both households and financial intermediaries have full access to pooled and un-pooled risky assets, there exists *restricted market participation* as long as the intermediaries are more efficient at pooling claims.

such condition holds, each firm breaks even for each k_t^i, its size is indeterminate, and it is willing to supply each market demand.[18]

Firms of type II, also a continuum, do not have an inter-temporal dimension, and produce perishable good y_t^i through a linear production function that has capital as input:

$$y_t^i = Ak_t^i. \tag{2.2}$$

The profit of each firm of type II at time t is thus simply $(A-p_t)k_t^i$. Therefore, in equilibrium, they always break even and their size is indeterminate.

2.2.2 Financial Sector and Households

The economy is populated by households and financial intermediaries. Households consist of a heterogeneous continuum of unit mass $\mathbb{H} := [0,1]$ indexed $h \in \mathbb{H}$. Similarly, intermediaries belong to $\mathbb{F} := (1,2]$ and are indexed $f \in \mathbb{F}$. Since the latter are homogeneous, they can be accounted for as a representative *financial sector*.

Intermediaries and households trade physical capital in a perfectly competitive market at the endogenous price q_t. Each agent has an initial endowment $e_0^i \neq 0$ and, over each time interval $[t, t+dt)$, she consumes at a rate $\frac{c_t^i}{e_t^i}$. Moreover, she allocates a fraction ω_t^i of what is left to risky claims, and a fraction $(1-\omega_t^i)$ to risk-less bonds.

All agents have log utility and discount the future at a common rate ρ; they are infinitely lived and chose c_t^i and ω_t^i to maximize their objective function

$$V_0 := \max_{\{c_t^i, \omega_t^i\} \in B^i} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \ln c_t^i dt \right], \quad i \in \{h, f\} \tag{2.3}$$

subject to

$$B_t^i : \quad \frac{de_t^i}{e_t^i} = \omega_t^i dR_t^i + \left(1-\omega_t^i\right) r_t dt - \frac{c_t^i}{e_t^i} dt, \tag{2.4}$$

where r_t is the risk-free interest rate, and the i^{th} agent has access to a different risky portfolio with return dR_t^i.[19]

The financial sector can invest the stock of physical capital at its disposal

[18]Type I firms' technology is non-linear in ι; however, linearity in k is maintained through the identification of $c = \iota k$ as expense for the perishable consumption and by splitting the decisions upon output and capital production over two types of firms. Details are in Appendix B.1.

[19]The derivation of the solution is in Appendix B.4.1. With a slight abuse of notation we use dR_t^i to denote the return to the agent i of firm $i = h$, and dR_t^f to denote the return of the aggregate portfolio that pools risky claims issued by all firms $i \in [0,1]$.

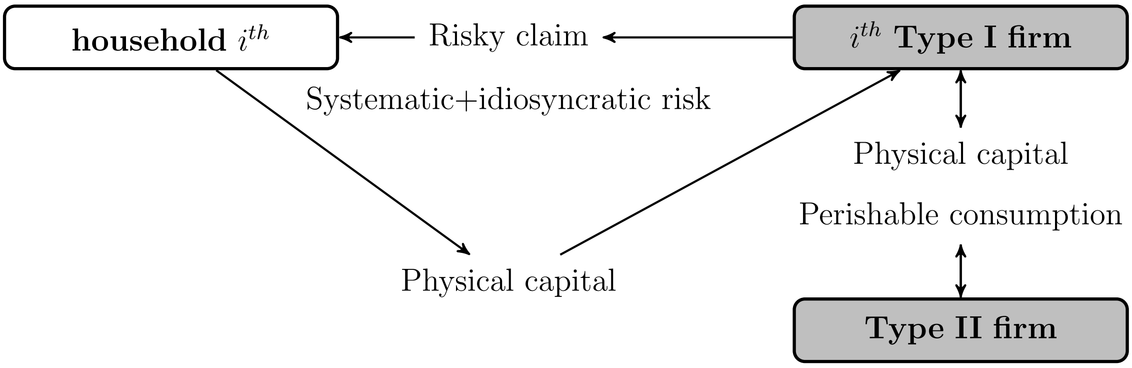

Figure 2.1: Micro-structure of production and risky claims for the households

across all type I firms, against the payment of an intermediation cost η per unit of capital.[20] The intermediation cost can be thought as a reduced form that represents the administrative costs that the intermediaries bear for screening and monitoring each firm, which the entrepreneur instead observes, as well as for operational purposes.

Conversely, due to *restricted market participation*, the households cannot diversify among firms, so that her investment opportunity set is restricted to the i^{th} firm only. Therefore, firms are financed by both households and intermediaries that, in our framework, provide venture capital services. Thus, the return on agent i risky assets holdings, dR_t^i, has the following structure:

$$dR_t^i = \underbrace{\mu_t^i dt}_{\text{Expected return}} - \underbrace{[\mathbb{I}_{i=f}]\,\frac{\eta}{q_t}dt}_{\text{Intermediation cost}} + \underbrace{\sigma_t dW_t}_{\text{Systematic risk}} + \underbrace{[\mathbb{I}_{i=h}]\,\tilde{\sigma} d\tilde{W}_t^i}_{\text{Idiosyncratic risk}}, \qquad (2.5)$$

where $\mathbb{I}_i$ is the indicator function, and both expected return μ_t^i and systematic risk σ_t are endogenous and determined in equilibrium as dependent on firms' optimizing behaviour.

The relationship between household i and its firm, is synthetically depicted in Figure 2.1. Similarly, Figure 2.2 displays the mechanism by which the financial sector may purchase a fraction of the households' physical capital versus the issuance of risk-free bonds.

It is relevant to highlight that restricted market participation, that we have

[20]A seminal paper that develops a theoretical framework where financial intermediation costs associate to a net advantage due to diversification is published by Diamond (1984). In an economy where all the agents are risk averse, the paper shows that financial intermediaries must have lower delegation costs than an entrepreneur to viably provide intermediation services. This intermediaries centralized monitoring structure will mean that there are no active markets for their pooled assets. This relates to the concept of restricted market participation, being the aggregate financial sector the only one supplying risk-mitigation instruments. From an empirical perspective, the side effect of risk pooling at financial institutions is treated, among the others, in Wolf (2010) and van Oordt (2014).

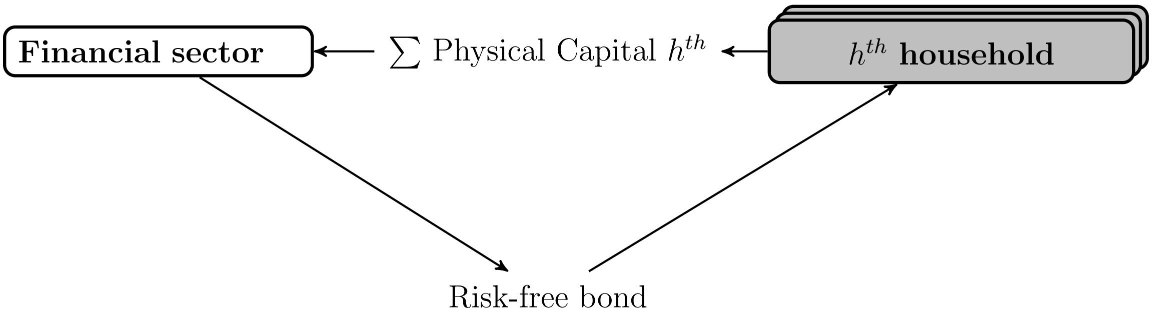

Figure 2.2: The financial sector and its purchase of a fraction of the households' physical capital versus the issuance of risk-free bonds.

assumed to be an exogenous financial friction, may emerge in equilibrium in presence of *transaction costs* on the households'. In Appendix B.1.1 we show that, even if both households and financial intermediaries have full access to risk-free bonds and both pooled and un-pooled risky assets, there does exist *restricted market participation* as long as the intermediaries are more efficient at pooling claims than households, and the transaction cost is not too large.[21]

2.3 The Equilibrium Dynamics

This section derives the *competitive equilibrium* of this economy (2.3.1). Then, Section 2.3.2 outlines the associated return on risky assets and characterizes the unique state variable: the relative capitalization of the financial sector. Henceforth, we denote all the aggregate variables with a capital letter.

2.3.1 Competitive Equilibrium

Informally, the equilibrium consists of maps from histories of systematic shocks to prices (capital prices, returns on risky claims, risk-free interest rates), production choices and consumption choices, as well as asset allocations such that firms maximize their profits, agents maximize their expected utility, and markets clear. The formal definition is given in Appendix B.2.

An equilibrium snapshot of agents' balance sheets at any instant of time t is in Figure 2.3. The dark grey boxes depict the asset allocation of each class of agents while the light grey boxes represent their liabilities.

The financial sector holds a long (leveraged) position in the aggregate portfolio of risky claims that is financed by both its own capital endowment $\int_{\mathbb{F}} e^f df = E^f$ plus a short position in risk-free bonds $\left(\omega^f - 1\right) E^f$. Conversely, each house-

[21]It is relevant to stress that the presence of capital markets is not self-sufficient to solve the monitoring problem, as long as the transaction cost is higher for the households than for the financial sector (see Diamond, 1984).

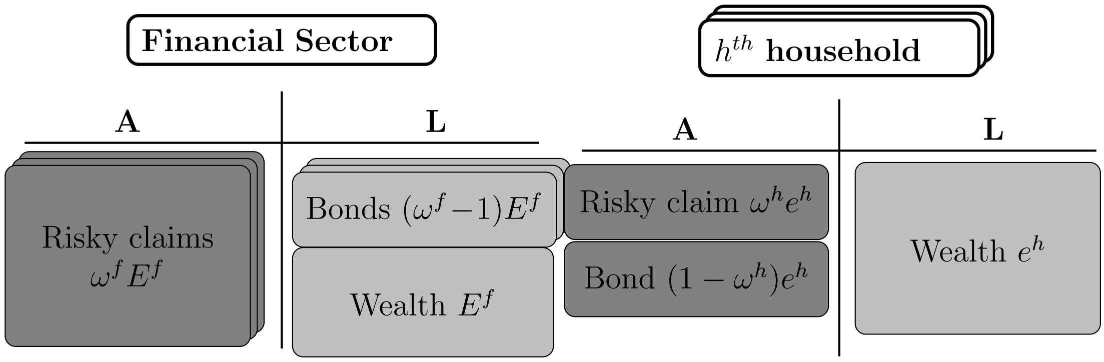

Figure 2.3: Synthetic agents' balance sheets at time t

hold allocates its wealth between a single risky asset and a risk-free bond. Market clearing conditions imply that risk-free bond is in zero net supply, while financial sector capital and households' wealth sum up to the aggregate (value of) capital within the economy $K_t q_t$. Accordingly, the stock of wealth that belongs to the aggregate of households holds as $\int_{\mathbb{H}} e^h dh = E^h$.

To place the last piece of the puzzle, Figure 2.4 shows the balance sheet of the j^{th} capital producing firms at any time t. As for the households' and financial sector balance sheets in Figure 2.3, the dark grey box represents the value of the firm's assets, whereas the light grey ones depicts the value of its liabilities. Basically, each capital producing firm is jointly financed by households' plus financial intermediaries' capital stock, that is they bear a fraction of all the risk of firms' assets. Therefore, firms neither do leverage nor default.[22]

In summary, each firm collects physical capital from both households' and intermediaries' (straight arrows) versus the issuance of risky claims written on its net revenues (dashed arrows). In particular, the j firm gathers capital $\omega^h e^{h,j}$ from the j^{th} household as well as from the financial sector, that evenly finances the continuum of firms, so that $\int_{\mathbb{F}} \omega^f e^{f,j} df = \int_{\mathbb{J}} \omega^f e^{f,j} dj = \omega^f E^f$.

2.3.2 Competitive Equilibrium: Characterization

In order to derive the equilibrium, we express optimal portfolios, drift, and diffusion of the stochastic process in (2.6) as functions of financial sector relative capitalization ψ_t, the state variable of our economy, defined as follows:

Definition 1. ***Relative Financial Capitalization*** *Let ψ_t be the financial sector's share of total capital value. Conversely, $(1 - \psi_t)$ represents the households'*

[22]The market price of the risky claim q and the dynamics of its returns, as related to the firms' optimal policies are discussed at length in Appendix B.1.

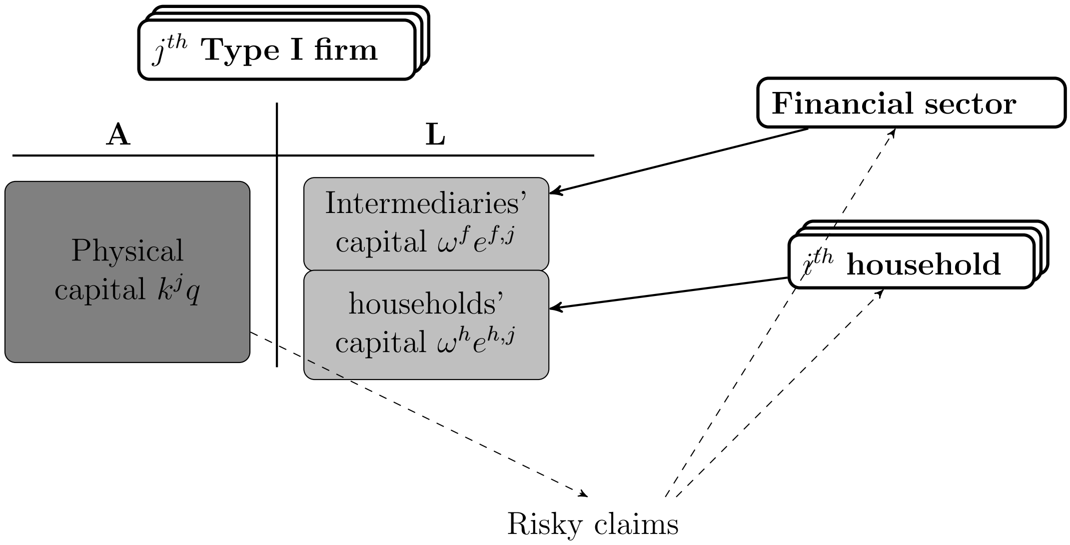

Figure 2.4: Synthetic balance sheet of the j^{th} capital producing firms at time t

share of aggregate capital value:

$$\psi_t := \frac{E^f_t}{K_t q_t}, \quad 1 - \psi_t := \frac{E^h_t}{K_t q_t}.$$

As we shall see, all relevant equilibrium quantities can be written as a function of ψ_t (see Appendix B.4.2).

Next, we restrict our search to the class of dynamically simple equilibria in the state variable ψ.[23] Moreover, we look for equilibria where the stochastic process that drives the price of physical capital q is an Itô diffusion.

Assumption 1. ***Price of Physical Capital Dynamics***
The price of physical capital evolves as an Itô diffusion:

$$dq_t := q_t \mu^q_t dt - q_t \sigma^q_t dW_t, \tag{2.6}$$

where μ^q_t and σ^q_t are $\mathcal{H}_t$-adapted functions.

According to Assumption 1, the dynamics of capital price is not affected by idiosyncratic shocks. Moreover, the minus sign to the diffusion term implies that positive shocks to capital stock affect negatively the price of unit of capital in consumption good.[24]

[23]The equilibrium is dynamically simple, i.e. it is time homogeneous and Markov in the state variable and it is such that there exists an associated stationary distribution. For a formal definition see Duffie et al. (1994)

[24]This choice is fundamental, as we shall see in the proof of Theorem 1 there does not exist an equilibrium with $\mathbb{C}ov_t \left[dk^i_t, dq_t\right] > 0$.

It can be shown that (see Appendix B.1) the total return on the i^{th} claim follows the dynamics in equation (3.4) with

$$\mu_t^i = \mu_t := \underbrace{\frac{A - \iota_t}{q_t}}_{\text{Dividend Yield}} + \underbrace{\Phi(\iota_t) - \delta + \mu_t^q - \sigma\sigma_t^q}_{\text{Capital Gain}}, \tag{2.7}$$

and

$$\sigma_t := \sigma - \sigma_t^q.$$

Accordingly, by equations (3.4) and (2.7), the expected return on risky assets of households is higher than for the financial sector, and so it is the associated risk. The difference is the absolute intermediation cost $\eta\omega^f E^f$ that the financial sector must pay in order to pool idiosyncratic risks $\tilde{\sigma}$ from different type I firms.

We now have all the ingredients to derive the dynamics of the state ψ in a competitive equilibrium. Moreover, we outline the conditions such that both classes of agents survive in the long-run and there exists a stationary density of the financial sector relative wealth share. Our results are summarised in the following theorem:

Theorem 1. ***Relative Capitalization Dynamics***
Given the law of motion of q in (2.6), there exists a unique (Markov) competitive equilibrium and it is characterized by the following:

1. *The relative capitalization dynamics follows the diffusion process*

$$d\psi_t = \underbrace{\psi_t\sigma_t^2\left[\left(\omega_t^f - 1\right)^2 - \frac{\omega_t^f\psi_t}{\sigma_t^2}\frac{\eta}{q_t}\right]}_{\psi_t\mu^\psi(\psi_t, q(\psi_t))} dt + \underbrace{\psi_t\sigma_t\left(\omega_t^f - 1\right)}_{\psi_t\sigma^\psi(\psi_t, q(\psi_t))} dW_t. \tag{2.8}$$

2. *The associated dynamics of price $q(\psi_t)$ satisfies Assumption 1 with*

$$\begin{cases} \sigma^q(\psi_t, q(\psi_t)) = -\epsilon_{q,\psi}\sigma^\psi(\psi_t); \\ \mu^q(\psi_t, q(\psi_t)) = \mathcal{A}q(\psi_t), \end{cases} \tag{2.9}$$

   where $\epsilon_{q,\psi}$ is the physical capital price elasticity to financial sector relative wealth share and $\mathcal{A}$ is the characteristic operator.

3. *As long as the intermediation cost η is positive and not too high, the left-hand side and right hand side boundaries, $\psi = 0$ and $\psi = 1$, are never attainable,*

$$\eta \in \left(0, \tilde{\sigma}^2\frac{1 + \theta A}{1 + \theta\rho + \theta\tilde{\sigma}^2}\right) \quad \Rightarrow \psi_t \in (0, 1)\ \forall t \in (0, \infty),\ \mathbb{P} - a.s. \tag{2.10}$$

   and there exists a unique (non trivial) stationary density $\pi(\psi)$.

4. *When the intermediation cost η lays outside the interval in (2.10), the economy drifts either to the right-hand or to the left-hand side boundary, respectively. In particular:*

   (a) *Full-risk-pooling economy:*

$$\eta = 0 \Rightarrow \mu^{\psi}(\psi_t) > 0 \Rightarrow \lim_{t\to\infty} \psi_t = 1 \ \mathbb{P} - a.s.;$$

   (b) *No-risk-pooling economy:*

$$\eta > \tilde{\sigma}^2 \frac{1+\theta A}{1+\theta\rho+\theta\tilde{\sigma}^2} \Rightarrow \mu^{\psi}(\psi_t) < 0 \Rightarrow \lim_{t\to\infty} \psi_t = 0 \ \mathbb{P} - a.s.$$

Proof. Points 1 and 2 are proved in Appendix B.4.3. The characteristic operator $\mathcal{A}$ is defined in **?**. Points 3, together with the characterization of the stationary density, are discussed in Appendix B.4.4. Point 4 (a) is proved by setting $\eta = 0$ in the consumption market clearing condition (B.22). It follows that $\mu_t^q = \sigma_t^q = 0$. By point 1, $\mu_t^{\psi} > 0$ and thus, $\psi_t \to 1$ when $t \to \infty$. Point 4 (b) is proved similarly. □

The core implication of Theorem 1 is that we are able to express the dynamics of all relevant equilibrium variables in the model as a function of the joint dynamics of intermediaries' relative capitalization ψ and physical capital price, q, as conjectured in Assumption 1. Given the relation between the two dynamics in point 2, we can solve for their drift and diffusion numerically.[25]

Another important result is that, provided intermediation costs are neither too low nor too high, the relative capitalization keeps floating *around* its long-run average where both classes of agents have positive relative capitalization (point 3). In this sense, heterogeneity is *persistent*.[26]

Instead, when intermediation costs are either null or too high (depending on the size of idiosyncratic volatility), the economy collapses in one of two "extreme" cases: the *full-risk-pooling* economy and the *no-risk-pooling* economy (see point 4 and Appendix B.6 for the characterization of prices and allocations in these benchmark cases).

In the intermediate case where both classes of agents coexist, henceforth an economy with *partial risk pooling*, it is interesting to outline the way exogenous systematic shocks affect equilibrium prices and relative wealth share dynamics altogether. To illustrate this relationship, Figure 2.5 plots the price level $q(\psi_t)$

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[25] See Appendix B.4.2 for details.

[26] Note that the long-run dynamics of ψ_t does not necessary coincide with the associated deterministic steady state where the drift is null. A discussion upon the relationship between steady-state and long-term average of the stochastic process describing the equilibrium is in Klimenko et al. (2017).

(left) as well as the associated drift (centre) and diffusion (right) of the financial sector relative capitalization (blue) and prices (red) dynamics as a function of the relative financial capitalization $\psi \in (0,1)$.[27] As far as the capital prices level is concerned, the larger the relative size of the financial sector the lower the price of physical capital (Figure 2.5, left). This negative relation is due to the higher incidence of intermediation costs on the average productivity of capital when the financial sector is large. For example, in the extreme case where the financial sector manages all the capital, the intermediation cost is paid on all units of capital.

In general, positive exogenous systematic shocks shift the size ψ (and thus q) to the right towards one, because in equilibrium, due to leverage and risk pooling, the total return of the financial sector portfolio is higher than of the households'. The opposite occurs for negative shocks. Importantly, the response of the relative size and capital price dynamics to exogenous shocks is state dependent: when the financial sector capitalization is small, its drift is positive (negative for q). When the financial sector capitalization is big enough, instead, the drift of its relative capitalization is negative (positive for q). This is because the benefit of leverage is reduced while the costs associated to intermediation (proportional to η/q) are higher. Overall, the relative capitalization of the financial sector shrinks (while q increase). The central panel of Figure 2.5 provides an illustration.[28]

2.4 Risk-free Rates and Macro Dynamics

In this section, we describe the equilibrium dynamics of leverage, risk-free interest rates, and macro-variables in an economy with *partial risk-pooling*. We shall characterise this case as a deviation from the benchmarks of full-risk-pooling and no-risk-pooling discussed in Appendix B.6.

The discussion is structured as follows: first, Section 2.4.1 investigates the mechanism that links financial sector relative capitalization to equilibrium lever-

[27] We solve the model numerically - details are in Appendix - by assuming the following parameters: $A = 0.5$, $\delta = 0.05$, $\tilde{\sigma} = 0.55$, $\sigma = 0.2$, $\eta = 0.05$, $\theta = 2$, and $\rho = 0.05$. According to Ang et al. (2006) and Fangjian (2009), reasonable values for the annualized systematic and idiosyncratic volatilities are approximately 20% and 55%, respectively. The remaining parametric specification is close to the one in Brunnermeier and Sannikov (2016a). Despite these choices do not come after calibration, they produce reasonable qualitative outcomes. To verify the model robustness, in Appendix B.7 we discuss the changes of equilibrium dynamics with respect to the key parameters in the model, namely the size of systematic and idiosyncratic risk as well as intermediation costs.

[28] In Appendix B.7, we show how drift and diffusion change with idiosyncratic risk $\tilde{\sigma}$ and systematic risk σ. When the financial sector is small (high leverage) they both increase with $\tilde{\sigma}$ (the higher the risk, the higher the demand for risk mitigation, the higher the leverage) and decrease with σ (the higher the systematic risk, the lower the Sharpe ratio, the lower the leverage). The last result is consistent with the *volatility paradox*: due to leverage, a lower systematic risk increase endogenous fluctuations.

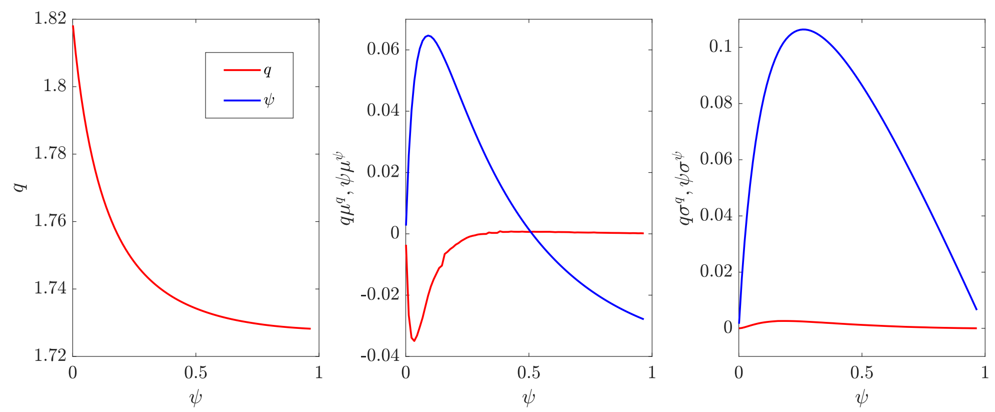

Figure 2.5: Left: Price level $q(\psi)$. Centre, right: Drift (centre) and diffusion (right) of the financial sector relative capitalization (blue) and prices (red) dynamics as a function of $\psi \in (0, 1)$.

age and risk-free interest rates. Then, Section 2.4.2 studies how those fluctuations affect real macro-variables, such as aggregate consumption and disposable output.

2.4.1 Leverage and Risk-free Rates

Having solved for the competitive equilibrium, we are able to address several questions upon the theoretical implications of our model, namely: how does financial leverage react to positive and negative exogenous (systematic) shocks, respectively? What is the relationship connecting the financial sector relative capitalization, and thus its leverage, to risk-free interest rates?

Figure 2.6 illustrates the drift (top-left) and diffusion (top-right) of the equilibrium process $d\psi_t$ over the state-space $\psi \in (0, 1)$. In red, we show the benchmark cases of the *full-risk-pooling* (solid) and the *no-risk-pooling* (dashed) economy. The same Figure (bottom row) displays the financial leverage ω^f (right) and the risk-free interest rate r (left) as functions of ψ.

Financial leverage The financial sector leverage, ω^f, is a decreasing function of ψ because the smaller the financial sector, the higher the demand of risk mitigation, the larger the leverage (Figure 2.6, bottom left). As shown in Appendix B.4.2, in equilibrium it holds

$$\omega_t^f = \frac{1}{\psi}\left[1 - \frac{\mu_t - r_t}{(\sigma - \sigma_t^q)^2 + \tilde{\sigma}^2}(1 - \psi_t)\right]. \tag{2.11}$$

Since in equilibrium leverage cannot be larger than $\frac{1}{\psi}$, which occurs when the financial sector holds all risky claims, the diminished financial sector leverage

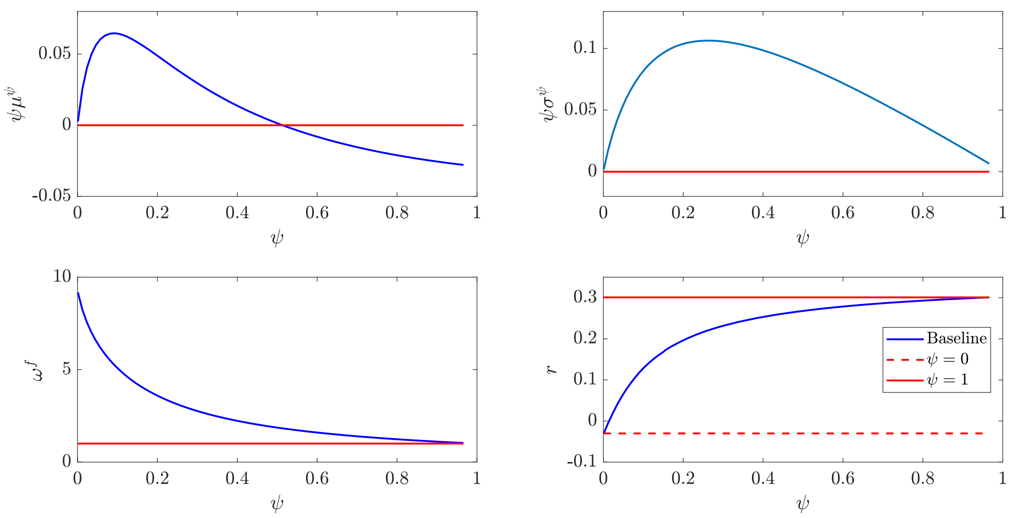

Figure 2.6: Top: Drift (left) and diffusion (right) of the equilibrium process $d\psi$ as a function of ψ. Bottom: Equilibrium leverage ω^f and risk-free interest rates r as a function of ψ. In red, the benchmark cases of *full-risk-pooling* (solid) and *no-risk-pooling* (dashed).

reflects its risk aversion. Note that, despite leverage is decreasing in ψ, the total holding of the financial sector, $\omega^f\psi$ is increasing in ψ, consistently with the equilibrium nature of the model.

How does leverage change with exogenous shocks? As confirmed by Theorem 1, positive (negative) exogenous systematic shocks deteriorate the financial sector assets and move its size towards one (zero). Stated differently, the diffusion term σ^{ψ} contributes positively to the size law of motion (see Figure 2.6, top-right panel). The latter, together with the fact that ω^f is a decreasing function of ψ, implies that negative shocks increase equilibrium leverage. This is the result of a relatively higher demand for risk-mitigation instruments by the households. The opposite holds as a response to positive shocks: when the financial sector increases its relative capitalization, its leverage reduces, and so does its supply of risk-mitigation instruments.

Overall, the size of the financial sector is pro-cyclical and financial leverage is *counter-cyclical* as also suggested by the recent empirical findings in Yepez (2017) and He et al. (2017).[29]

[29]Note that (see Equation 2.11) intermediaries' leverage is decreasing in the systematic volatility σ (the lower the systematic risk, the higher the risky claim demand of the financial sector, the higher the equilibrium leverage) and increasing in the idiosyncratic risk volatility $\tilde{\sigma}$ (the higher the idiosyncratic risk, the higher the demand for risk mitigation, the higher the leverage). Moreover, given that financial leverage is associated with high fluctuations of the financial sector size, decreasing systematic risks also increases the relative size of endogenous fluctuations, an effect consistent with the *volatility paradox*. Appendix B.7 provides a graphical representation of these results.

Risk-free interest rates As far as the equilibrium risk-free interest rate r_t is concerned (see Figure 2.6, bottom row, right), the risk-free return on bonds is increasing in financial sector capitalization, due to a declining demand/increasing supply of mitigation instruments, making interest rates pro-cyclical (the stylized fact of pro-cyclical risk-free rates is documented in Fatih Guvenen, 2006, among the others).

For low value of financial sector capitalization, r turns negative. Since both sides are equally risk-averse, with a high demand/low supply of bonds, households are willing to pay the financial sector to offload some of their risky claims to its balance sheet. This effect does not require any "crisis" contingency to take place, rather it is generated by restricted market participation jointly with the allocation of capital (and risk) among heterogeneous classes of agents.

As it is not the main concern of this paper, the asset pricing implications of our model (state dependent financial assets returns and Sharpe ratios) are discussed at length in Appendix B.3. Nonetheless, in the light of our results, it is relevant to highlight that our theoretical framework implies that: i) The link between financial leverage, Sharpe ratios, and risk-free interest rates strictly relates to the pooling capacity of the financial sector, and can be decomposed into two different components: first, higher financial leverage corresponds to lower (even negative, depending on the parameters) interest rates. Second, higher leverage corresponds to higher aggregate marginal productivity, and thus higher risky assets returns, since a smaller share of aggregate wealth is spent after pooling; ii) The size of idiosyncratic risks fundamentally contributes to financial sector risk premiums, despite the fact they can be pooled, and therefore eliminated via diversification; iii) As long as there does exist residual (un-pooled) idiosyncratic risk, this is accounted for in the equilibrium risk-free rates; iv) There is no need of binding constraints to link financial leverage to Sharpe ratios: in this terms, it is an inherent effect of financial intermediation (unlike in He and Krishnamurthy, 2013).[30]

Note that the connection between higher risk premiums and *restricted participation* models is well known (see Fatih Guvenen, 2006), and dates back to Basak and Cuoco (1998). In the original model the limitation is extreme, since households have access to risk-free assets only. As a result, the equilibrium interest rate adjusts such that stockholders borrow the *entire* wealth owned by non-stockholders and make interest payments every period, which sustains the consumption of the latter group. Our contribution is to implement the aforementioned mechanism in a fully-fledged general equilibrium model of a production

[30]The argument that, as long as agents are able to adjust their leverage, Sharpe ratios are counter-cyclical, i.e. assets that covary with leverage are riskier and earn a proportionally larger risk premium can be found in Brunnermeier and Pedersen (2008), Adrian et al. (2014), and Dell'Ariccia et al. (2014).

economy and, in particular, to draw the relationship between financial and real macro-dynamics.

2.4.2 Consumption and the Business Cycle

The relationship between the size of the financial sector, its leverage, and the business cycle is a long-standing issue. In particular, the nature of such a connection is explored in several studies: In Denizer et al. (2002), for example, countries with more developed financial sectors are shown to experience less fluctuations in output, consumption, and investment growth. More recently, Beck et al. (2014) show that intermediation activities increase growth and reduce volatility in the long-run. Nevertheless, they argue that an over-sized financial sector could result in miss-allocation of resources. What follows is that the over-development of auxiliary financial services may lead the financial sector to grow too large relative to its *social optimum*. In the light of these empirical findings, we dispose of our theoretical framework to highlight the mechanism that relates the size of the financial sector to the equilibrium behaviour of real macro-variables such as aggregate consumption and disposable output.

In equilibrium, the aggregate output Y_t can be decomposed as the sum of consumption C_t, investments I_t, and what is spent as intermediation costs due to pooling, G_t. We denote as *disposable output* $\tilde{Y}_t$ the fraction of total output that is either consumed or invested to generate new capital, $\tilde{Y}_t = C_t + I_t$, or, equivalently, $\tilde{Y}_t = Y_t - G_t$. $\tilde{Y}_t$ is the share of output that contributes at generating welfare. The dynamics of total output is

$$dY_t = AdK_t = \underbrace{dC_t + dI_t}_{d\tilde{Y}_t} + dG_t.$$

where $G_t = \eta K_t^h$, $I_t = \iota_t K_t$ and thus $C_t = (A - \iota_t) K_t - \eta K_t^f$ (note that K_t^f represents the financial sector's physical capital holdings in equilibrium, i.e. $K_t^f := \omega_t^f E_t^f$).[31]

Disposable output In Figure 2.7 we plot the drift (left) and the normalized diffusion (right) of the (aggregate) disposable output growth process $\frac{d\tilde{Y}}{\tilde{Y}}$. In red, we depict the benchmark cases of the *full-risk-pooling* (solid) and the *no-risk-pooling* (dashed) economy.

Both drift and diffusion depends on the financial sector relative wealth share and they always remain within the bounds set by the two benchmarks. In

[31] For the purpose of our analysis, we focus on consumption and disposable output only. In Appendix B.4.7, we show that the dynamics of disposable output and consumption growth rates evolve as Itô's processes whose drifts and diffusions are function of both state ψ and prices $q(\psi)$.

particular, $\mu^{\tilde{Y}}$ is decreasing in ψ (increasing in financial leverage), whereas the (normalized) diffusion term $\frac{\sigma^{\tilde{Y}}}{\sigma}$ is a convex function of the financial relative capitalization ψ.

In our model, the output drift is decreasing in the relative size of the financial sector due to a *pecuniary externality*: the larger the financial sector capitalization, the lower the aggregate productivity of capital (due to high intermediation costs per unit of capital), the lower the cost of capital, the lower the investments in new capital. The reduction of output volatility implies that $\sigma^{\tilde{Y}}$ can be read as a mitigation with respect to the width of exogenous fluctuations due to the volatility of capital stock σ. This feature highlights a positive effect of the *pecuniary externality* that stems from the financial sector activity: having a large fraction of idiosyncratic risks that are pooled by the financial sector implies that capital is less productive (due to intermediation costs) and thus, being the size of the financial sector positively related to capital, a negative relationship exists between intermediaries' capital holdings its productivity. The latter reduces the width of capital growth rates fluctuations as driven by systematic shocks. This result is summarised in the following Lemma:

Lemma 1. ***Mitigation*** *The diffusion terms of disposable output growth can be written as mitigation with respect to the exogenous systematic shocks volatility* σ*. In particular*

$$\sigma_t^{\tilde{Y}}(\psi_t) = \sigma \left[1 - \underbrace{\frac{\eta}{\sigma} \frac{\partial_\psi \left(\psi_t \omega_t^f \right)}{A - \eta \psi_t \omega_t^f} \psi_t \sigma_t^\psi}_{Mitigation} \right]. \tag{2.12}$$

Figure 2.7 (right panel) shows that the mitigation is a concave function of intermediaries' size. Indeed, the negative correlation between $(A - \eta\psi\omega^f)$ and K implies mitigation to be maximal when the state volatility σ^ψ is high, i.e. for relative small values of the financial sector size. This suggests that there exists an optimal size of the financial sector.

The mitigation of disposable output volatility is in line with the empirical findings in Beck et al. (2014) suggesting that, in the long-run, intermediation-based services negatively associate with growth volatility.[32]

Consumption To understand the connection between financial relative capitalization and the consumption dynamics, in the top panels of Figure 2.8 we plot the drift (left) and the normalized diffusion (right) of the aggregate consumption growth rate $\frac{dC_t}{C}$ as a function of the state $\psi \in (0, 1)$. In red, we depict the

[32]Conversely, non-intermediation services increase the output volatility of high income countries. Nevertheless, the role that intermediation and non-intermediation financial activities play in the growth process of countries is not yet fully disentangled.

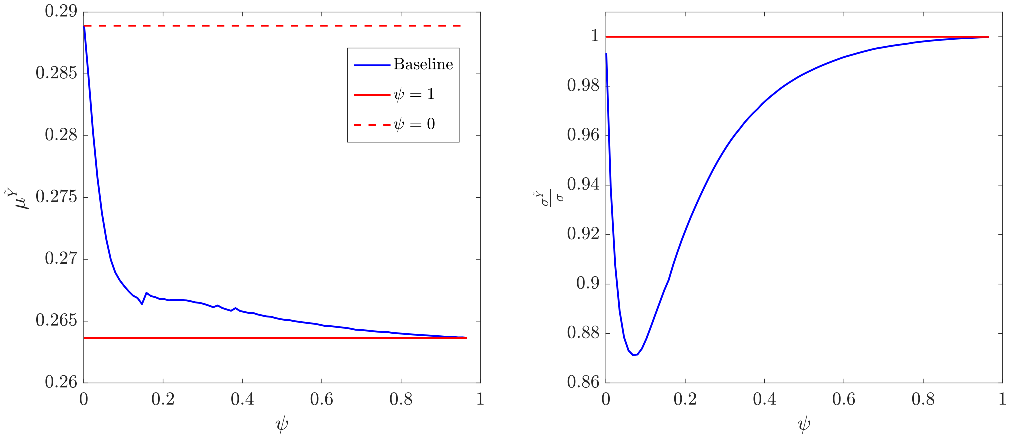

Figure 2.7: Diffusion (left) and normalized diffusion (right) of the (aggregate) disposable output growth rate $\frac{d\tilde{Y}}{\tilde{Y}}$. In red, the benchmark cases of the *full-risk-pooling* (solid) and the *no-risk-pooling* (dashed) economy.

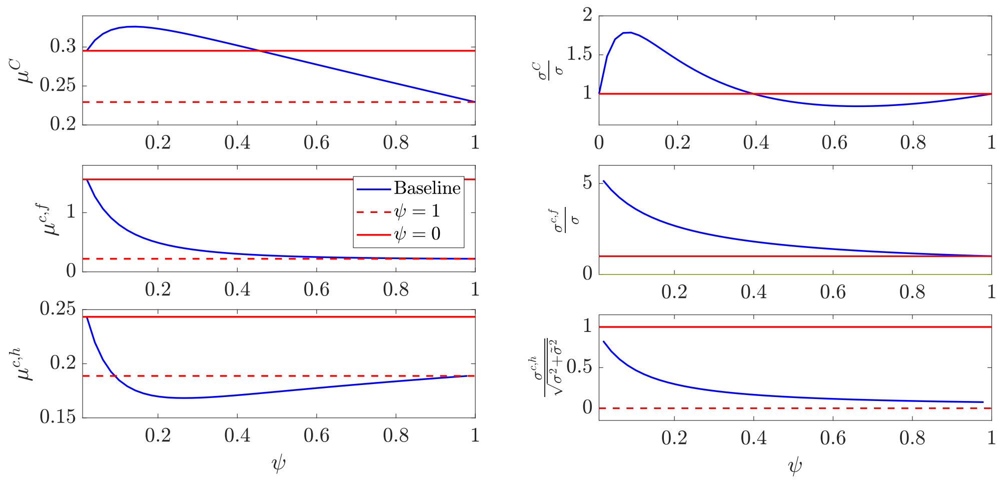

Figure 2.8: Top: Drift (left) and normalized diffusion (right) of the equilibrium aggregate consumption growth rate $\frac{dC}{C}$ as a function of ψ. Middle: Drift (left) and normalized volatility (right) of the financial sector consumption growth rate $\frac{dc^f}{c^f} \propto \frac{de^f}{e^f}$ as a function of ψ. Bottom: Drift (left) and normalized volatility (right) of the households' consumption growth rate $\frac{dc^h}{c^h} \propto \frac{de^h}{e^h}$ as a function of ψ. In red, the benchmark cases of the *full-risk-pooling* (solid) and the *no-risk-pooling* (dashed) economy.

benchmark cases of the *full-risk-pooling* (solid) and the *no-risk-pooling* (dashed) economy.

In the aggregate, the financial sector relative capitalization negatively affects the drift of consumption growth μ^C. Moreover, as long as ψ is small enough, the consumption drift lays above the upper benchmark where $\psi = 0$ (Figure 2.8, top left panel). This result is also due to the *pecuniary externalities*: when the financial sector manages capital, it reduces aggregate productivity, making physical capital relatively cheaper. The fact that the dynamics of physical capital prices q inversely relates to the dynamics of ψ implies that lower financial relative wealth share (higher financial leverage) relates to higher prices, investments, and thus consumption growth.

As far as consumption volatility is concerned, it features both an amplification and a mitigation term. However, the mitigation term always dominates in magnitude. Thus, similar to disposable output, consumption volatility exhibits a U-shape pattern, although its fluctuations are narrower than those of $\tilde{Y}$. This result is summarised in the following Lemma:

Lemma 2. ***Amplification and Mitigation*** *The diffusion terms of aggregate consumption growth can be written as with respect to the exogenous systematic shocks volatility σ as sum of an amplification plus a mitigation term. In particular*

$$\sigma_t^C(\psi_t) = \sigma \left[1 + \underbrace{\frac{\sigma_t^q}{\sigma} \frac{q_t}{\theta} \frac{1}{A - \iota_t - \eta \psi_t \omega_t^f}}_{Amplification} - \underbrace{\frac{\eta}{\sigma} \frac{\partial_\psi \left(\psi_t \omega_t^f \right) \psi_t \sigma_t^\psi}{A - \iota_t - \eta \psi_t \omega_t^f}}_{Mitigation} \right]. \tag{2.13}$$

Perhaps the most compelling feature is that σ^C can be decomposed as the sum of an *amplification* plus a *mitigation* term. It follows that the magnitude of consumption volatility with respect to σ depends on what component dominates. Within the framework of our model, the mitigation term always overtakes the amplification counterpart (Equation 2.8, top panel, right). This result is consistent with Denizer et al. (2002), whose empirical findings suggest that risk management services provided by financial intermediaries may be particularly important in reducing consumption volatility.

The remaining panels of Figure 2.8 consider separately the growth rates of financial sector (centre) and households' consumption (bottom). Both drift and volatility of financial sector consumption growth declines with its size. A somehow similar effect occurs for the volatility of consumption growth rates of households'. Here, however, idiosyncratic risks play a big role: the larger the financial sector, the higher the share of pooled idiosyncratic risk, the lower the entrepreneurs' consumption growth rate volatility. The drift is first sharply declining in the financial sector size, reflecting the shape of households' wealth drift when the financial sector is small, and the slowly increasing when the financial sector is too large.

2.5 Leverage and Welfare

In this section, we study how the relative capitalization of the financial sector, and so its leverage, relates to the agents' welfare. First, Section 2.5.1 derives the welfare of both households' and financial sector conditional on the relative capitalization of the latter. Second, Section 2.5.2 explores the effect of a static leverage constraint on the equilibrium dynamics and, in turn, on intermediaries' and households' unconditional welfare. Finally, being the leverage constraint related to the minimal size of the financial sector, we investigate the role of a redistributive taxation policy. Finally, Section 2.5.3 studies the relationship between households' welfare, leverage constraint, and such a redistributive taxation.

Our purpose is to investigate whether a too small (or too big) financial sector is detrimental for the households' welfare; this would suggest that there exists a *"welfare optimal"* size of the financial sector, and so that leverage constraints and redistributive taxation may be welfare improving.

2.5.1 Welfare Analysis

In general, the welfare W^i of the agent i conditional on the state ψ equals its value function V^i.[33] This result is summarised in Proposition 1:

Proposition 1. ***Conditional Welfare***
The conditional *welfare of the i sector, for unitary capital, can be expressed as*

$$V^i := W^i(\psi) \propto \ln \rho q(\psi) v^i(\psi) + H(\psi)^i, \quad i \in \{h, f\}, \tag{2.14}$$

where $v^i(\psi)$ is the i^{th} class relative wealth share, that is, $v^f(\psi) = \psi$ and $v^h(\psi) = 1 - \psi$. The function H^i solves the HJB equation and summarises the expected dynamics of the i^{th} agent's wealth.

Proof. See Appendix B.4.5. □

The welfare function in (2.14) is the sum of two components: the former is *static*, and accounts for the current benefit due to the ownership of a certain share of the aggregate capital $v(\psi)^i$ valued $q(\psi)$. The latter is *dynamic*, and

[33]Since the model is scale invariant in aggregate capital stock, we set $K = 1$.

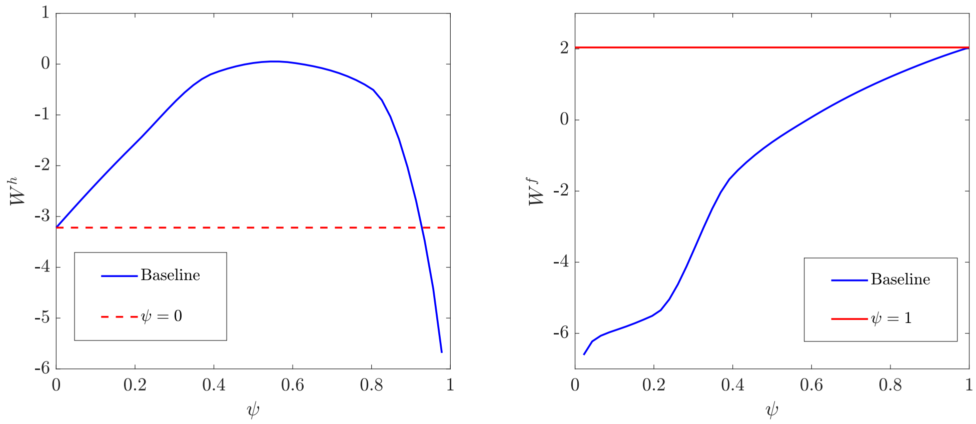

Figure 2.9: Conditional welfare of households' (left) and financial sector (right). In red, the benchmark cases $\psi = 0$ (dashed) and $\psi = 1$ (solid).

summarizes the expected discounted benefit of future consumption conditional on an initial state ψ (further details and derivation are in Appendix B.4.5).

In the left panel of Figure 2.9 we show the welfare of households' contingent to relative wealth share ψ (blue line). In red, we display the benchmark case $\psi = 0$.[34] What stands out is that the households' conditional welfare W^h is an *inverted U-shaped* function of the financial relative capitalization ψ. For low level of ψ, W^h is increasing: the larger the financial relative capitalization, the higher the equilibrium risk-free interest rate (see Figure 2.6, bottom right), the more risk mitigation of aggregate output fluctuations is provided (see Figure 2.7, right), the higher households' welfare. Conversely, the W^h turns decreasing when the financial sector is relatively too large. In such a case, even if the supply of risk mitigation is quite large, the small relative size of households' diminishes their consumption growth rate (see also Figure 2.8, bottom panel), since a greater share of wealth is spent after the payment of intermediation costs. This result suggests that there exists a *"welfare optimal"* size of the financial sector.

In the right panel of Figure 2.9 we repeat the same exercise with respect to financial sector welfare W^f. In blue, we plot W^f contingent to the relative financial capitalization ψ. In red, we display the benchmark case when $\psi = 1$.

Overall, the financial sector conditional welfare is increasing in its own relative capitalization, and it is maximal when its relative capitalization approaches one. This is due to relative wealth share effect on the price of physical capital (the *static* terms of Equation 2.14).

[34]We compute the value of the function H numerically by Monte Carlo simulations (see also Appendix B.4.5). In particular, we simulate $N = 2,000$ paths of ψ_t for $t = 400$ periods over a equally spanned grid of initial values ψ_0. We then interpolate the results over the solution grid by means of a cubic function.

2.5.2 Leverage Constraints

The analysis of the previous section suggests that controlling the size of the financial sector may improve households' welfare. When leverage is countercyclical, as captured by our model, this can be achieved by imposing a static leverage constraints. Our contribution is to provide theoretical evidence of the role that such constraints may have at determining the fluctuation of disposable output, consumption and, in turn, welfare.

Hereafter, we solve the model assuming an additional constraint to the financial sector leverage. Then, we discuss the effect of such a constraint over the equilibrium dynamics. The financial sector optimization problem is now written to take into account the additional constraint $\omega_t^f \leq LC$. Finally, we compute the welfare in presence of a static Leverage Constraint (LC).

With a static LC, the HJB equation of the financial sector becomes

$$\rho V_t = \max_{\left\{\omega_t^f, c_t\right\}} \left\{ \ln c_t^f + \frac{1}{dt}\mathbb{E}_t\left[dV_t\right] - \lambda_t \left(\omega_t^f - LC\right) \right\},$$

where λ_t is the Lagrangian multiplier, and the transversality condition $\lim_{s\to\infty} \mathbb{E}_t e^{-\rho s} V_s = 0$ holds. The problem is solved in Appendix B.4.6. It is relevant to highlight that, since agents are risk averse, the LC is not always binding. It follows that the motion through which the equilibrium shifts in and out the constrained area is state contingent: both its drift and volatility depend on how restrictive the LC is.

Constrained dynamics In Figure 2.10 (top), we show the drift (left) and diffusion (right) of the (constrained) equilibrium relative wealth share process. In particular, we consider bounded (green) and unbounded (blue) LC. In red (solid), we plot the benchmark case of the *full risk pooling* economy. What stands out is that, when the constraint binds, it reduces both drift and diffusion of the state process. This result is intuitive since, when the financial sector leverage is exogenously capped by the prudential policy, so it is the supply of risk mitigation instruments to the households when financial capitalization is scarce. This can be seen through the portfolio choices of the agents in the states where the LC is binding.

In the same Figure (bottom), we repeat a similar analysis with respect to both equilibrium financial sector leverage ω^f (left) and households portfolio share in risky claims ω^h (right). What is relevant is that binding constraints oblige households to keep a higher share of their wealth allocated in risky assets. Accordingly, the speed at which the system drifts back towards the high capitalization phase is weakened.

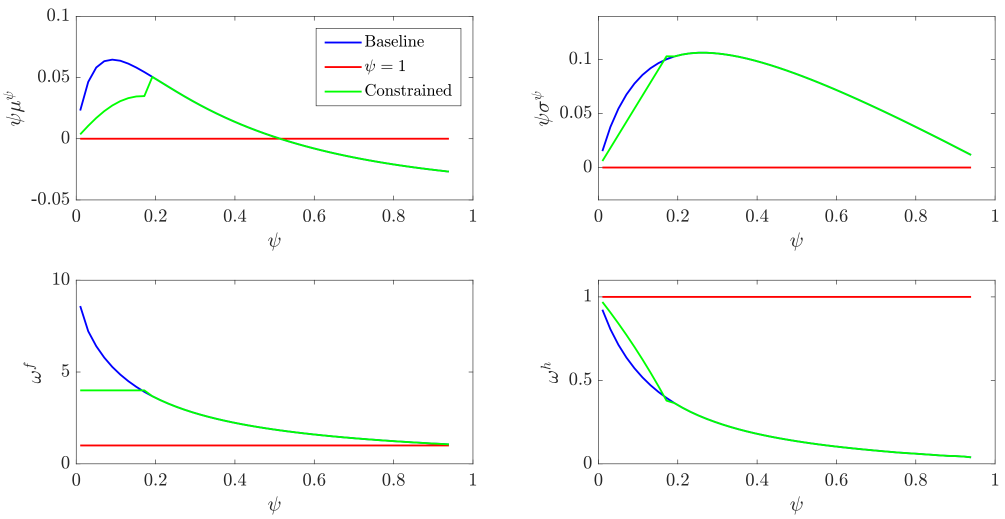

Figure 2.10: Top: Drift (left) and diffusion (right) of the process $d\psi_t$ for bounded (green) and unbounded (blue) constraints LC. Bottom: Equilibrium financial sector's (right) and households' (left) portfolio shares for bounded (green) and unbounded (blue) LC. In red, the benchmark case of the *full-risk-pooling* economy.

Disposable output If we look to the effect of leverage constraints through the lenses of the business cycle (see Figure 2.11) we find out that bounded LC to the financial sector slightly increases the drift $\mu^{\tilde{Y}}$ (left) of disposable output growth. This is because, in our model, the productivity of households' is higher than the financial sector's (due to intermediation costs). On the other hand, the constraint also impairs intermediaries' positive externality - mitigation - of aggregate output volatility $\sigma^{\tilde{Y}}$ (Figure 2.11, left).

Consumption Similar to what we observed for the dynamics of aggregate (disposable) output, in Figure 2.12 (top panel) we plot the drift and diffusion of aggregate consumption for bounded (green) and unbounded (blue) leverage constraints. As for aggregate output, what stands out is that LC contribute at increasing consumption growth rate, while dampening the mitigation that comes after intermediaries' activity. In particular, the mitigation may be compromised to the point that the amplification component dominates (see Lemma 2).

Instead, a rather different picture emerges if we look at the dynamics of intermediaries' and households' consumption apart from each other. From the perspective of the financial sector, biding constraints reduce the growth rate of its consumption as well of its volatility, due to the limited leverage. Conversely, the growth rate of households' consumption is higher, due to the price effect of a higher share of risky capital in their portfolio (Figure 2.12, middle panel, left). At the same time, the households suffer a scarce supply of risk-free bonds when

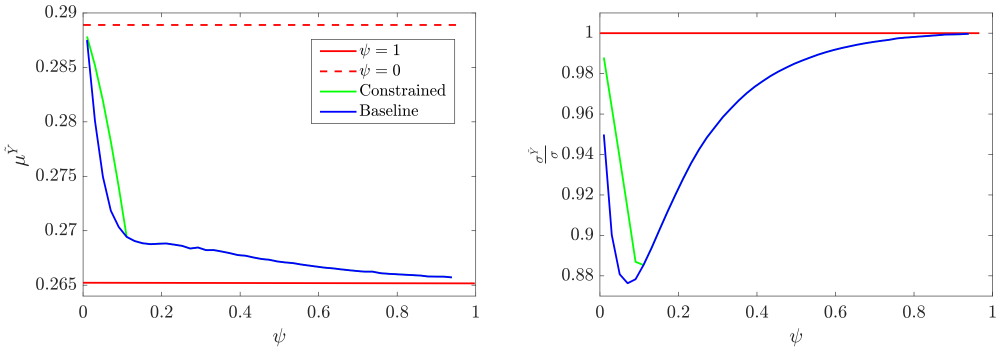

Figure 2.11: Diffusion (left) and normalized diffusion (right) of the (aggregate) disposable output growth rate $\frac{d\tilde{Y}}{\tilde{Y}}$ for bounded(blue) and unbounded LCs. In red, the benchmark cases of the *full-risk-pooling* (solid) and the *no-risk-pooling* (dashed) economy.

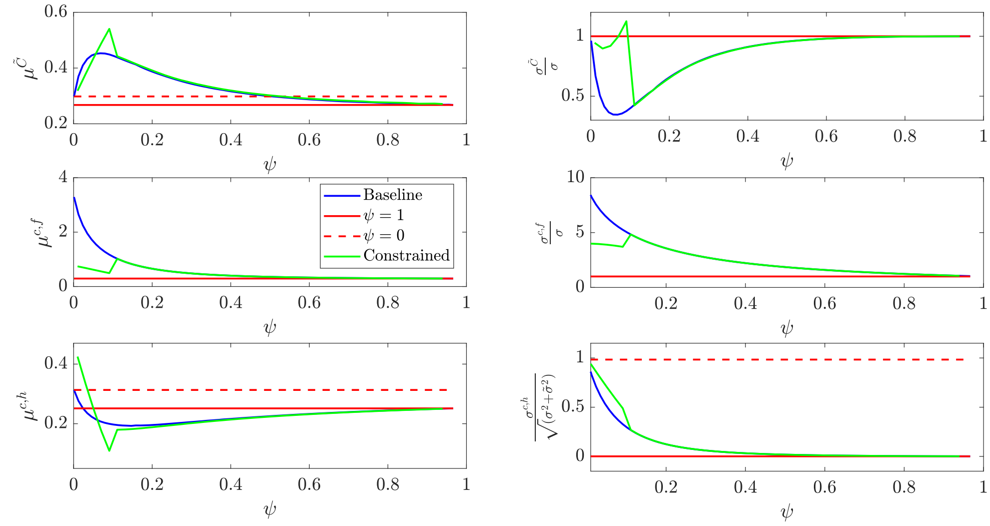

Figure 2.12: Consumption growth dynamics for bonded (green) and unbounded (blue) LC. Top: Drift (left) and normalized diffusion (right) of the equilibrium aggregate consumption growth rate $\frac{dC}{C}$ as a function of ψ. Middle: Drift (left) and normalized volatility (right) of the financial sector consumption growth rate $\frac{dc^f}{c^f} \propto \frac{de^f}{e^f}$ as a function of ψ. Bottom: Drift (left) and normalized volatility (right) of the h/entrepreneurs' consumption growth rate $\frac{dc^h}{c^h} \propto \frac{de^h}{e^h}$ as a function of ψ. In red, the benchmark cases of the *full-risk-pooling* (solid) and the *no-risk-pooling* (dashed) economy.

they are needed the most; when financial capitalization is scarce (the leverage constraint is binding) and they sustain an extra exposure to idiosyncratic risks (bottom panel, right).

In summary, the most relevant pattern is that imposing limits to leverage hinders the mitigation of aggregate consumption fluctuations by the financial sector, as it reduces the volatility of relative wealth share. However, at the same time, it increases the aggregate consumption growth due to a higher share of capital allocated to the - most productive - households.

In a similar fashion, the *LC* contribute at reducing the mitigation of the disposable output fluctuations, since the constraint limit the positive externality due to the financial sector. Moreover, a restrictive policy hinders the optimal allocation of risk by setting an upper bound to the equilibrium supply of risk-mitigation instruments.

Welfare after LCs In order to evaluate the welfare effect of imposing leverage constraints to the financial sector (and later on of having a redistributive policy) we may build an *unconditional* measure of welfare. To do so, we weight the conditional value of (2.14) by the associated stationary density $\pi(\psi)$:

$$\mathbb{E}\left[W^i(\psi)\right] = \int_0^1 W^i(\psi)\pi(\psi)d\psi. \tag{2.15}$$

Accordingly, the *aggregate* welfare equals sum of expected constrained welfare of households and financial sector weighed by a function $\Gamma(\psi)$, and is defined as:

$$W^\Gamma = \sum_i \mathbb{E}\left[W^i(\psi)\Gamma(\psi)^i\right]. \tag{2.16}$$

Table 2.1 reports the constrained and unconstrained *unconditional* (aggregate) welfare (as defined in Equation 2.16) for different weighting functions Γ^i.[35]

We start by focusing on the households' and financial sector welfare apart from each other: once we look at the unconditional welfare of the households' before and after *LCs*, we find out that the constraints may be welfare improving ($\Gamma^h = 1$ and $\Gamma^f = 0$, Table 2.1, second row). Conversely, when only the financial sector is considered, we find *LC*s to be welfare detrimental ($\Gamma^f = 1$ and $\Gamma^h = 0$, Table 2.1, third row). These results may be better understood by looking at the agents' conditional welfare before and after imposing the constraint along with the stationary distribution of the state variable. In Figure 2.13 we show the wel-

[35] We compute the aggregate welfare in (2.16) numerically. In particular, we approximate the unconditional welfare over an evenly spaced grid $[0, 0.2, 0.4, 0.6, 0.8, 1]$ and interpolate it by using a cubic polinomial. Then, we integrate by trapezoid method $W^h(\psi)$ over the an evenly matched stationary density $\pi(\psi)$ weighted by $\Gamma(\psi)$.

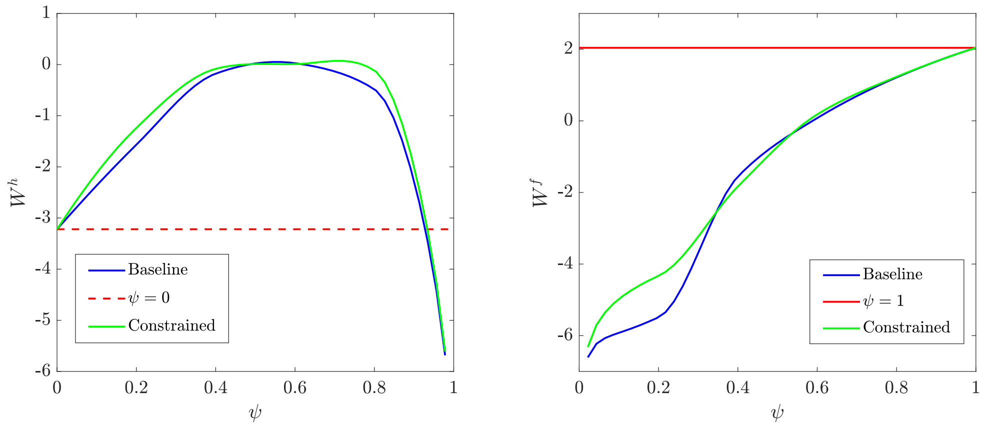

Figure 2.13: Conditional welfare of the households' (left) and of the financial sector (right) for bounded (green) and unbounded (blue) *LC*. In red, the benchmark cases of the *no-risk-pooling* (dashed) and the *full-risk-pooling* (solid) economy

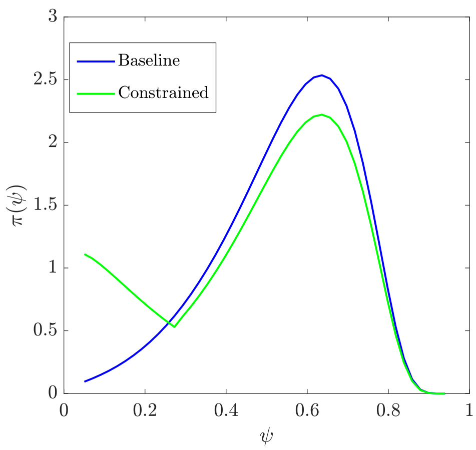

Figure 2.14: Stationary density of the financial sector relative capitalization $\pi(\psi)$ as a function of the state for unbounded (blue) and bounded (green) *LC*.

fare of households' (left panel) and the financial sector (right panel) contingent to relative wealth share ψ (blue line) or subject to a LC (green line). In red (dashed line), we display the benchmark case $\psi = 0$ (dashed) and $\psi = 1$ (solid). In Figure 2.13 depicts instead the stationary density of relative financial sector capitalization $\pi(\psi)$ for unbounded (blue) and bounded (green) *LC*.

What stands out is that *LCs* increase the households' conditional welfare when the financial sector is either under or over-capitalized, while it decreases for intermediate values of ψ.

Similarly, leverage constraints benefit the financial sector in case of low or high capitalisation, while they reduce its conditional welfare for intermediate states. Nonetheless, as intermediaries welfare holds strictly increasing in its relative capitalization, and binding constraints increase the likelihood of "lower" states. Therefore, in general, leverage constraints are welfare detrimental for the financial sector. Finally, in the last two rows of Table 2.1 we report the aggregate when the weighting function Γ is either constant and even, or proportional to each class relative wealth share. In either cases, the leverage constraints are welfare detrimental. In the former case this means that, at this level of leverage constraint, the welfare gain of the households' less than compensate the welfare loss of the financial sector. Not surprisingly, the same result holds when the weighting function is proportional to the agents' relative share of wealth.

Obviously, the result of the leverage constraints being welfare detrimental when jointly considering households and intermediaries is fundamentally tied up to the arbitrary choice of the weighting function Γ. For this reason, in the next section we focus on the households' welfare only.

Weights, $\Gamma(\psi)^i$	Welfare		
	LC Unbounded	LC Bounded (4)	% Gain
$\Gamma(\psi)^f = 0; \Gamma(\psi)^h = 1$	-0.2847	-0.2188	+0.3
$\Gamma(\psi)^f = 1; \Gamma(\psi)^h = 0$	-0.4143	-0.5838	-0.29
$\Gamma(\psi)^f = \Gamma(\psi)^h = 0.5$	-0.3495	-0.4012	-0.13
$\Gamma(\psi)^f = \psi; \Gamma(\psi)^h = 1 - \psi$	-0.0252	-0.04	-0.25

Table 2.1: Unconditional aggregate welfare for different weighting functions.

2.5.3 Constraints, Redistributive Taxation, and Welfare

Now that we have pointed out how leverage constraints influences households' and financial sector welfare, we conclude by addressing two further issues, namely: i) Since LCs may be beneficial, how does the households' unconditional welfare change for different levels of constraints? ii) What is the role of a *redistributive taxation* that aims at reducing the relative capitalization of the financial sector?

To answer these questions, Figure 2.15 plots the unconditional welfare of the households' as a function of the leverage constraint LC.[36]. We then interpolate the obtained points by a cubic function (blue, solid line). What stands out is that, according to our previous results, constraints to the financial sector leverage may be welfare improving for the households. In particular, the effect on W^h is positive as long as LC is not too high. Conversely, the level of the constraint compromises the equilibrium supply of risk mitigation instruments to

[36]The welfare is approximated numerically with $T = 300$, and $N = 4,000$ over a evenly 8-spaced grid over $LC \in [2, 8]$

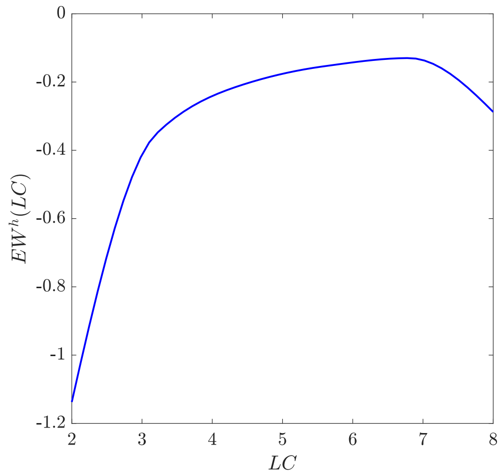

Figure 2.15: Unconditional households' welfare as a function of the leverage constraint LC.

the economy, and so the positive effect of financial sector's activity *pecuniary externality*.

There exists a growing literature regarding this aspect (see Blum and Hellwig, 1995; Blum, 2008; DeAngelo and Stulz, 2015; Myerson, 2014, among the others), however, the arguments considered for a lower leverage are mostly based on either partial equilibrium models or focusing on information asymmetries. To our knowledge, our paper is the first that stresses the connection between leverage and the real as well as financial macro-dynamics, and that explicitly highlight the mechanism that links the agents' welfare to the size of the financial sector in a general, although extremely stylized, equilibrium model.

Redistributive taxation Having established the way leverage constraints affect the agents' welfare, we now investigate how *tax transfers* from the financial sector to the households' may alter their welfare.

This is relevant because, being the LC related to the minimal size of the financial sector only, it does not prevent it to grow too large when the constraint is slack. In this term, the role of a *redistributive taxation* is to reduce the relative financial capitalization, and so the amount of resources it spent after the payment of intermediation costs.

Let τ be the constant (tax) rate at which the stock of wealth is evenly redistributed from the financial sector to all the households'. It is possible to show (the derivation is in Appendix B.4.8) that, accounting for the policy, the state variable ψ evolves as

$$\frac{d\psi_t^\tau}{\psi_t^\tau} = \frac{d\psi_t}{\psi_t} - \tau\frac{\psi_t}{1-\psi_t}dt,$$

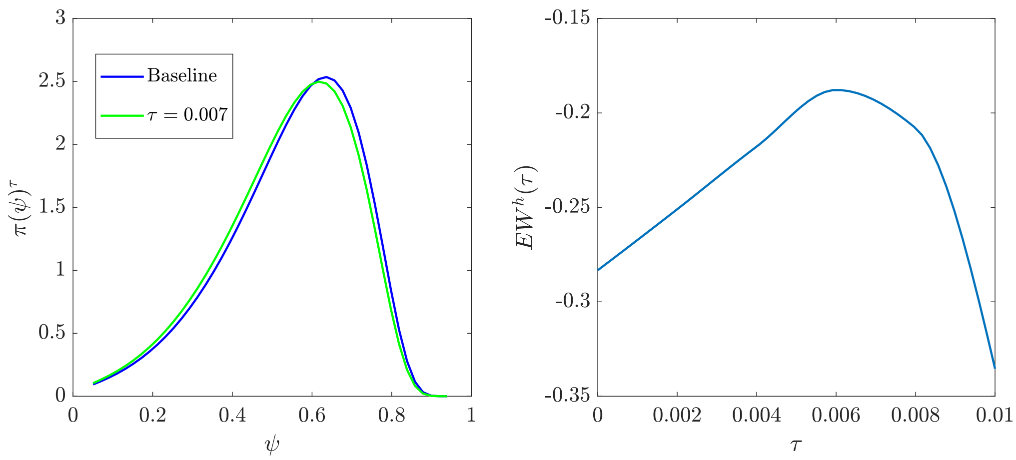

Figure 2.16: Left: Stationary state density $\pi(\psi)^\tau$ before (blue) and after (green) a redistributive taxation policy. Right: Unconditional households' welfare as a function of the tax rate τ.

where the term $\frac{d\psi_t}{\psi_t}$ has dynamics as in (2.8).

For our purposes, we look at the effect of different tax rates τ on the households' conditional welfare and on the stationary density of the state. In Figure 2.16 we show the households' unconditional welfare W^h as a function of the tax rate τ (right panel) and the stationary density $\pi(\psi)^\tau$ (left panel) for no (blue) and positive tax rate (green).[37]

Not surprisingly, the redistributive policy shifts the stationary density to the left, where the financial sector has a lower relative capitalization. In general, the higher ψ, the more effective the policy (the redistribution is hyperbolically increasing in the state). This is because, the absolute redistribution is directly proportional to the wealth stock of the financial sector.[38]

As far as the households' welfare is concerned, our numerical results suggest that the redistributive taxation may be welfare improving for a moderate tax rate τ. This is because the financial capitalization is more likely to float through states where the positive (mitigation) effect of the financial sector *pecuniary externality* is maximal, and fewer resources are spent after intermediation costs. Conversely, when τ is too high, the tax negatively affects the households' welfare since the financial sector is hindered from growing big enough, and so from supplying - cheap - risk mitigation to the economy.

[37] As for the results in Figure 2.15, the welfare function is approximated numerically over an evenly spaced grid and interpolated by a cubic polinomial.

[38] Note that, at the boundaries: $\lim_{\psi\to 1} \frac{\partial}{\partial\psi}\left(\tau\frac{\psi_t}{1-\psi_t}\right) = \infty$, while $\lim_{\psi\to 0} \frac{\partial}{\partial\psi}\left(\tau\frac{\psi_t}{1-\psi_t}\right) = \tau$.

2.6 Conclusions

This paper investigates the mechanism through which the risk pooling capacity of an aggregate financial sector relates to the economic macro-dynamics in a general equilibrium model of a productive economy with financial frictions.

In order to mitigate the idiosyncratic risk in their portfolios, the households exchange physical capital versus risk-free bonds issued by the financial sector, who finances its risky assets by leveraging its balance sheet. The risk mismatch between intermediaries' assets and liabilities, together with a positive cost of risk pooling, stems into an equilibrium where agents' heterogeneity is persistent.

The equilibrium allocation of risk generates state-dependent counter-cyclical leverage and, in turn, mitigation of aggregate consumption and disposable output fluctuations in response to exogenous systematic shocks. The endogenous dynamics of financial leverage stems from agents' asymmetric exposure to risk that generates, in turn, structural demand for risk-mitigation instruments. In this terms, the equilibrium macro-dynamics is inherently related to the risk pooling capacity of the financial sector. Accordingly, equilibrium risk-free interest rates are decreasing in financial leverage, and low (even negative) rates associate to high intermediaries' leverage.

Finally, we study the relationship between intermediaries' size, leverage, and agents' welfare. From this perspective, our model suggests that there exists a trade-off between the welfare gain from aggregate consumption growth and the cost from its fluctuations. Therefore, imposing leverage constraints may be welfare-improving for the households. On the other hand, we show that preventing the financial sector to grow too large, and so to destroy too many resources after intermediation costs, may also be welfare improving for the households.

Overall, these results suggest that there exist welfare improving leverage constraints and redistributive taxation policies such that the size of the financial sector remains within an "optimal" range.

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Chapter 3

Banks Recapitalization, Bailout, and Long-run Welfare

"We are inheriting the worst financial system since the Depression. We're inheriting a situation - when people go back and study major banking crises a quarter century from now, the one that America developed in 2007 and 2008 is going to be one of those crises."

- Lawrence Summers, *2018*

Abstract

This paper studies the mechanism that relates banking resolution regimes to households' long and short-run welfare in a DSGE model of a productive economy with financial frictions. Due to their cost advantage at monitoring capital producing firms, banks issue short-term risk-free liabilities and, jointly with their own equity endowment, purchase risky claims issued by firms. Banks optimally choose dividends payouts and equity issuance to maximise their own market value. In equilibrium, it is individually optimal for each bank to be recapitalized by her own shareholders. However, banks are homogeneous and evenly exposed to common systematic shocks. Thus, all banks always issue new equity simultaneously. As the whole banking sector is at stake, we show that a tax financed resolution (bailout) that tops up individual recapitalization is social optimal, as it may improve long-run households' welfare. This is because, in a perfectly competitive environment, economic actors do not internalize the positive externality of banking sector aggregate capitalization over equilibrium prices and, in turn, aggregate investments.

Keywords Banks, Bailout, DSGE, Financial Frictions, Recapitalization.

JEL Classification D51, G21.

3.1 Introduction

Over the last five decades, numerous countries dealt with a stern crisis of their banking sector. Many of those crises stemmed into an impaired supply of those countries' financial services, and massive recapitalization decisions had to be taken. Most times, what followed was a major overhaul of a relevant share of the countries' banking sector, often finance by public money (*bailout*). In the EU only, no less than 114 banks benefited from government support during the period 2007-2013.[1] In recent years, especially after the sub-prime financial crisis, this resulted into public discontent against the policy of *"privatizing profits and socializing losses"*.

As a response, with the intent of minimizing the stock of taxpayers' money depleted after recapitalization purposes, the regulators developed new tools to root out the drawbacks of a distressed banking sector, such as the commitment to retain dividends or coercive conversion of subordinate debt and deposits into equity (bail-in).[2] In this regard, both public and academic debate struggled upon the relative convenience of different recapitalization regimes: should the cost of banks' distress be a burden to their own shareholders only, or should it be tax-financed? Are individual banks' recapitalization choices also social optimal?

To address these questions, we develop a suitable DSGE model of a productive economy with a banking sector and financial frictions. Then, we use the model to explore the interlink between different resolution regimes, here individual banks recapitalization and bailouts, and households' welfare. In this framework, we propose a mechanism that associates long-run positive externality to banks aggregate capitalization. As our focus is on the aggregate banking sector, we exclusively consider those cases when the whole banking capitalization is at stake. Thus, we assume that there exists a unique source of aggregate risk.

We model an infinite-horizon continuous-time economy, populated by homogeneous households, banks, and productive firms. Households are risk-averse and maximise the inter-temporal utility of their consumption. Each household is initially endowed with the ownership of one bank's equity, i.e. she is the bank unique shareholder, and is willing to allocate the residual share of its wealth to short-term (risk-free) bank liabilities. Therefore, households utterly mandate risky investments to banks.

Due to their cost advantage at monitoring capital producing firms with respect to households, banks are willing to issue short-term liabilities and, jointly with their own equity, purchase risky claims from firms. Banks optimally choose

[1]Over the same period, the European Commission (2019) reports that about 3% of EU 2012 GDP has been provided as new capital to ailing banks by member states.

[2]In a broad sense, a bail-in may be defined as any resolution imposing losses on private stakeholders. An overall review of the EU resolution framework as compared to the US one is in Philippon and Salord (2017).

their dividends payouts and (costly) equity issuance to maximise their own market value. Note that, as banks are run in the best interest of their shareholders, there are no managerial agency conflicts. Thus, the model intentionally focuses on the interlink between banks' capital, households' welfare, and firms' investment choices as channelled through equilibrium prices (Tobin's Q mechanism).

In equilibrium, banks' strategies associate to their market-to-book value and, in turn, to the capitalization thresholds at which either they pay out dividends or issue new equity. What follows is that, from their individual perspective, it is optimal for each bank to be recapitalized by its own shareholders as long as the marginal value of her equity (market-to-book value) exceeds the cost of recapitalization. Conversely, banks pay out dividends when the marginal value of their equity shrinks below one. Else, in all the residual intermediate states, banks do not issue equity, while progressively re-building their capital buffer by retaining dividends.[3]

A fundamental assumption of the model is that banks are homogeneous and uniformly exposed to the unique common source of systematic risk. Therefore, distress contingencies to each bank (equity issuance) are always synchronous to all other banks', i.e. recapitalization happens across the whole banking system. As such, individual recapitalization strategies turn out to be socially sub-optimal. In this framework we show that, as a response to systematic banking distress, a tax-financed bailout regime that tops up individual recapitalization policies may improve households' long-run welfare, notwithstanding the additional short-run cost due to taxation. Intuitively this happens because all agents act in competitive markets; accordingly, they fail at internalizing the pecuniary externality of banking sector activity, through its aggregate capitalization, over equilibrium outcomes. In this term, additional equity issuance financed by taxation reduces banks' leverage, thereby stabilizing their recovery path, increases capital prices and, in turn, investments. Moreover, it affects the marginal value of banks' equity, thereby increasing the likelihood of those states where banks pay out dividends.

In summary, even when all economic actors are homogeneous, and all are subject to a common source of aggregate risk, there exists a trade-off between costs and benefits of banks bailout recapitalization. On the one hand, additional costs imposed to households by taxation associate to a reduction of their short-run welfare. On the other hand, those extra resources smoothen the transition through "bad states" where banks' leverage is high. This reduces the likelihood of additional recapitalization due to high volatility-leverage, and allows the banks

[3]Note that in a general equilibrium setting where all households are homogeneous and keep their whole disposable stock of wealth allocated to banks' short-term liabilities, banks' issuance of new equity is always equivalent to a bail-in, i.e. debt is converted into equity to keep each bank solvent.

to rapidly rebuild their own equity by themselves. In the long-run, this *positive feedback loop* prompts the banks' recovery path towards "good states", where prices and investments are higher, and dividends payouts are more frequent.

The paper proceeds as follows. In Section 3.1.1, we frame our contribution within the incumbent literature. Section 3.2 introduces the baseline model and discusses the characteristics of: a) Productive technologies and the financial frictions (3.2.1); b) Households' (3.2.2) and banks' (3.2.3) problems. Section 3.3 describes the competitive equilibrium of this economy and characterize its features. Finally, Section 3.4 introduces the bailout policy and dissects the mechanism that connects it to the households' welfare. Section 3.5 concludes.

3.1.1 Related Literature

From a broad perspective, this paper belongs to the body of literature studying resolution policies, in particular recapitalization versus bailout regimes, aimed at restoring distressed financial institutions while countering the associated (macro) consequences. The main, although controversial, stylized facts brought to light by the incumbent literature can be summarised as follows:

a) On the one hand, it is by now common knowledge that bailouts may lead to *moral hazard*, and eventually prompt excessive risk taking by the institutions that shall be virtually rescued (Hryckiewicz, 2014). On top of that, this is relevant because unlimited open ended liquidity support and repeated recapitalization with no control upon consequential risk taking may hinder the process of recovery (Honohan and Klingebiel, 2000).

On the other hand, it is argued that bailout regimes may result in higher franchise values because they reduce funding costs and, hence, discourage risk taking (Sarin and Summers, 2016).[4] This squares nicely with an earlier paper by Cordella and Yeyati (2003) showing that, by announcing and committing ex-ante to bailout insolvent institutions in times of adverse conditions, the risk-reducing that comes after the so-called "value effect" outweighs the moral hazard component of the policy, thus lowering bank risk;

b) Alternative shareholder burdening recapitalization regimes, such as bail-ins, contribute at discouraging those behaviours, although they may dissuade investments to the banking sector, consequently hindering long-run economic growth (Dewatripont, 2014);

[4]Accordingly, Gropp et al. (2010) argue that there is no evidence that public guarantees increase the protected banks' risk-taking, except for banks that have outright public ownership. In this regard, Lambrecht and Tse (2019) recently propose a theoretical model where, even without considering to role of bailouts at containing systemic risk, from a micro-prudential perspective, banks create the most value net of any recapitalization costs under bailout regimes.

c) All in all, the short-run cost of recapitalization shall be accounted jointly with the long-run benefit of a more stable and profitable financial system; as long as the reallocation of resources within the economy acts as engine to foster growth, there is room to consider a trade-off between costs of taxation-based recapitalizations and benefits from a healthy banking sector (Hoggarth et al., 2002; Bernanke, 2009). In this spirit, Homar and van Wijnbergen (2017) argue that early interventions preserve the functions of the financial system and mitigates the *macro* consequences of a crisis. Recent empirical evidence that bailout policies in the EU were able to enhance economic conditions is also in Barucci et al. (2019).[5]

To the best of our knowledge, while the moral hazard problem after banking resolution, a) and b), has been already extensively discussed, the trade-off between long-run benefits and short-run costs of bailouts, c), still deficits a proper treatment in the theoretical banking literature. This paper aims at filling that gap and, in particular, we emphasize two contributions of our analysis: first, we provide theoretical evidence supporting bailout resolution regimes as welfare improving when the working capacity of the banking system is jeopardised by a systematic crisis.[6]

Second, this paper develops a novel micro-foundation of the apparatus underlying the trade-off between (long-run) benefits and (short-run) costs of bailouts recapitalization policies.

This paper also contributes to the theoretical literature on banks recapitalization. Two seminal contributions studying the efficiency of financial recapitalization in a general equilibrium, although static, setting are Gorton and Huang (2004) and Philippon and Schnabl (2013). In Gorton and Huang (2004), they show that there is room for the government to supply liquidity financed by tax revenue. This is because, in their model, private liquidity provision is socially beneficial, since it allows valuable reallocations.

In our model, we reach similar conclusion, although in a very different framework. The allocation efficiency that comes after bailout capital injections pass through prices exclusively, and so it fosters the firms' investments. Whereas in our paper inefficiency comes after scarce capitalization and high leverage, in Gorton and Huang (2004) it stems from liquidity issues, i.e. to the amount of

[5]Further evidence that, when there are too many banks to liquidate, the regulatory intervention in the form of bailing out some banks may be optimal in order to avoid allocation inefficiencies is in Caprio and Klingebiel (1996) and Acharya and Yorulmazer (2007)

[6]For this reason, we model the banking sector as homogeneous. Conversely, a recent working paper that focus on the relationship between default and banks' heterogeneity is Nuno and Rey (2017).

readily available resources that can be used to purchase claims on projects when they are offered for sale at later dates: *"[...] liquidity considerations result in prices that deviate from efficient market prices"*.

Philippon and Schnabl (2013) study governmental interventions to recapitalize a banking sector that restricts lending to firms because of its debt overhang. They find that efficient recapitalization policies request equity versus cash injections rather than other common forms of intervention, such as asset purchases and debt guarantees. This is because the former requires that banks share their upside with the government who financed them, that gradually reduces its participation in the supported banks. This is the same mechanism proposed in our setting, since the households' benefit after the banks' recapitalization due to the expected value of the future dividends. In their paper, government interventions generate two sources of rents: *macroeconomic*, occurring because of general equilibrium effect, and *informational*. In our model, we focus on the former only, although we loosely account for the latter within the monitoring cost advantage of the banking sector.[7]

Another important contribution addressing the relationship between banking regulation (PCA, Prompt Corrective Actions), welfare, and efficiency is Nicolo et al. (2014). From a general point of view, we essentially set apart, as we consider a general equilibrium productive economy rather than a partial equilibrium model. Moreover, as far as our research question if concerned, we focus on the complementary role of individual banks recapitalization and bailout recapitalization regimes rather than on banks micro-prudential regulation. In this term, our setting allows to measure the banks' efficiency as related to the market/enterprise value for its shareholders (as in Nicolo et al., 2014), but also to the positive externality it produces, by its monitoring activity, over equilibrium prices.

Recent relevant papers modelling a banking sector in a dynamic general equilibrium setting are Sandri and Valencia (2013), Phelan (2016), Nuno and Rey (2017), Hugonnier and Morellec (2017), Van Der Ghote (2017), and Gale et al. (2018). However, none of them models the trade-off between the long-run benefit of bailout resolutions after systematic banking crisis versus their short-term costs.

In Sandri and Valencia (2013), they study a DSGE model where financial frictions are introduces by a Financial Accelerator (FA) mechanism. The paper shows that recapitalizing the financial sector as a response to large losses in its net worth may be welfare improving, since it relates to the fluctuations of

[7]To this respect, the mechanism we propose also relates to Hennessy (2004). The paper incorporates debt in a dynamic real options framework, and shows that underinvestment stems from truncation of equity's horizon to default. Similarly, in our model equilibrium investments depends on prices (*Tobin's* Q), themselves a function of the banking sector recapitalization.

aggregate output. Moreover, they argue that the welfare gain are larger when recapitalization funds are raised from the households.

We distinguish under several aspects: first, in our model financial leverage is due to the cost advantage of intermediaries at monitoring firms, rather than on a FA. Moreover, we do not consider any idiosyncratic risk. Second, the paper analyses recapitalization policies in a very general way, by simply considering the impact of redistributing net worth across sectors. Conversely, we argue on the general convenience of individual recapitalization versus bailout regimes.

In Phelan (2016), the attention is on the role of leverage constraints at stabilizing the business cycle, and the possibility of default is not considered explicitly. Moreover, the agents being risk neutral, the households' demand for deposits is not explicitly modelled as endogenous. Conversely, we allow households to be risk-averse.

In Nuno and Rey (2017) the focus is on the heterogeneity within the banking sector. In particular, the paper studies joint role of the banks' *extensive* and *intensive* margins at determining, trough their leverage, the business cycle fluctuations. The relationship, as well as the banks' heterogeneity, comes after a non uniform distribution of VaR constraints over the banking sector.

In Hugonnier and Morellec (2017), the banking sector is modelled to assess the effects of liquidity and leverage requirements on banks' financing decisions and insolvency risk. They show that liquidity requirements lead to milder bank losses when defaulting, at the cost of an increased likelihood of default. On the other hand, higher leverage requirements reduces both the likelihood of default and the magnitude of bank losses after defaults. They conclude that the optimal policy is a combination of the two. The main differences with this paper is that they take the dynamics of the risky claim in which the bank may invest their assets as given (it is not a productive economy), and thus the firms' investments do not relate to the banking sector capitalization. An important common element is instead the costly issuance of new capital.

With concern to the *inter-temporal* trade-off between costs and benefits of banking regulation policies, this work partially relates to a recent insightful paper by Mendicino et al. (2019) that studies the relationship between transition costs and the long-run benefits of banking capital requirement. The paper shows that capital requirements make the banks safer in the long-run, since they successfully address stability risks. At the same time, the associated (short-run) costs negatively impact aggregate demand. However, they address the issue as related to monetary policy: as long as it nominal rates do not hit the lower bound, monetary policy is effective at dampening the real effect of those costs.

We basically differentiate since we focus on the *positive feedback loop* (long-run benefit) that associates to the short-run cost of recapitalization policy, as it is channelled by the positive externality of the banking sector aggregate capitalization through equilibrium prices. Moreover, we highlight the discrepancy between individual and social optimal recapitalization policies.

Finally, from the technical point of view, we relate to the macro-finance literature introducing financial frictions in dynamic models with a financial sector such as He and Krishnamurthy (2011, 2013, 2019), Brunnermeier and Sannikov (2014) as well as Klimenko et al. (2016).[8]

3.2 The Model

In this section, we discuss the building blocks of our theoretical framework. First, Section 3.2.1 introduces the technologies of two different types of productive firms. Second, it outlines the role of banks at monitoring capital producing firms. Third, it discusses the relationship between firms' problem and the return on their risky claims issuances. Finally, Section 3.2.2 and 3.2.3 describe households' and banks' problems, respectively. We begin with a general overview of the model economic environment.

Time is continuous, the horizon infinite, and there exists a unique source of (systematic) risk common to all economic actors. We consider a production economy with two non-fungible goods: physical capital (such as a tree) and perishable good (such as apples). Each good is produced by a specific type of firms, and the perishable good acts as numéraire.

The economy is populated by households and banks, henceforth indexed $h \in \mathbb{H} := [0, 1)$ and $b \in \mathbb{B} := [1, 2)$, respectively. Household h is born at time zero with an initial endowment e_0 in physical capital, exogenously split between bank's b equity e_0^b and disposable wealth e_0^h so that $e_0 = e_0^b + e_0^h$. For simplicity, household h is the unique shareholder of bank b; consequently, the aggregate banking sector is owned by the collectivity of households.

All actors are price takers and operate on a perfectly competitive market for physical capital, with equilibrium price q_t per traded unit. The total wealth within the economy at each instant of time $t \in [0, \infty)$ consists of the aggregate stock of physical capital K_t valued $K_t q_t$, and it equals the sum of capital stock within banks' equity (book value) and households' disposable wealth (henceforth, by capital letters we denote aggregate variables)[9]

[8]In particular, Klimenko et al. (2016) aim at explaining the dynamics of bank capital as related to the fluctuations of lending and output, in particular with regards to the role of capital requirement at generating credit crunches. On the contrary, we focus on the relationship between recapitalization policies and long-run welfare.

[9]As we shall see, in equilibrium, banks pay out dividends and issue equity from/to their shareholder and the book value of their equity is never fully redeemed.

$$\int_{\mathbb{H}} e_t^h dh + \int_{\mathbb{B}} e_t^b dh = E_t^h + E_t^b = K_t q_t.$$

Households consume and choose upon the allocation of their disposable wealth e_t^h between risky claims (issued by capital producing firms) and short-run banks' liabilities to maximize their inter-temporal utility. Banks issue short-term liabilities and, jointly with the capital value of their equity, invest those resources into risky claims (firms' equity) issued by capital producing firms. They optimally choose their leverage, dividends payouts, and equity issuance policies (recapitalization) to maximize their *market value* for their shareholders; namely, the expected discounted value of the future dividends minus (costly) recapitalization. In these term, banks are run in the best interests of their shareholders, the households, and there are no managerial agency conflicts. What is relevant to stress is that banks never choose a capital structure that is completely equity or debt. This is because, even though banks' equity provides a buffer against insolvency and pays out dividends in good states only, its issuance is costly. On the contrary, short-term debt provides attractive financing because it earns a liquidity yield (risk-free rate), and its issuance is costless.

To summarise the relationship between households and banks, Figure 3.1 synthetically represents the balance sheet positions of two "paired" bank-shareholder (b-h and k-j, respectively) at time t; in dark grey we depict assets, in light grey liabilities. Household h is the unique shareholder of bank b, as household j is for bank k. Household h (j) holds a fraction of her assets e_t^b (e_t^k) invested in bank's b (k) equity, while she deposits the remainder of her wealth e_t^h (e_t^j) to bank's k (b) short-term liabilities (debt). Therefore, each household is the unique shareholder of one bank, while it holds her disposable wealth in any other bank's short-term liabilities[10]. All in all, total banks' assets sum up to aggregate households' wealth endowments, and ω_t^i represents the ratio between bank i assets and its equity endowment, i.e. her leverage. Banks' assets are invested in risky claims (equity) issued by productive firms.

3.2.1 Technologies, Monitoring, and Risky Returns

Production technologies There exist two types of firms: capital and output producing firms. *Capital producing firms* (type I), act as lessors, and collect capital stock (trees) from either households and/or banks. As we shall see, in equilibrium they will collect capital from banks only, i.e. households mandate risky investments to banks. By managing the collected resources, type I firms rent capital stock to output producing firms at an instantaneous rate p_t, and choose the re-investment rate ι_t to inter-temporally generate new capital at time

[10] This could be loosely interpreted as taste for variety

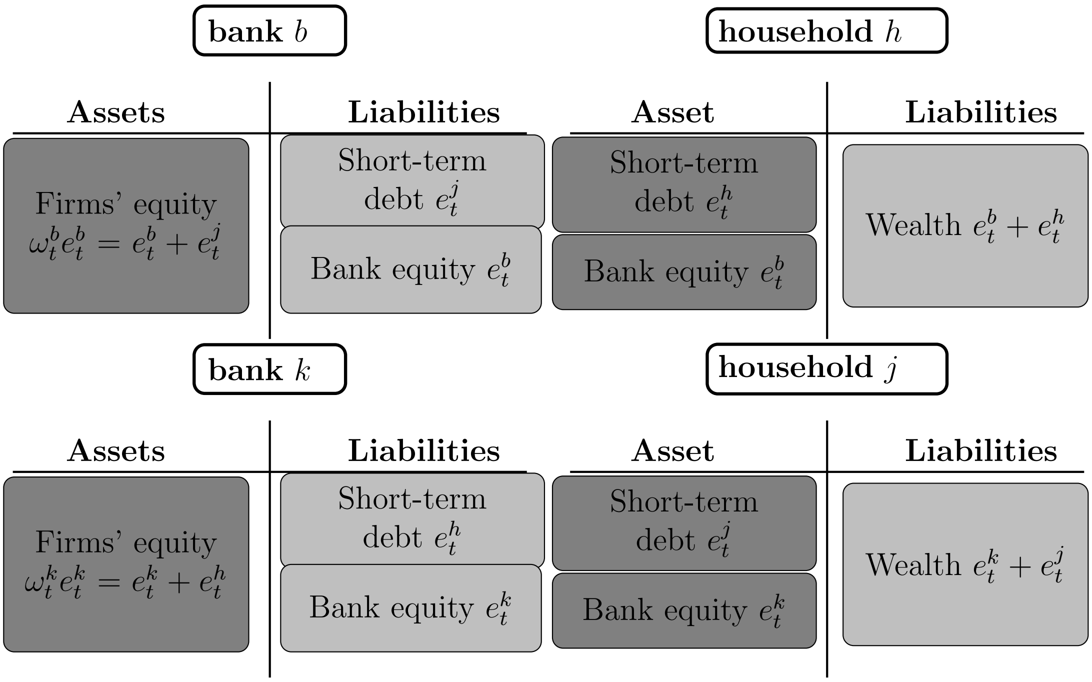

Figure 3.1: Banks' (left) and households' (right) synthetic (cross) balance sheets.

$t + dt$ by means of a stochastic technology. Let dW_t be a standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{H}, \mathbb{P})$, where $\{\mathcal{H}_t, t > 0\}$ is the natural filtration over the measurable space $(\Omega, \mathcal{H})$. The capital stock k_t managed by type I firms evolves as an Itô diffusion:

$$T_t : \frac{dk_t}{k_t} = \Phi(\iota_t)dt + \sigma dW_t, \tag{3.1}$$

where $\Phi(\iota_t)$ is an increasing and concave function that represents the technological illiquidity that comes after turning perishable good into new capital. What is relevant to stress is that dW_t represents the unique source of aggregate risk within the economy, and it is common across firms. In equilibrium, capital producing firms always break even and earn no profits.[11]

Output producing firms (type II) rent physical capital (trees) k_t at the instantaneous rate p_t to produce perishable consumption (apples) y_t by a linear technology

$$y_t = Ak_t, \tag{3.2}$$

[11]This aspects, and the overall micro-foundation of the production process jointly with the issuance of risky claims, is discussed at length in Dindo et al. (2019). As in Brunnermeier and Sannikov (2016a), we assume the functional form for the capital producing technology to be $\Phi(\iota) = \frac{\log(1+\theta\iota)}{\theta}$, where θ is a parameter summarizing technological illiquidity. This functional form is equivalent to having quadratic adjustment costs, but is has nicer analytical properties such that the re-investment rate ι is an affine transform of the price q.

so that their profits at time t equal $(A - p_t)k_t$. As for capital producing firms, type II firms in equilibrium always break even.

Monitoring costs In this model, neither banks nor households directly hold physical capital, that is managed on their behalf by capital producing firms. In equilibrium, type I firms manage the whole stock of capital within the economy and issue equity shares against the present discounted value of their profits. By exerting costly effort, they can increase the productivity of output producing firms, to whom they rent capital; this gives rise to a moral hazard problem. The problem can be tackled by implementing costly monitoring of effort decisions; as those activities prevent the possibility of not exerting effort. Firms' equity shareholders, either households or banks, may monitor the activities of risky claims issuers to induce it to provide its services. When conducted by banks, monitoring is cheaper than for households. In either cases, the costs scales down the risky assets absolute return by a fixed amount $\eta^i, i \in \{h, b\}$ for unit of capital invested. Accordingly, we define the *Banking Premium* (BP) as

$$\eta := \eta^h - \eta^b \geq 0. \tag{3.3}$$

The BP can be read as a reduced form that summarizes the advantage cost of the banking sector at monitoring firm that supplies capital services. This feature, aside from its capacity of supplying liquidity through its short-term liabilities, is the main reason that motivates the existence of banks in our model. Henceforth, for the purpose of our analysis, we set $\eta^b = 0$.[12] By (3.3), it follows that $\eta^h = \eta$.

Return on risky assets Type I firms finance their activity by issuing risky claims (equity) against the present discounted value of their net profits. As the capital producing technology is stochastic, so it is the return on risky claims issued by firm j, dR_t (equal across firms), with dynamics

$$dR_t = \mu_t dt + \sigma_t dW_t, \tag{3.4}$$

where the drift μ_t and diffusion σ_t terms are endogenous, and will be jointly determined in equilibrium. Note that all risky claims issued by all firms are equal and uniformly exposed to the unique common source of risk dW_t. To

[12] Similarly, in Van Der Ghote (2017) the banking premium can be rationalized as a cost advantage that originates from a moral hazard problem in the firms equity market (see also Diamond, 1984; Tirole, 2010). Likewise, in Dindo et al. (2019) it associates to the capacity of the financial sector to monitor heterogeneous capital producing firms and pool their issuance of risky claims.

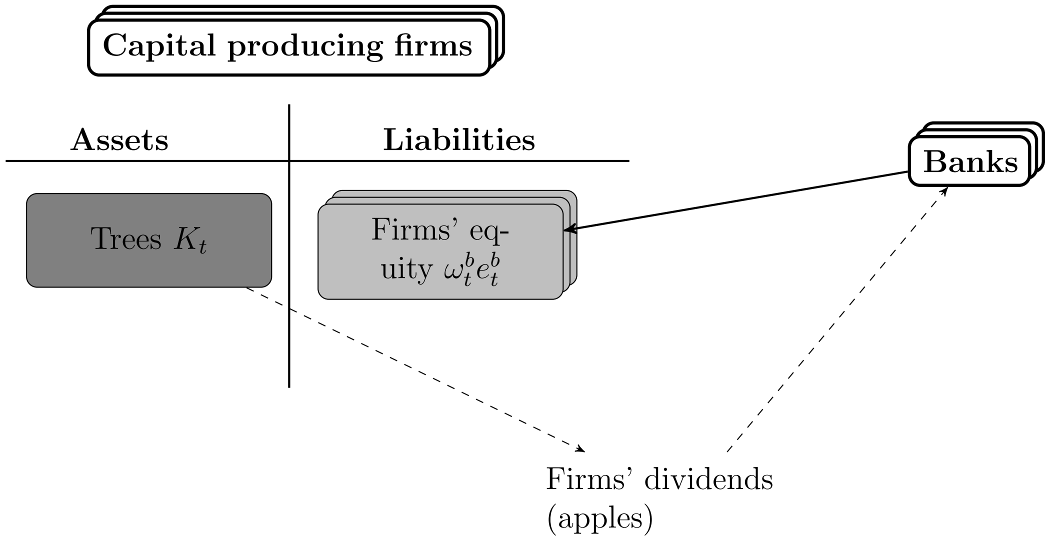

Figure 3.2: Capital producing firms' synthetic balance sheets at time t.

summarise the relationship between banks and capital producing firms, Figure 3.2 represents the balance sheet of the latter as related to their activity as capital lessors.

3.2.2 Households

There exists a continuum of households defined over the space $\mathbb{H}$. They are risk-averse, infinitely lived, and discount the future at a constant rate ρ. Households are born at time zero with an initial endowment e_0, exogenously split between bank's b equity e_0^b and disposable wealth e_0^h so that $e_0 = e_0^b + e_0^h$. Household h mandates the management of those resource allocated as bank equity e_t^b to bank b, consumes, and allocates her idle wealth stock e_t^h between risky claims issued by capital producing firms and short-term banks' liabilities to maximise the inter-temporal utility of her consumption.

The households' wealth share allocated in short-term bank liabilities is remunerated at the instantaneous (endogenously determined) risk-free rate r_t. The return on wealth share allocated in risky claims is uncertain, and evolves as in (3.4).

As household h is the unique owner and shareholder or bank b, she receives dividends flows $d\delta_t^b$ from her own bank, and pays the equity issuance $d\pi_t^b$ necessary to her recapitalization. Issuing bank equity is costly, and requires $1 + \lambda$ unit of capital value, i.e. the flow of resources depleted after bank b recapitalization equals $d\pi_t^b(1 + \lambda)$ (see also Klimenko et al., 2016).[13] Formally, households'

[13] In short, λ summarizes banks' administrative and organizational costs for issuing new equity. Loosely speaking, this friction may be also thought as a reduced form that represents the impaired market liquidity that the banks' face when they need to issue

problem reads as follows:

$$H_0 := \max_{\left\{c_t^h, \omega_t^h\right\}_{t\in[0,\infty)} \in G_t^h} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \ln c_t^h dt, \tag{3.5}$$

s.t.

$$G_t^h : \frac{de_t^h}{e_t^h} = \omega_t^h dR_t - \underbrace{\omega_t^h \frac{\eta}{q_t} dt}_{\text{Monitoring cost}} + \left(1 - \omega_t^h\right) r_t dt - \frac{c_t^h}{e_t^h} dt + \underbrace{\frac{d\delta_t^b - d\pi_t^b (1 + \lambda)}{e_t^h}}_{\text{Trasfers from/to bank } b}, \tag{3.6}$$

where ω_t^h is the portfolio share allocated in risky claims, η represents the instantaneous monitoring expenditure after each unit of capital invested in firms' equity, and $d\tau_t^b := d\delta_t^b - d\pi_t^b (1 + \lambda)$ is the net transfer from/to the banking sector (dividends or equity issuance purchase, respectively).

Under the optimal strategy $\left\{c_t^h, \omega_t^h\right\}$, we postulate the functional form of the value H_t such that it satisfies the following:

$$h_0 + \bar{b} \log e_0^h := \mathbb{E}_0 \int_0^\infty e^{-\rho t} \ln c_t^h dt, \tag{3.7}$$

where the unknown function $h(\bullet)$ captures the effect of the (aggregate) banking sector capitalization on the households' value function. To further characterize the households' optimal strategy, we may define the state variable (aggregate) banking sector *relative wealth share* as

$$\psi_t := \frac{\text{Aggregate banking sector book-value}}{\text{Aggregate wealth}} \in (0, 1), \tag{3.8}$$

and thus, $h(\psi_t)$. The banks HJB is characterized in Lemma 3:

Lemma 3. ***The Households' HJB Equation***
According to the ansatz in (3.7) the households HJB equation holds as

$$\rho h(\psi_t) + \ln e_t^h := \max_{\left\{c_t^h, \omega_t^h\right\}_{t\in[0,\infty)}} \left\{ \log c_t^h + \frac{1}{dt} \mathbb{E}\left[dH_t(\psi_t)\right] \right\}, \tag{3.9}$$

subjected to (3.6), with transversality condition $\lim_{t\to\infty} \mathbb{E}_0 \left[e^{-\rho t} H_t\right] = 0$.

The optimal strategy $\left\{c_t^h, \omega_t^h\right\}$ that solves problem (3.5) and the features of the value function H_t are summarised in Proposition 2. We conjecture that it is always optimal (welfare improving) for the households to pay the cost of recapitalization $(1 + \lambda) d\pi^b$, i.e. the total cost of equity issuance is always less or equal to the market value of bank b after recapitalization. In Section 3.2.3, we show that this is actually the case.

securities at the moment of distress. Empirical evidence of the relationship between stock liquidity and its issuance cost is in Butler et al. (2005).

Proposition 2. ***The Households: Optimal Strategy and Value Function***

Given Lemma 3, conditional on the couple $\{d\delta_t^b, d\pi_t^b\}$,

1. *The optimal controls* $\{c_t^h, \omega_t^h\}$ *of problem (3.5) equal:*

$$c_t^h = \rho e_t^h, \quad \omega_t^h = \frac{\mu_t^h - r_t}{\sigma_t^2}, \tag{3.10}$$

   where $\mu_t^h = \mu_t - \frac{\eta}{q_t}$;

2. *The households' value* H_t *is such that*

$$H_t = h_t(\psi_t) + \frac{1}{\rho}\ln e_t^h, \tag{3.11}$$

   where h_t *satisfies the following ODE:*

$$\rho h_t = a\left(\psi_t, d\delta_t^b, d\pi_t^b\right) + h_\psi b\left(\psi_t\right) + \frac{1}{2}h_{\psi\psi}c\left(\psi_t\right), \tag{3.12}$$

   with Cauchy boundary conditions $h(0) = \frac{1}{\rho}\left[\log\rho + \frac{r(0)}{\rho} - 1\right]$ *and* $h_\psi(0) = 0$.

Proof. See Appendix C.3.1. □

3.2.3 Banks and Financial Markets

There exists a continuum of banks defined over the space $\mathbb{B}$. They are risk-neutral, and discount the future by households' *Stochastic Discount Factor* (SDF).[14]

Bank's b has an initial equity endowment e_t^b, and supplies liquidity services by issuing short-term liabilities to the households (remunerated at the risk-free interest rate). Henceforth, we refer to banks' *book value* and *equity* as the same object. Bank b optimally chooses her dividends payouts $d\delta_t^b$, equity issuances $d\pi_t^b$, and asset allocation (financed by equity plus short-term liabilities) to maximize her *market value*. Formally, the banks' problem reads as follows:

$$J_0 := \max_{\left\{d\delta_t^b, d\pi_t^b, \omega_t^b\right\}_{t\in[0,\infty)} \in B_t^b} \mathbb{E}_0 \int_0^\infty \Lambda_t \underbrace{\left[d\delta_t^b - (1+\lambda)d\pi_t^b\right]}_{d\tau_t^b}, \tag{3.14}$$

[14]Since the asset structure of this economy includes risk-free bank deposits and one risky asset exposed to a unique source of aggregate risk, financial markets are complete, and the SDF Λ_t has dynamics

$$\frac{d\Lambda_t}{\Lambda_t} = -r_t dt - \xi_t dW_t, \tag{3.13}$$

where ξ_t is the market price of risk.

s.t.

$$B_t^b : \frac{de_t^b}{e_t^b} = \omega_t^b dR_t + \left(1 - \omega_t^b\right) r_t dt + \frac{d\pi_t^b}{e_t^b} - \frac{d\delta_t^b}{e_t^b}, \tag{3.15}$$

where ω_t^b represents the ratio of bank b assets over equity, i.e. her leverage.

Similar to Brunnermeier and Sannikov (2014), we guess the value J_t to satisfy, under the optimal strategy $\left\{d\delta_t^b, d\pi_t^b, \omega_t^b\right\}$,

$$v_t e_t^b =: \mathbb{E}_0 \int_0^\infty \Lambda_t \left[d\delta_t^b - (1+\lambda) d\pi_t^b\right]. \tag{3.16}$$

In general, the value $J_t := v_t e_t^b$ can be read as the market value of the b^{th} bank, as it represents the maximal future expected value of future pay-offs, dividends minus recapitalization flows, that the bank can attain conditional on having book value e_t^b.

Market-to-book value The term v_t is a proportionality coefficient that summarizes the way market conditions other then the bank own capital endowment affect the bank market value per unit of book value. v_t can be interpreted as the *market-to-book value* of bank b (see also Phelan, 2016; Klimenko et al., 2016). In particular, it represents the marginal value of the banks' stock of wealth, i.e. the marginal value of their book value. As we shall see, this implies that risk-neutral banks (with respect to the dividends and recapitalization flows) act *as if* they were risk-averse, since when they solve their problem they take into account the co-variance between the fluctuations of their market value and of their risky assets. The bank' HJB equation is characterized in Lemma 4:

Lemma 4. ***The Banks' HJB Equation***
Conditional on the ansatz in (3.16), the banks HJB equation reads as

$$r_t v_t e_t^b dt = \max_{\left\{d\delta_t^b, d\pi_t^b, \omega_t^b\right\}_{t\in[0,\infty)}} \left\{d\delta_t^b - (1+\lambda) d\pi_t^b + \mathbb{E}_t^{\mathbb{Q}} \left[d\left(v_t e_t^b\right)\right]\right\}, \tag{3.17}$$

subject to (3.15), with transversality condition $\lim_{t\to\infty} \mathbb{E}_0 \left[\Lambda_t v_t e_t^b\right] = 0$*, where* $\mathbb{Q}$ *is the risk-neutral probability induced by the households preferences.*

We now move forward and derive the competitive equilibrium of the economy as it relates to the banks' market-to-book value and, in turn, to their optimal strategies.

3.3 Competitive Equilibrium

This section is structured as follows: first, we outline the steps to derive the competitive (Markov) equilibrium of this economy. Second, Section 3.3.1 characterises the banks' optimal strategy in equilibrium, as it relates to their market-to-book value, and outline the economic implications of it. Third, in Section 3.3.2 we solve the model and discuss the main features of the competitive equilibrium.

Equilibrium The competitive equilibrium of this economy consists of maps from histories of systematic shocks $\{dW_t\}$ to prices $\{q_t\}$, returns on risky claims $\{dR_t\}$, risk-free rates on banks' short-term liabilities $\{r_t\}$, production $\{K_t, \iota_t\}$ and consumption $\left\{c_t^h : h \in \mathbb{H}\right\}$, asset allocations $\left\{\omega_t^h, \omega_t^b : h \in \mathbb{H}, b \in \mathbb{B}\right\}$ as well as dividends payouts and equity issuance strategies $\left\{d\delta_t^b, d\pi_t^b : b \in \mathbb{B}\right\}$ so that:

1. Firms, capital and output producing, maximise their profits;
2. Households $h \in \mathbb{H}$ maximise their utility;
3. Banks $b \in \mathbb{B}$ maximise their market value;
4. All markets clear (risky assets/physical capital, deposits, and consumption).

The formal statement of the competitive equilibrium is in Appendix C.3.2.

3.3.1 Equilibrium Dynamics: Characterization

To further characterize the equilibrium dynamics, we must relate it to the banks' strategy so that it satisfies the HJB Equation (3.17). To do so, we postulate a stochastic process that describe the dynamics of the banks' market-to-book value v_t, so that we can analytically derive the expectation over the stochastic differential $d\left(v_t e_t^b\right)$. As the only source of aggregate uncertainty within the economy is the systematic risk component dW_t, we guess v_t to evolve as Itô's process

$$\frac{dv_t}{v_t} = \mu_t^v dt - \sigma_t^v dW_t, \tag{3.18}$$

where μ_t^v and σ_t^v are $\mathcal{H}$-adapted stochastic processes whose values are endogenous and determined in equilibrium. By the ansatz (3.18), we restrict our search to those equilibria where after good (bad) aggregate shocks, the market-to-book value of banks' is decreasing (increasing) since, due to its leverage, the banks' assets grows (reduces) relatively more than its liabilities. This is equivalent to say that banks' marginal value of equity increases after negative shocks, because the fraction of assets finance thought short-term liabilities increases with respect the remaining share financed by equity.

Proposition 3 summarizes the conditions under which the banks' strategy $\left\{d\delta_t^b, d\pi_t^b, \omega_t^b\right\}$ is an optimal control for problem (3.14), conditional on the dynamics postulated in (3.18).

Proposition 3. ***Individual Banks' Strategy***
Conditional on the ansatz (3.18) and on the dynamics of Λ_t (3.13), the optimal controls $\left\{d\delta_t^b, d\pi_t^b, \omega_t^b\right\}$ of problem (3.14) are such that $1 \leq v_t \leq 1+\lambda$ and

1. *The dividend flow $d\delta_t^b$ is such that*
$$d\delta_t^b \geq 0 \Longleftrightarrow v_t = 1, \tag{3.19}$$
and (3.19) holds with equality when $v_t > 1$;

2. *The individual recapitalization flow $d\pi_t^b$ is such that*
$$d\pi_t^b \geq 0 \Longleftrightarrow v_t = 1+\lambda, \tag{3.20}$$
and (3.20) holds with equality when $v_t < 1+\lambda$;

3. *The banks' leverage is such that the risk premium satisfies*
$$\mu_t - r_t \geq -\frac{1}{dt}\left[\mathbb{C}ov_t\left(d\Lambda_t, dR_t\right) + \mathbb{C}ov_t\left(dv_t, dR_t\right)\right], \tag{3.21}$$
and (3.21) holds with equality when bank b is indifferent between holding risky assets and short-term liabilities;

4. *The banks' HJB equation holds as*
$$-\mu_t^v = \underbrace{\omega_t^b\left(\mu_t^b - r_t\right)}_{\frac{1}{dt}\mathbb{E}_t\left[de_t^b - r_t dt\right]} - \underbrace{\omega_t^b\sigma_t\sigma_t^v}_{-\frac{1}{dt}\mathbb{C}ov_t\left[de_t^b, dv_t\right]} - \underbrace{\omega_t^b\sigma_t\xi_t}_{-\frac{1}{dt}\mathbb{C}ov_t\left[de_t^b, dZ_t\right]} + \underbrace{\sigma_t^v\xi_t}_{\frac{1}{dt}\mathbb{C}ov_t\left[dv_t, dZ_t\right]}, \tag{3.22}$$
where $Z_t := \frac{d\mathbb{Q}}{d\mathbb{P}}$ is the price kernel (Radon-Nikodym derivatives) associated to Λ_t.

Proof. See Appendix C.3.3. □

The first implication of Proposition 3 is that banks are willing to pay out dividends $d\delta_t^b$ as long as the marginal value of their equity (market-to-book value) v_t holds lower (or equal) than one. Else, paid out dividends equal zero.[15] On the contrary, banks are willing to issue new equity, proportionally to $d\pi_t^b$, as long as the marginal value of their equity holds lower or equal to the marginal cost of recapitalization. Else, equity issuance equal zero. As we shall see, in equilibrium, equity issuance takes place when bank b is not able to remunerate its short-term liabilities. When this happens, as the marginal value of bank b equity is higher than the cost of her recapitalization, the bank b shareholder

[15]In fact, v_t can never be less than one because banks can always pay out the full value of equity instantaneously, guaranteeing a value of at least e_t^b.

withdraws the counter-value $d\pi_t^b$ from short-term liabilities "deposited" in any other bank $k \in \mathbb{B}\backslash b$ in order to keep her own bank solvent. This mechanism grants the absolute safety of short-term bank liabilities over the time interval $[t, t+dt]$, and so their liquidity benefit (as for example in Stein, 2012). In this term, the capital structure of banks' is not trivial because they face the fixed cost λ for issuing equity and households benefit of the liquidity services granted by their short-term liabilities.

In summary, banks finance (recapitalize) themselves by retaining dividends/issuing equity as long as those resources marginally contribute to higher expected future dividends, by short-term liabilities otherwise.

The second result in Proposition 3 is that, to hold in equilibrium when banks' leverage is unconstrained, the risk premium $\mu_t - r_t$ must be such that banks are indifferent between holding risky claims (firms' equity) and short-term liabilities. Accordingly, the risk-free interest rate r_t adjusts so that households hold no risky claims themselves and, by market clearing conditions, banks' collects as short-term liabilities the whole residual stock of capital that is not already within their equity so that (3.21) holds with equality. Therefore, households utterly mandate investments in firms' equity to banks and it holds that

$$\frac{\frac{1}{dt}\mathbb{E}dR_t - r_t}{\sigma_t} \propto \sigma_t^v.$$

As far as asset pricing is concerned, what is relevant to stress is that, in the spirit of He and Krishnamurthy (2013), this is an *Intermediary Asset Pricing* model, as the equilibrium risk premium is determined by the (here negative) covariance between banks' market-to-book value v and return on risky claims. It follows that, ceteris paribus, the higher the correlation between dv and dR, the higher the risk premium.[16]

The third and last result of Proposition 3 concerns the drift of the banks' market-to-book value. In particular, Equation (3.22) shows that μ_t^v depends on the covariance between the dynamics of banks' equity and both their market-to-book value v_t and price kernel Z_t. As we shall see, μ^v holds negative in equilibrium, i.e. the banks' market-to-book value v decreases as the banking aggregate capitalization ψ, on average, increases.

Before we define the dynamics of the (unique) state variable of the equilibrium, the banking sector relative wealth share ψ, we may specify the equilibrium return of the agents' risky assets as well as the aggregate dividends and recap-

[16]Empirical evidence that the marginal value of the financial sector wealth provides relevant information for asset pricing is in Adrian et al. (2014). The paper argues that the leverage of intermediaries represents a good proxy for the marginal value of their wealth, and that it can be used to characterize their SDF. In particular, they show that when funding conditions tighten, so that the exposure to intermediaries' leverage factor alone explains the excess returns on a wide range of assets.

italization flows. Hereafter, we define as *banking sector* the aggregate of banks within the economy.

Risky assets in equilibrium As we already stressed, all agents are price takers and exchange physical capital in a perfectly competitive market at the equilibrium price q_t per traded unit. To characterize the equilibrium return on risky claims, we may postulate a stochastic process that describe the dynamics of capital price. As the only source of uncertainty in our economy is the systematic risk dW_t, we conjecture q_t to evolve as an Itô diffusion, such that

$$\frac{dq_t}{q_t} = \mu_t^q dt + \sigma_t^q dW_t, \tag{3.23}$$

where the drift μ_t^q and diffusion σ_t^q terms are $\mathcal{H}$-adapted stochastic processes, whose values are endogenous and determined in equilibrium. By Itô's lemma, given the stochastic capital producing technology (3.1), the output producing technology (3.2), and the conjectured processes in (3.23), the return on risky assets evolves as (3.4), where

$$\mu_t := \underbrace{\frac{A - \iota_t}{q_t}}_{\text{Dividend Yield}} + \underbrace{\Phi(\iota_t) + \mu_t^q + \sigma_t^q \sigma}_{\text{Capital Gain}}; \quad \sigma_t := \sigma + \sigma_t^q. \tag{3.24}$$

As in Brunnermeier and Sannikov (2014), the left-hand side component of the drift μ_t can be read as the *dividend yield*, that is paid off in consumption good, of the risky claim issued by the capital producing firm.[17] The right-hand side component of (3.24), the *capital gain*, summarizes instead the fraction of the expected risky assets return that accounts for the additional capital stock generated over the instantaneous time interval dt jointly with its change in price.

Dividends and recapitalization flows Let the aggregate dividends and recapitalization flows from/to the banking sector be defined as

$$d\Delta_t := \int_{\mathbb{B}} d\delta_t^b, \quad d\Pi_t := \int_{\mathbb{B}} d\pi_t^b. \tag{3.25}$$

Note that, since there does not exist any idiosyncratic risk that effects the b^{th} bank wealth, the banks $b \in \mathbb{B}$ are homogeneous in terms of their market-to-book value v (that is function of the relative wealth share of the aggregate banking sector ψ), and so are their recapitalization and dividend strategies (See Proposition 3). What follows is that, when one bank either issue new equity

[17]The former addend is proportional to the marginal productivity of the output producing firm (A), and summarises the stock of output that is not re-invested to generate new capital at time $t + dt$.

or pays out dividends, the whole banking sector does so. Thus, in this model, banking recapitalization is always systemic (synchronous to all banks) and, by the optimal strategy in Equation (3.20), it happens when the banks' market-to-book value reaches the upper threshold $v = 1+\lambda$. What is also relevant to stress is that, due to the banks homogeneity, the upper bound of v uniquely relates to a lower threshold of banks' relative capitalization $\underline{\psi}$, so that recapitalization happens when $\psi_t \leq \underline{\psi}$. The natural threshold $\underline{\psi}$ equals zero (see also Klimenko et al., 2016).[18] At this stage, we set the level $\underline{\psi} = 10^{-4}$ arbitrarily close to zero. The threshold $\underline{\psi}$ can be read as a regulatory provision that establishes the minimum required equity that is necessary for the banking sector to work. Note that this choice does not affect the qualitative results of the model.

In the same fashion, a specular condition holds for the lower threshold $v = 1$ (see Equation 3.19). In this second case, dividends are paid out because banks' hold "too much" capital, so to keep the marginal value of their equity within at or below one. Thus, dividends are only paid when the banking sector is "well capitalized" in the aggregate, that is, when $\psi_t > \overline{\psi}$. It is relevant to highlight that the upper threshold is endogenously determined by equilibrium conditions, and fundamentally depends on η, σ, and λ.

We now have all the necessary elements to formally define the (unique) state variable, the banking sector *relative capitalization* (relative wealth share), as well as the stochastic process that describes its dynamics.

The state and its dynamics Let the banks' relative capitalization be

$$\psi_t := \frac{E_t^b}{E_t^b + E_t^h}, \tag{3.26}$$

where E_t^b is the aggregate equity of the banking sector and E_t^h is the aggregate stock of wealth in short-term bank liabilities at time t. As well shall see, all relevant equilibrium quantities can be expressed as functions of ψ, whose dynamics is driven by a regulated diffusion process. The result is summarised in Proposition 4.

---

[18]Note that, in the aggregate, by smooth pasting, it must hold that

$$\frac{\partial J(\underline{\psi})}{\partial E^b} = 1 + \lambda,$$

this implies that, given the guess $E^b v(\psi)$ and $\psi = \frac{E^b}{E^h + E^b}$,

$$v_\psi \frac{E^b}{E^b + E^h} + v = 1 + \lambda \to \frac{E^b}{E^b + E^h} = \psi \approx 0,$$

for any finite value of v_ψ.

Proposition 4. ***Equilibrium Dynamics***

1. *The relative wealth share (3.26) of the aggregate banking sector evolves as a regulated diffusion with dynamics*

$$\frac{d\psi_t}{\psi_t} = \mu_t^{\psi} dt + \sigma_t^{\psi} dW_t + d\Xi_t, \tag{3.27}$$

where

$$\mu_t^{\psi} = (1-\psi_t)\left[\rho + \omega_t^b \frac{\overline{\eta}}{q_t} + \omega_t^b \sigma_t^2 \omega_t^h - \left(\omega_t^h \sigma_t\right)^2\right] - \sigma_t^2\left(\omega_t^b - 1\right), \tag{3.28}$$

$$\sigma_t^{\psi} = \left(\omega_t^b - 1\right)\sigma_t; \tag{3.29}$$

and the impulse variable $d\Xi_t$ is such that

$$d\Xi_t = \begin{cases} -\frac{d\Delta_t}{K_t q_t}\frac{1}{\psi_t}, & \psi_t = \bar{\psi}, \\ \frac{d\Pi_t}{K_t q_t}\left(\frac{1-\psi_t \lambda}{\psi_t}\right), & \psi_t = \underline{\psi}, \\ 0, & \underline{\psi} < \psi_t < \bar{\psi}; \end{cases} \tag{3.30}$$

2. *$\bar{\psi}$ and $\underline{\psi}$ are the upper and lower thresholds that determine whether the aggregate recapitalization and dividends flows are positive or zero, respectively, so that*

$$v(\underline{\psi}) = 1 + \lambda, \quad v(\bar{\psi}) = 1. \tag{3.31}$$

Proof. See Appendix C.3.4 □

What stands out from Proposition 4 is that, as long as banks' relative wealth share lays between the upper and lower bounds $\psi \in (\underline{\psi}, \bar{\psi})$, then ψ evolves as an Itô diffusion with drift μ^{ψ} and diffusion σ^{ψ}. Accordingly, the regulatory term $d\Xi$ must hold equal to zero.

Conversely, when ψ lays outside its boundary values, $d\Xi_t$ is an impulse that adjusts the process (3.27) by creating a regulated diffusion.[19] The "adjustment" takes place instantaneously when either the banking sector pays dividends and/or issues new equity, respectively. The upper and lower thresholds of ψ uniquely relate to the upper and lower bounds of v jointly with the cost of new equity issuance λ.

Another relevant point is that, as long as the recapitalization friction holds

[19] In this regard, the banks of our model pay out dividends and issue equity after observing the stochastic increment dW_t. Therefore, we consider left-continuous processes in the definition of banks' portfolio choices, so that their right-hand limits to define cash in (recapitalization) and out (dividends) flows.

greater than zero ($\lambda > 0$), then $v(\overline{\psi}) = 1 < v(\underline{\psi}) = 1 + \lambda$, and banks never pay dividends and issue equity at the same time.[20]

3.3.2 Solution

In this section, we numerically solve the model for its competitive equilibrium when the banking sector is unconstrained. By matching the asset pricing condition (3.21) with the equilibrium leverage ω_t^b and with the diffusion process in (3.18), it can be shown that the banks' market-to-book value v satisfies the following ODE

$$-\frac{\eta}{q_t} = \frac{\partial v_t}{\partial \psi_t}\frac{\psi_t}{v_t}\sigma_t^{\psi}\sigma_t, \tag{3.32}$$

with boundary condition $v(\underline{\psi}) = 1 + \lambda$ (see Appendix C.3.5).

The equilibrium level $\bar{\psi}$ that defines the threshold of relative wealth share above which the banking sector pays out dividends, jointly with the dynamics of the state ψ_t are summarised in the following corollary of Proposition 4:

Corollary 1. *Dividends Threshold*
Conditional on the exogenous lower bound $\underline{\psi}$ below which the banks issue new equity, given the boundary condition (3.31):

1. *The upper threshold $\bar{\psi}$ above which the banking sector pays out dividends equals*

$$\overline{\psi} = 1 - \left(1 - \underline{\psi}\right)\left(\frac{1}{1+\lambda}\right)^{\chi}, \tag{3.33}$$

where $\chi = \sigma\frac{1+A\theta}{\eta}$;

2. *When $\underline{\psi} < \psi < \bar{\psi}$, then the banking sector relative wealth share ψ_t has dynamics as in (3.27), where the drift and diffusion terms satisfy:*

$$\mu_t^{\psi} = (1-\psi_t)\left[\rho + \frac{\eta}{\psi_t}\frac{1+\theta\rho\,(1-\psi_t)}{1+\theta A}\right] - \sigma^2\left[1+\theta\rho\,(1-\psi_t)\right]^2\frac{1-\psi_t}{\psi_t}, \tag{3.34}$$

$$\sigma_t^{\psi} = \sigma\frac{1-\psi_t}{\psi_t}\left[1+\theta\rho\,(1-\psi_t)\right]. \tag{3.35}$$

[20] In the limit case where $\lambda = 0$, there is no friction after recapitalization flows. It follows that the FOCs for $d\Delta_t$ and $d\Pi_t$ are such that

$$d\Delta_t > 0, d\Pi_t > 0 \Longleftrightarrow v(\underline{\psi}) = v(\overline{\psi}) = 1,$$

which means that the banks pay dividends and issue equity to keep $\psi = \underline{\psi} = \overline{\psi}$, where the marginal value of the banks' equity equals 1. We briefly discuss this benchmark case, where there are no banks and the economy is populated by the households only in Appendix C.1.

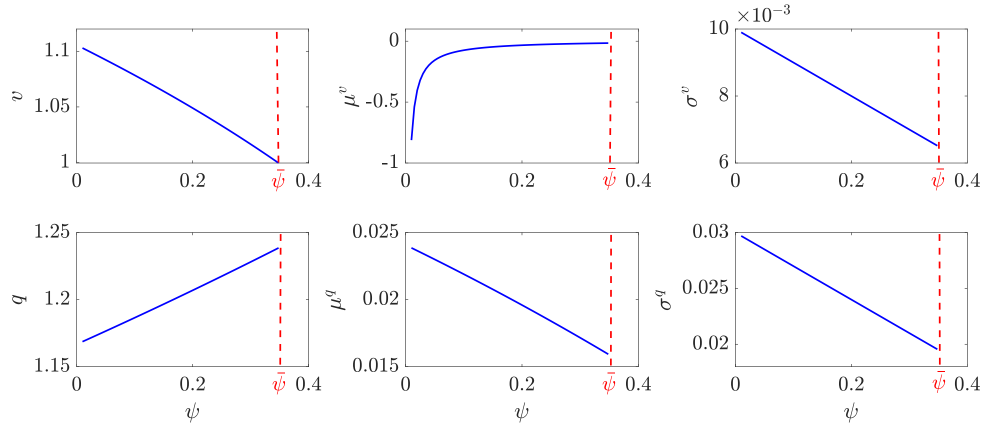

Figure 3.3: Top: Banking sector market-to-book value (left), drift (centre), and diffusion (right) as a function of the state. Bottom: Price of physical capital value (left), drift (centre), and diffusion (right) as function of the state. In red, the endogenous dividend threshold $\bar{\psi}$. Baseline parameters: $A = 0.1$, $\eta = 0.1$, $\sigma = 0.15$, $\theta = 4$, $\lambda = 0.1$ and $\rho = 0.05$.

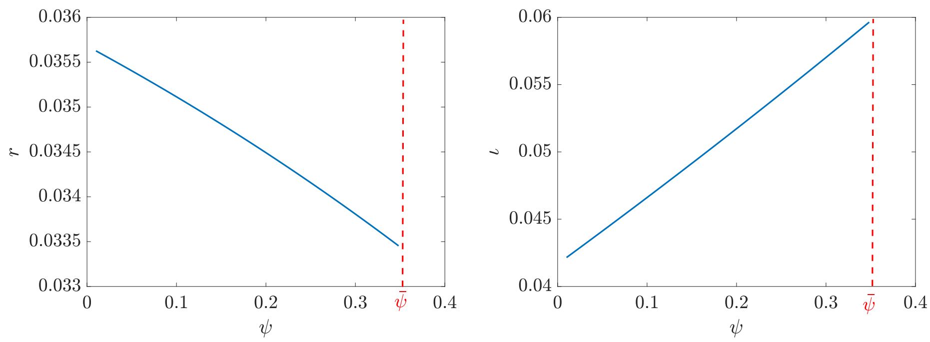

Figure 3.4: Risk-free interest rate (left) and re-investment rate (right) as a function of the state. In red, the endogenous dividend threshold $\bar{\psi}$. Baseline parameters: $A = 0.1$, $\eta = 0.1$, $\sigma = 0.15$, $\theta = 4$, $\lambda = 0.1$ and $\rho = 0.05$.

Proof. See Appendix C.3.5. A more comprehensive discussion upon the features of long-run equilibrium dynamics is in Appendix C.2.1. □

Figure 3.3 plots the value (left), the drift (centre), and the diffusion (right) of the equilibrium price of capital q (top) jointly with the banks' market-to-book value v (bottom). In red, we show the endogenous upper threshold above which banks pay out dividends. Henceforth, all the equilibrium quantities are computed considering the following parametric values: $A = 0.1$, $\eta = 0.1$, $\sigma = 0.15$, $\theta = 4$, $\lambda = 0.1$ and $\rho = 0.05$. The implied upper threshold equals $\bar{\psi} \approx 0.35$.[21]

Market-to-book value and price of capital The banks' market-to-book value v is determinant for the equilibrium dynamics. In fact, conditional on the lower bound threshold $\underline{\psi}$, it determines the upper ones, where the banking sector pays off dividends. As a result the value of v, as well the drift μ^v (in absolute value) and diffusion σ^v of its dynamics, are decreasing in banking sector capitalization. This is mainly due to banks' leverage, and so to the speed at which banks reconstitute their equity E_t^b with respect to the value of their short-term liabilities E_t^h.

On the contrary, the price level q (Figure 3.3, bottom left) is increasing in banking relative capitalization (decreasing in short-term liabilities). This is because, when banks are small, a greater fraction of aggregate output is consumed by households. Conversely, a greater share of wealth allocated to banks' equity associates to lower aggregate consumption (stock, as households' consumption rate is constant and equals ρ) and, accordingly, higher investments. As q positively relates to the level of ψ, so it does to its dynamics. In particular, the growth rate of banks' relative capitalization is increasing in both households' consumption rate ρ and banking premium (see Equation 3.34), as and so does the drift of price dynamics μ^q (Figure 3.3, bottom centre).

Another relevant aspect is that, as the financial leverage ω^b holds decreasing in ψ, and so it is the endogenous volatility σ^q which is generated by the banks' leverage (Figure 3.3, bottom right). This is because the higher the financial leverage (ω^b), the higher the sensitivity of the state dynamics ψ to exogenous systematic shocks (Equation 3.35).[22]

[21] While these choices are not the result of calibration, they produce reasonable qualitative results. Nonetheless, from the analysis of equilibrium results it is straightforward that the model dynamics strictly relates to the banks' market-to-book value, and so on the upper so that banks pay out dividends. As such, further questions may arise: how does the upper threshold changes with respect to the main parametric values? How does that relate to the banks' market-to-book value v? To answer these questions, a comparative static analysis is in Appendix C.2.

[22] Remind that, in equilibrium, the volatility of physical capital relates to the volatility of the relative wealth share by

$$\sigma_t^q = \epsilon_{q,\psi}\sigma_t^\psi, \tag{3.36}$$

Risk-free interest rates and re-investment rate In Figure 3.4 we plot the risk-free interest rate on the deposits r (left) and the re-investment rate ι (right) as a function of the state. What stands out is that, in equilibrium, r is decreasing in the banking sector capitalization. This is because the risk-free interest rate on deposits is proportional to both drift and diffusion of the stochastic process that drives the price of physical capital q. Due to the financial sector leverage jointly with households' consumption, the states where the banking sector is small are also those in which it grows faster, and the volatility of its relative capitalization is higher. Accordingly, those are also the states where the price of physical capital grows the most.

The increasing relationship between state ψ and price q has its counterpart in the re-investment rate ι (Figure 3.4, centre), that is increasing in the banks' relative share of wealth. Thus, a bigger financial sector associates to lower aggregate consumption and higher investments.

3.4 Recapitalization, Bailout, and Welfare

So far, we have considered the aggregate recapitalization flow as indirectly given by individual banks' optimal strategies, so that households transfer resources from their wealth allocation in short-term bank liabilities to keep solvent their bank. In equilibrium, households are willing to purchase banks' equity issuance as long as the marginal value of that equity is greater or equal to the cost of recapitalization (new equity issuance, $1 + \lambda$. Is the individual strategy of banks also optimal from the aggregate (social) standpoint?

To address this question, we consider the possibility of a complementary recapitalization regime that may top up banks' individual optimal equity issuance. This latter policy will by financed by taxing the aggregate of households; therefore, it can be loosely interpreted as a *bailout* over the whole banking sector. The aim of bailouts is to maximizing long-run households' welfare. Within this setting, due to the *pecuniary externality* that comes after a well capitalized banking sector, we show that when a systemic recapitalization takes place, it is welfare improving to recapitalize the banking sector well above the individual optimum that is consistent with banks' choices upon equity issuance. Intuitively, this is because all economic actors act in a perfectly competitive environment and fail

where the $\epsilon_{q,\psi} \geq 0$ is the elasticity of q with respect to ψ. This is the well know *amplification* mechanism introduced by Brunnermeier and Sannikov (2014), and

$$\sigma_t^q = \sigma \frac{\epsilon_{q,\psi}\left(\frac{1}{\psi_t} - 1\right)}{1 - \epsilon_{q,\psi}\left(\frac{1}{\psi_t} - 1\right)}. \tag{3.37}$$

at internalizing the effect of the aggregate banking sector capitalization upon equilibrium outcomes.

This section develops as follows. First, in 3.4.1 we define the short-run welfare function of the aggregate household, i.e. conditional on the state ψ, and describe its main components. Second, in 3.4.2 we introduce the long-run welfare, i.e. unconditional on the state, and characterize the bailout policy topping up the individual recapitalization regime. Finally, we discuss the economic mechanism underneath the existence of such a policy and the reasons why it is socially optimal.

3.4.1 Short-run Welfare

According to the results summarized in Proposition 2, the households' value H_t (conditional on the state ψ), equals

$$H_t(\psi_t) = h_t(\psi_t) + \underbrace{\frac{1}{\rho} \ln E_t^h}_{H^e(\psi)} . \tag{3.38}$$

Equation (3.38) can be interpreted as the *short-run* households' welfare, as it represents their value conditional on the state ψ_t and, by continuity, approximates it in its neighbourhood.[23] Note that, when the aggregate recapitalization flow is uniquely determined by individual banks' strategies (Equations 3.19 and 3.20), then the impulse $d\Xi_t$ is such that it instantaneously adjusts the process $d\psi_t$ to keep it within the upper and lower thresholds $\bar{\psi}$ and $\underline{\psi}$, respectively. Thus, $d\Xi_t$ holds as

$$d\Xi_t := \frac{\underline{\psi} - \psi_{t+dt}}{\psi_t} \mathbb{I}_{\psi_{t+dt}<\underline{\psi}} + \frac{\bar{\psi} - \psi_{t+dt}}{\psi_t} \mathbb{I}_{\psi_{t+dt}>\bar{\psi}}, \tag{3.41}$$

and regulate upward (downward) the diffusion process $d\psi_t$ when it is above (below) the upper (lower) bound.[24] In Figure 3.5 we plot the first (left) and

[23] As in equilibrium households do not retain investment in firms' equity (risky claims), h_t holds as a solution to the ODE in (3.12) where

$$a(\psi) := \log \rho + \frac{r(\psi)}{\rho} - 1 + \frac{1}{\rho} \left(\frac{d\Delta_t}{E^b} - (1+\lambda) \frac{d\Pi_t}{E^b} \right) \frac{\psi}{1-\psi} + \psi d\Xi, \tag{3.39}$$

while

$$b(\psi) := \psi \mu^{\psi}; \quad c(\psi) := \frac{1}{2} \left(\psi \sigma^{\psi} \right)^2 . \tag{3.40}$$

[24] Equation (3.41), together with the point (2) of Proposition 4, imply that

$$\frac{d\Pi_t}{K_t q_t} = \frac{\underline{\psi} - \psi_{t+dt}}{1 - \psi_t \lambda}, \quad \frac{d\Delta_t}{K_t q_t} = \psi_{t+dt} - \underline{\psi}. \tag{3.42}$$

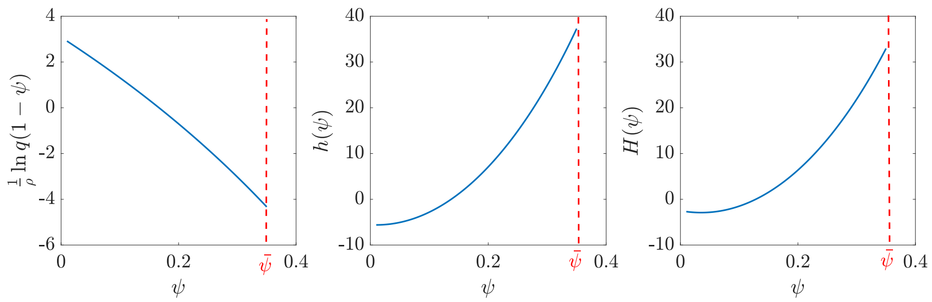

Figure 3.5: First (left) and second (centre) components of households' *short-run*. Right: Aggregate households short-run welfare $H(\psi)$. In red, the endogenous threshold $\bar{\psi}$. Baseline parameters: $A = 0.1$, $\eta = 0.1$, $\sigma = 0.15$, $\theta = 4$, $\lambda = 0.1$, and $\rho = 0.05$.

second (centre) components of households' value (welfare) function (3.38), as well as the overall H_t (right) as a function of the state ψ.[25] What stands out is that the first component of (3.39) is an increasing function of the state. This is because h_t summarizes two fundamental aspects of the equilibrium dynamics, both affecting households' welfare positively. First, $h(\psi)$ accounts for the fact that higher banking capitalization channels higher investment (through capital prices) cheaper future consumption. Second, $h(\psi)$ accounts for state contingent dividend flows $d\Delta$ (higher ψ associates to higher dividends payouts). On the contrary, the term $H^e(\psi)$ is strictly decreasing in ψ. In particular, it can be decomposed in two sub-components with opposite effects:

$$H^e(\psi) \propto \underbrace{\ln q(\psi)}_{+} + \underbrace{\ln(1-\psi)}_{-}. \tag{3.44}$$

The first component of (3.44) is increasing in ψ (through the price q, see Figure 3.3, bottom left), while the latter is decreasing. This means that, for households, the negative effect of a relatively lower share of aggregate wealth, and thus of a lower aggregate consumption, dominates the benefit of a more valuable stock of wealth.

All in all, when accounting for both $h(\psi)$ and $H^e(\psi)$, the conditional (henceforth, short-run) households' welfare H_t is a convex function of the state and it

Note that the unique state variable is banks' relative capitalization ψ, then the equilibrium dynamics of the model is scale invariant in the aggregate stock of physical capital K_t. Thus, we consider households aggregate wealth at time t as proxied by $q_t(1-\psi_t)$, i.e. for unitary aggregate capital stock.

[25]We solve the ODE (3.39) numerically by Matlab *ode45*, with Cauchy initial conditions:

$$h(0) = \frac{1}{\rho}\left[\log\rho + \frac{r(0)}{\rho} - 1\right], \quad h(0)_\psi = 0. \tag{3.43}$$

is maximal when banks are well capitalized and pay out dividends.

3.4.2 Bailout and Long-run Welfare

What is the welfare effect of an additional tax financed issuance of banks' equity further than what implied by individual banks' strategy? The aim of this analysis is to study what is the effect of an additional tax-based transfer of capital dO_t from the aggregate of households to the banking sector. The bailout is such that, contingent on banks' equity issuance ($\psi_t \leq \underline{\psi}$), it tops up the individual recapitalization so that banks' relative wealth share is enhanced to a higher level $\psi^* \leq \underline{\psi}$ than how it would be according to individual optimal strategies (hereafter, bailout threshold). As we shall see, choosing the level of tax transfers that maximises households' welfare is equivalent to choose the social optimal bailout threshold.

By considering the bailout transfer dO_t, the aggregate households' dynamic budget constraint in equilibrium modifies as follows

$$G_t^{h,O}: \quad \frac{dE_t^{h,O}}{E_t^{h,O}} = \underbrace{(r_t - \rho)dt + \frac{\int_{\mathbb{B}} d\tau_t^b}{E_t^h}}_{\frac{dE_t^h}{E_t^h}} - \underbrace{\left(1 + \lambda^G\right)\frac{dO_t}{E_t^h}}_{\text{Bailout term}},$$

where the term $\left(1 + \lambda^O\right)\frac{dO_t}{E_t^h}$ represents the gross bailout tax transfer. The term λ^O, similarly to the recapitalization cost λ, is a reduced form that represents the market illiquidity at the moment of bailout. For our purposes, we set $\lambda^O = \lambda$, meaning that the cost of raising capital for bailouts equals the cost of individual recapitalizations. Accordingly, the banking sector equity in equilibrium evolves as

$$B_t^{b,O}: \quad \frac{dE_t^{b,O}}{E_t^{b,O}} = r_t dt + \frac{1}{\psi_t}\left(dR_t - r_t dt\right) - \frac{d\Delta_t}{E_t^b} + \frac{1}{E_t^b}\left(d\Pi_t + dO_t\right).$$

Starting from the result summarized in Proposition 4 (Equation 3.30), it is straightforward to show that the impulse term that regulates the dynamics of relative wealth share while accounting for the bailout policy $d\Xi_t^O$ equals

$$d\Xi_t^O\big|_{\psi_{t+dt}\leq\underline{\psi}} = \frac{d\Pi_t + dO_t}{K_t q_t}\left(\frac{1-\psi_t\lambda}{\psi_t}\right). \tag{3.45}$$

Accordingly, by matching (3.45) to (3.41),

$$d\Xi_t^O\big|_{\psi_{t+dt}\leq\underline{\psi}} = \frac{\psi^* - \psi_{t+dt}}{\psi_t}, \tag{3.46}$$

where $\psi^* \geq \underline{\psi}$ is the bailout threshold.

From Equation (3.46), it is unambiguous that, conditional on the state ψ and on the banks' individual recapitalization strategies $d\Pi_t$, choosing the bailout flow dO_t is equivalent to choosing the threshold level ψ^*. Now that we have defined the impulse process in presence of bailout (3.45), we aim to find dO_t so that the long-run households' welfare, i.e. the short-run welfare H_t as integrated over the (long-run) stationary density of ψ, is maximal. Formally, the long-run welfare reads as

$$\mathbb{E}H = \int_{\underline{\psi}}^{\bar{\psi}} H(\psi)f(\psi)d\psi := W. \tag{3.47}$$

Equation (3.47) represents the long-run welfare since, as $t \to \infty$, the state variable ψ_t visits every state within the support $[\underline{\psi}, \bar{\psi}]$ with density given by its stationary distribution $f(\psi)$, notwithstanding the initial point ψ_0. We account for the long-run transition by integrating the short-run welfare function $H(\psi)$ weighted by $f(\psi)$. In this setting, the bailout recapitalization policy is defined as follows:

Definition 1. ***Bailout Recapitalization Policy***
Contingent on the state that requires a recapitalization of the banking sector, $\psi_{t+dt} \leq \underline{\psi}$, the bailout recapitalization policy dO, and so the associated bailout threshold ψ^*, is defined as

$$\psi^* = \arg\max_{dO} \underbrace{W^*}_{\mathbb{E}_t^*[H_t]}, \tag{3.48}$$

where the expected value $\mathbb{E}_t^*$ measures the long-run welfare given given the state density $f(\psi)$ under the policy (3.48), and the impulse term $d\Xi_t^O$ is so that

$$d\Xi_t^O := \frac{\psi^* - \psi_{t+dt}}{\psi_t}\mathbb{I}_{\psi_{t+dt}<\underline{\psi}} + \frac{\bar{\psi} - \psi_{t+dt}}{\psi_t}\mathbb{I}_{\psi_{t+dt}>\bar{\psi}}, \tag{3.49}$$

To intuitively evaluate the effect of bailouts over households' long-run welfare $\mathbb{E}_t^*[H_t]$, in Figure 3.6 (left) we plot the households' short-run welfare H_t (left) and the long-run state density $\pi(\psi)$ before (blue) and after (green) the bailout policy that recapitalize the banking sector up to $\psi^* = 0.15$.

What stands out is that the bailout top up shifts the short-run welfare H_t downward, due to the higher amount of resources that is depleted as a recapitalization cost after distress contingencies. Conversely, the same policy shrinks the density mass around the left-hand side tail of the distribution, and the stationary density $f(\psi)$ shifts upward-left. This is because the transition through small banking capitalization phases "from below" is avoided by means of the enhance issuance of new equity, that "jumps" up to the state ψ^*. This also implies that the probability of being in "good" states, where the banking sector is well capitalized, is higher. This suggests that there exists a trade-off between

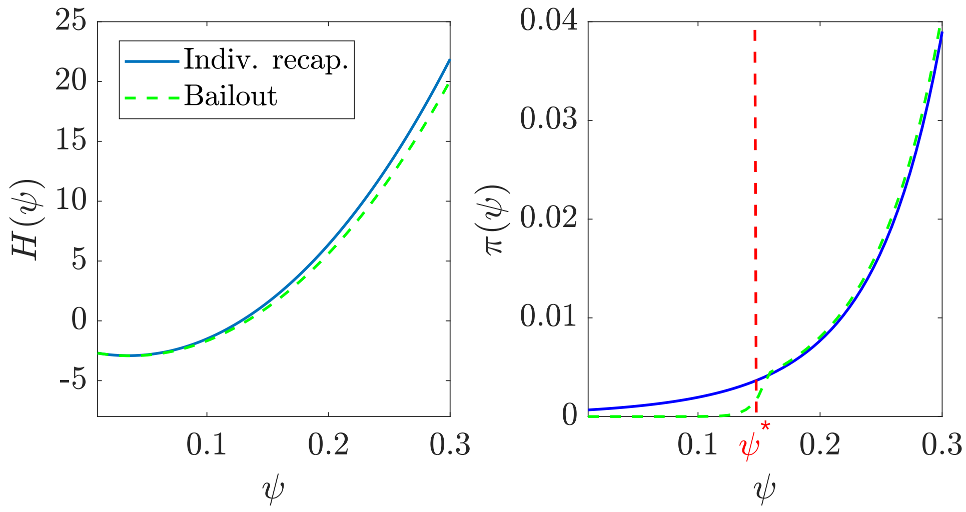

Figure 3.6: Long-run relative wealth share density (left) and aggregate welfare function (right) before (blue) and after (green) the bailout policy $\psi^* = 0.15$. In red, the bailout threshold ψ^*. Baseline parameters: $A = 0.1$, $\eta = 0.1$, $\sigma = 0.15$, $\theta = 4$, $\lambda = 0.1$, and $\rho = 0.05$.

the negative effect of reduced short-run welfare over the interval $\psi \in (\underline{\psi}, \bar{\psi})$ and the positive effect due to the shift of the long-run density $f(\psi)$. Therefore, there exists a long-run welfare maximizing policy level ψ^*.

To conclude, Table 3.1 reports the numerical computation of the maximizer of (3.48), as well as its sensitivity with respect to the models core parameters: the banking premium η and the recapitalization cost λ, respectively. Not surprisingly, a higher λ reduces the optimal bailout threshold; this is because the higher short-run cost dampens the long-run benefits, making the trade-off relatively less convenient. Accordingly, higher values of λ associate to a greater threshold $\bar{\psi}$. This is because the banks react to harsher recapitalization costs by increasing their capital buffer before they begin their dividend payouts. Less intuitively, a higher banking premium reduces the optimal bailout level ψ^* jointly with the level $\bar{\psi}$ above which the banks pay out dividends. This happens because an increasing banking premium makes the banks' management of capital relatively more valuable. As such, the volatility of the banks' market-to-book value is lower while its drift is higher. Therefore, a lower capital buffer is required to withstand aggregate fluctuations.

Due to our model extremely stylized nature, its quantitative implications are not to be taken by the book. Going beyond the figures, our results provide a few important takeaways concerning the mechanism that interlinks different banks' resolution policies, individual recapitalization versus bailouts, and their aggregate consequences. In this regard, we provide theoretical evidence that the

	η	λ	$\underline{\psi}$	$\bar{\psi}$	ψ^*
Baseline	0.1	0.1	0	0.35	0.092
Hihger recap. Cost	-	0.15	=	0.48	0.079
Higer bank premium	0.15	-	=	0.29	0.065

Table 3.1: Optimal policy threshold ψ^* as a function of the financial friction parameters: the banking premium and the recapitalization cost. Baseline parameters: $A = 0.1$, $\sigma = 0.15$, $\theta = 4$, and $\rho = 0.05$.

volume of banks' equity, and so its supply of financial services may be fundamental at determining equilibrium prices and investments. If follows that, as the marginal value of banks' capitalization relates to their dividends payout and recapitalization strategies, so it does to the price externality implicit in equilibrium outcomes.

Through the lenses of our model we show that, in a perfectly competitive environment with institutionally heterogeneous agents, here households and banks, economic actors may fail at internalizing the externality of aggregate capital allocation between classes over equilibrium outcomes. In such a framework, those choices that holds as optimal from the perspective of individual agents, may not be so from the social standpoint.

In out model, this happens because banks' and households take prices as given, and do not internalize the positive externality that a reallocation of resources to the financial sector would generate by fostering future growth. In such a framework, tax-based bailout policies may be welfare improving in the long-run, since the benefits of a well functioning financial sector more than compensate the short-run cost of those additional resources depleted for recapitalization purposes.

3.5 Concluding Remarks

In this paper, we study the mechanism through which banking resolution regimes, here individual optimal recapitalization and bailouts, affect households' short and long-run welfare in a dynamic general equilibrium model of a productive economy with financial frictions.

We show that, in equilibrium, banks' optimal strategies associate to their market-to-book value. This, in turn, relates to the upper (lower) capitalization thresholds above (below) which banks pay out dividends (issue new equity); in this theoretical framework, it is individually optimal for each bank to be recapitalized by its own shareholder. Nonetheless, from the social point of view, a bailout topping up individual recapitalization policy is shown to improve households' long-run welfare. This implies that there exists a trade-off between the costs of recapitalization (short-run) and the benefits of a well functioning banking sector (long-run).

In summary, this paper provides theoretical evidence that bail-out recapitalization regimes may be justified as a last resort, exclusively when financial stability is gravely threatened, and fine-tuned to mitigate the associated costs. This is because banks who act in a perfectly competitive environment do not internalize the positive externality of their aggregate capitalization over equilibrium prices dynamics and, in turn, on aggregate investments.

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Appendices

Appendix A

Chapter 1

A.1 Capital and Output: Sketch of the Solution

By standard stochastic control arguments, the households' HJB reads as (we omit the time down-script for sake of clear notation)

$$\rho H^{g,q} = \max_{\{C,\omega,\iota\}} \left\{ \frac{C^{1-\gamma}}{1-\gamma} + \frac{\partial H^{g,q}}{\partial t} + \frac{\partial H^{g,q}}{\partial E}\mu^e + \frac{\partial H^{g,q}}{\partial q} q\mu^q + \right.$$

$$\left. + \frac{1}{2}\frac{\partial^2 H^{g,q}}{\partial E^2}\left(\sigma^e\right)^2 + \frac{1}{2}\frac{\partial^2 H^{g,q}}{\partial q^2}\left(q\sigma^q\right)^2 + \frac{\partial^2 H^{g,q}}{\partial q \partial E} q\sigma^q\sigma^e \right\},$$

where μ^e and σ^e are drift and diffusion of the households' absolute wealth process

$$dE = \mu^e dt + \sigma^e dW.$$

By taking FOCs, considering an ansatz of the form $H^{g,q} = \kappa(q)^{-\gamma}\frac{K^{1-\gamma}}{1-\gamma}$, it follows that h_t satisfies the following non-linear ODE

$$\frac{\rho}{1-\gamma} + \gamma\epsilon_{h,q}\left(\mu_t^q + \omega_t\sigma_t\sigma_t^q\right) = h_t\left(\frac{\gamma}{1-\gamma}\right) + r_t + \omega_t\left(\mu_t - r_t\right) + \frac{1}{2}\left(\omega_t\sigma_t\right)^2 + h_{qq}\frac{\gamma(\gamma+1)}{2}\left(\frac{q_t\sigma_t^q}{h_t}\right), \tag{A.1}$$

with transversality $\mathbb{E}_t\left[e^{-\rho t}h_t E_t\right] \longrightarrow 0$. By considering the equilibrium where $q_t = q$ is constant, jointly with the market clearing conditions for

1. Physical capital (risky claims):

$$\omega_t E_t = K_t q_t;$$

2. Risk-free bond (zero-net supply):

$$(1-\omega_t)\,E_t = 0;$$

3. Consumption:

$$(A-\iota_t)\,K_t = C_t;$$

it is straightforward that (1.12)-(A.1) imply

$$\frac{C}{E} = \frac{\rho}{\gamma} - \frac{1-\gamma}{\gamma}\left[\frac{A-\iota}{q} + \ln q - \delta + \frac{1}{2}\sigma^2\right]; \quad \omega = 1; \quad \iota = \frac{q-1}{\theta},$$

where q solves the following non-linear equation

$$q = \frac{1+\theta A}{1+\theta\left[\frac{\rho}{\gamma} - \frac{1-\gamma}{\gamma}\left(\frac{1+\theta A-q}{\theta q} + \ln q - \delta + \frac{\sigma^2}{2}\right)\right]},$$

A.2 Idiosyncratic Risk and Financial Frictions: Sketch of the Solution

The household h HJB reads as follows (we omit the time down-script for sake of clear notation)

$$\rho H^p = \max_{\{C,\omega^p,\omega,\iota\}}\left\{\frac{c^{1-\gamma}}{1-\gamma} + \frac{\partial H^p}{\partial t} + \frac{\partial H^p}{\partial E}\mu^e + \frac{\partial H^p}{\partial q}q\mu^q + \right. \tag{A.2}$$

$$\left. + \frac{1}{2}\frac{\partial^2 H^p}{\partial E^2}(\sigma^e)^T(\sigma^e) + \frac{1}{2}\frac{\partial^2 H^p}{\partial q^2}(q\sigma^q)^2 + \frac{\partial^2 H^p}{\partial q \partial E}q\sigma^q(\sigma^e)^T\right\}.$$

μ^e and $(\sigma^e)^T$ are drift and diffusion of the households' absolute wealth dynamics

$$dE = \mu^e dt + (\sigma^e)^T d\mathbf{W},$$

where

$$\frac{\mu^e}{e} = \frac{1}{dt}\omega_t^{h,p}\mathbb{E}dR_t^h + \frac{1}{dt}\omega_t^h\mathbb{E}dR_t - c_t^h,$$

$$(\sigma^e)^T = \left[\ (\omega^p+\omega)\,\sigma_t; \quad \omega_t\tilde{\sigma}\ \right]; \quad d\mathbf{W} = \left[\begin{array}{c} dW \\ d\tilde{W} \end{array}\right].$$

By considering that, in equilibrium, households' will not keep their wealth in risk-free assets ($\omega_t^{h,p} + \omega_t^h = 1$), the problem can be re-written in term of excess returns from the pooled portfolio. Moreover, we look for those equilibria where the price of capital is fixed. Then, problem (A.2) reduces to

$$\rho H^p = \max_{\{C,\omega,\iota\}} \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \frac{\partial H^p}{\partial t} + \frac{\partial H^p}{\partial E}\mu^e + \frac{1}{2}\frac{\partial^2 H^p}{\partial E^2}\left(\sigma^e\right)^T\left(\sigma^e\right) \right\}$$

where ω denotes ω^h, the portfolio share invested into the un-pooled risky claim, and

$$\frac{\mu^e}{e} = \frac{A-\iota}{q} + \Phi(\iota^h) - \delta - \frac{c_t^h}{e_t^h} - \left(1-\omega^h\right)\frac{d\eta}{dt},$$

$$\frac{\left(\sigma^e\right)^T}{e} = \left[\begin{array}{cc} \sigma; & \omega\tilde{\sigma} \end{array}\right].$$

By consider a proper ansatz as $H^p := h^{-\gamma}\frac{e^{1-\gamma}}{1-\gamma}$, the FOCs read as

$$\frac{c}{e} = h; \quad \Phi_\iota(\iota^h) = \Phi_\iota(\iota) = \frac{1}{q}; \quad \omega = \frac{\eta\frac{1}{\gamma}}{\tilde{\sigma}^2},$$

while h satisfies

$$\rho\frac{1}{1-\gamma} = h\frac{\gamma}{1-\gamma} + \frac{A-\iota}{q} + \Phi(\iota) - \delta - (1-\omega)\,\eta - \frac{\gamma}{2}\left(\sigma^2 + \omega^2\tilde{\sigma}^2\right).$$

A.3 Leverage and Restricted Market Participation

Equilibrium Since the expert holds the whole capital stock of the economy, the market clearing condition for risky claims (capital) implies that $\omega_t^e E_t^e + (1-\omega_t^e)E_t^e = K_t q_t \Longrightarrow K_t q_t = E_t^e$. In the same fashion, aggregate consumption is so that $(A-\iota_t)\,K_t = C_t = \rho(E_t^e + E_t^l)$. Thus,

$$A = \rho q_t + \iota_t,$$

and, given the FOC on ι, $1+\theta\iota = q$, we have that

$$q = \frac{1+\theta A}{1+\theta\rho}; \quad \iota = \frac{q-1}{\theta}.$$

Accordingly, $\mu^q = \sigma^q = 0$.

State dynamics By substituting the optimal policies $\frac{E_t^e}{E_t^e} = \rho$ and $\omega_t^e = \frac{1}{\psi_t}$ into the experts' dynamic budget constraint (1.46), we write her wealth dynamics as (for sake of clear notation, we omit the upper-script e)

$$dE_t = r_t dt + \frac{1}{\psi_t}\left(\mu_t - r_t\right)dt - \rho + \frac{1}{\psi_t}\sigma dW_t.$$

In the same fashion, the dynamics of aggregate wealth reads as

$$\frac{d(q_t K_t)}{q_t K_t} = \left(r_t + \frac{\sigma^2}{\psi_t} - \rho \right) dt + \sigma dW_t.$$

By Itô's Lemma

$$\begin{aligned} d\psi_t = \frac{\partial \psi_t}{\partial E_t} dE_t + \frac{\partial \psi_t}{\partial (K_t q_t)} d(q_t K_t) + \\ + \frac{1}{2} \frac{\partial^2 \psi_t}{\partial (K_t q_t)^2} (d(q_t K_t))^2 + \frac{1}{2} \frac{\partial^2 \psi_t}{\partial (E_t)^2} (dE_t)^2 + \\ + \frac{\partial^2 \psi_t}{\partial (K_t q_t) \partial (E_t)} dE_t d(q_t K_t) \end{aligned}$$

from which, rearranging

$$\frac{d\psi_t}{\psi_t} = \frac{(1-\psi_t)^2}{\psi_t^2} \sigma^2 dt + \frac{(1-\psi_t)}{\psi_t} \sigma dW_t. \tag{A.3}$$

A.4 Persistent Heterogeneity: Price volatility

By Itô's Lemma, it is possible to link the dynamics of capital prices q to the dynamics of the state ψ:

$$dq_t = \underbrace{\left[\frac{\partial q}{\partial \psi} \psi_t \mu_t^{\psi} + \frac{1}{2} \frac{\partial^2 q}{\partial \psi^2} \left(\psi \sigma_t^{\psi} \right)^2 \right]}_{q_t \mu_t^q} dt + \underbrace{\psi \sigma_t^{\psi} \frac{\partial q}{\partial \psi}}_{q_t \sigma_t^q} dW_t. \tag{A.4}$$

By the market clearing condition for consumption and capital it follows that

$$\frac{A - \iota_t}{q_t} = \rho^h \left(1 - \psi_t\right),$$

so that

$$q_t = \frac{1 + \theta A}{1 + \theta \rho^h \left(1 - \psi_t\right)}, \tag{A.5}$$

and

$$\frac{\partial q}{\partial \psi} = \theta \rho^h \frac{1 + \theta A}{\left[1 + \theta \rho^h \left(1 - \psi_t\right)\right]^2}. \tag{A.6}$$

Finally, by matching (A.4), (A.5), and (A.6)

$$\sigma_t^q = \psi \sigma_t^{\psi} \frac{\theta \rho^h}{1 + \theta \rho^h \left(1 - \psi_t\right)}.$$

A.5 Money as Risk-free Assets: State Dynamics

In equilibrium, we know that the aggregate wealth of experts' and intermediaries evolve as

$$\frac{dE_t^e}{E_t^e} = \mu_t^m dt + \omega_t^e \left(\mu_t^k - \mu_t^m\right) - \rho dt + \sigma dW_t,$$

$$\frac{dE_t^i}{E_t^i} = \left(\mu_t^m - \rho\right) dt + \sigma dW_t.$$

By Itô's Lemma, the state has dynamics

$$\frac{d\psi_t}{\psi_t} = \frac{d\left(\frac{E_t^e}{E_t^e+E_t^i}\right)}{\left(\frac{E_t^e}{E_t^e+E_t^i}\right)} = \left(1-\psi_t\right)\omega_t^e\left(\mu_t^k - \mu_t^m\right) dt +$$
$$- \left(1-\psi_t\right)^2 \left[\sigma^2 + \left(\omega_t^e \tilde{\sigma}\right)^2\right] dt + \left(1-\psi_t\right)^2 \sigma^2 dt + \left(1-\psi_t\right)\left(2\psi_t - 1\right)\sigma^2,$$

and, by considering $\omega_t^e = \frac{\mu_t^k - \mu_t^m}{\tilde{\sigma}^2}$ and rearranging, (1.62) holds. The experts' portfolio share (1.64) can be found by normalizing the market clearing condition for capital stock over the overall wealth within the economy, so that

$$\frac{E_t \omega_t^e}{K_t(p_t+q_t)} = \frac{K_t q_t}{K_t(p_t+q_t)} \Longrightarrow \psi_t \omega_t^e = \frac{q_t}{p_t+q_t}.$$

A.6 Aggregate Bank Capital and Credit Dynamics

The banks' problem The banks' HJB equation holds as (we omit up and down-script for sake of clear notation)

$$\rho B dt = \max_{\{d\delta, d\pi, \omega\}} \left\{ d\delta - (1+\lambda) d\pi + \mathbb{E} dB \right\}.$$

By considering an ansatz for the bank' value of the form $B := eb(E)$, it follows that

$$dB = eb\frac{de}{e} + eb_E dE + \frac{1}{2} eb_{EE} dE^2.$$

Thus, given the dynamic budget constraint (1.74), it is easy to see that the problem is separable in all controls,

$$\rho b = \frac{1}{dt}\max_{\{d\delta\}}\{d\delta - bd\delta\} + \frac{1}{dt}\max_{\{d\pi\}}\{bd\pi - (1+\lambda)d\pi\} + \\ + \max_{\{\omega\}} b\left[r + \omega\left(q - p - r\right) + \frac{b_E}{b}\omega\sigma E\sigma^E\right] + b_E E\mu^E + \frac{1}{2}b_{EE}\left(E^2\sigma^E\right)^2.$$

By taking FOCs

$$d\delta \geq 0 \iff b \leq 1,$$

$$d\pi \geq 0 \iff b \geq 1 + \lambda,$$

and

$$q = p + r + \Omega_t \frac{E}{b} b_E \sigma^2.$$

A.7 I Theory of Money

A.7.1 Productivity of Capital

The marginal contribution of ϵ additional input to technology a equals

$$(K_t + \epsilon)A\left[\left(\frac{K_t\phi + \epsilon}{K_t + \epsilon}\right)^r + \left(\frac{K_t\left(1-\phi\right)}{K_t + \epsilon}\right)^r\right]^{\frac{1}{r}} = Y\left(\frac{K_t\phi + \epsilon}{K_t + \epsilon}\right)(K_t + \epsilon).$$

Taking the limit of the derivative with respect to ϵ for $\epsilon \to 0$ we have

$$\lim_{\epsilon \to 0} \frac{\partial\left[Y\left(\frac{K_t\phi+\epsilon}{K_t+\epsilon}\right)(K_t+\epsilon)\right]}{\partial \epsilon}$$

$$\frac{1}{r}(K_t+\epsilon)A\left[\left(\frac{K_t\phi+\epsilon}{K_t+\epsilon}\right)^r + \left(\frac{K_t\left(1-\phi\right)}{K_t+\epsilon}\right)^r\right]^{\frac{1}{r}-1} r\times \\ \times\left[\left(\frac{K_t\phi+\epsilon}{K_t+\epsilon}\right)^{r-1}\frac{-(K_t\phi+\epsilon)+(K_t+\epsilon)}{(K_t+\epsilon)^2} + \frac{-K_t\left(1-\phi\right)}{(K_t+\epsilon)^2}\left(\frac{K_t\left(1-\phi\right)}{K_t+\epsilon}\right)^{r-1}\right] + \\ + A\left[\left(\frac{K_t\phi+\epsilon}{K_t+\epsilon}\right)^r + \left(\frac{K_t\left(1-\phi\right)}{K_t+\epsilon}\right)^r\right]^{\frac{1}{r}}$$

$$A\left[(\phi)^r + (1-\phi)^r\right]^{\frac{1}{r}-1} \times K_t\left[(\phi)^{r-1}\frac{(1-\phi)}{K_t} + \frac{-(1-\phi)}{K_t}(1-\phi)^{r-1}\right] + Y(\phi)$$

$$A\left[(\phi)^r + (1-\phi)^r\right]^{\frac{1}{r}-1} \times (1-\phi)\left[(\phi)^{r-1} - (1-\phi)^{r-1}\right] + Y(\phi),$$

from which

$$Y^a(\phi) = Y(\phi) + (1-\phi)\,Y'(\phi).$$

Similarly, for good b

$$Y^b(\phi) = Y(\phi) - \phi Y'(\phi).$$

A.7.2 Risk Premiums

By subtracting (1.115) from (1.116) and rearranging, the inside equity premium on money asset equals

$$\mu_t^b - \mu_t^m = \chi_t \left(\xi_t^b \sigma_t^b + \tilde{\xi}_t{}^b \tilde{\sigma}^b - \xi_t \sigma_t^b \right).$$

In the same fashion, the outside equity premium on money holds as

$$\mu_t^I - \mu_t^m = \xi_t^I \sigma_t^b - \xi_t^m \sigma_t^m. \tag{A.7}$$

Note that, as the systematic source of risk to which the intermediary is exposed holds unique, according to (A.7) it must be that $\xi_t^I = \xi_t^m$. By log utility, we also know that the price of each risk source is proportional to the agents' exposure to it. By substituting (1.110) and (1.110) in the dynamics budget constraints (1.106) we have that, for $j \in \{a, b\}$

$$\frac{de_t^j}{e_t^j} = (...)dt + \left[\omega_t^j (\sigma_t - \sigma_t^m) + \sigma_t^m \right] dW_t + \omega_t^j \tilde{\sigma}^b d\tilde{W}_t^j,$$

$$\frac{de_t^I}{e_t^I} = (...)dt + \left[\omega_t^I (\sigma_t - \sigma_t^m) + \sigma_t^m \right] dW_t.$$

Therefore,

$$\mu_t^a - \mu_t^m = \left[\omega_t^a (\sigma_t^q + \sigma^a - \sigma_t^m) + \sigma_t^m \right] (\sigma_t^q + \sigma^a - \sigma_t^m) + \omega_t^a \left(\tilde{\sigma_t^a} \right)^2, \tag{A.8}$$

and

$$\mu_t^I - \mu_t^m = \left[\omega_t^I \left(\sigma_t^b - \sigma_t^m \right) + \sigma_t^m \right] \left(\sigma_t^b - \sigma_t^m \right), \tag{A.9}$$

while

$$\mu_t^b - \mu_t^I = \chi_t \left[\left(\omega_t^b - \omega_t^I \right) \left(\sigma_t^b - \sigma_t^m \right) \sigma_t^b + \tilde{\xi}_t{}^b \tilde{\sigma}^b \right].$$

A.7.3 State Dynamics

The intermediary's wealth share has dynamics (we omit unnecessary upscripts for sake of clear notation)

$$\frac{dE_t}{E_t} = dR_t^m - \rho dt + \omega_t \left[dR_t^I - dR_t^m \right],$$

where

$$dR_t^I - dR_t^m = \underbrace{\left(\frac{Y^b(\phi) - \iota_t}{q_t} + \mu_t^q - \mu_t^p + \sigma_t^q \sigma^a - \sigma_t^p \sigma_t^K\right)}_{\mu_t^{I-m}} dt + \underbrace{\left(\sigma_t^q + \sigma^b - \sigma_t^M\right)}_{\upsilon_t^b} dW_t$$

and, by (A.9),

$$\frac{dE_t}{E_t} = dR_t^m - \rho dt + \omega_t \left(\omega_t \upsilon_t + \sigma_t^m\right) \upsilon_t dt + \omega_t \upsilon_t^b dW_t.$$

Likewise, the dynamics of aggregate total wealth $K_t(p_t + q_t)$ equals

$$\frac{d\left(K_t\left(p_t + q_t\right)\right)}{K_t\left(p_t + q_t\right)} = dR_t^m - \rho dt +$$
$$+\phi_t\left(1-\pi_t\right)\left(\chi_t\left[\left(\omega_t^b \upsilon_t^b + \sigma_t^m\right)\upsilon_t + \omega_t^b\left(\tilde{\sigma_t^b}\right)^2\right] + \left(1-\chi_t\right)\left[\left(\omega_t^I \upsilon_t^b + \sigma_t^m\right)\upsilon_t\right]\right) dt +$$
$$+\left(1-\phi_t\right)\left(1-\pi_t\right)\left(\left(\omega_t^b \upsilon_t^b + \sigma_t^m\right)\upsilon_t + \omega_t^b\left(\tilde{\sigma_t^b}\right)^2\right) dt +$$
$$+\phi_t\left(1-\pi_t\right)\left(\sigma_t^q + \sigma^b - \sigma_t^M\right) dW_t + \left(1-\phi_t\right)\left(1-\pi_t\right)\left(\sigma_t^q + \sigma^b - \sigma_t^m\right) dW_t,$$

whose diffusion term can be arranged as

$$\left(1-\pi_t\right)\left(\sigma_t^q + \underbrace{\phi\sigma^b + \left(1-\phi_t\right)\sigma^a}_{\sigma_t^K} - \underbrace{\sigma_t^m}_{\sigma_t^p + \sigma_t^K}\right) dW_t = \left(1-\pi_t\right)\left(\sigma_t^q - \sigma_t^p\right) dW_t.$$

Therefore, the overall dynamics equals

$$\frac{d\left(K_t\left(p_t + q_t\right)\right)}{K_t\left(p_t + q_t\right)} = \left(1-\pi_t\right)\sigma_t^m \underbrace{\left[\phi \upsilon_t^a + (1-\phi)\upsilon_t^b\right]}_{\left(\sigma_t^q - \sigma_t^p\right)} dt + \left(1-\pi_t\right)\left(\sigma_t^q - \sigma_t^p\right) dW_t +$$
$$+\underbrace{\left(1-\pi_t\right)\left[\phi_t \chi_t \omega_t^a\left(\upsilon_t^2 + \left(\tilde{\sigma_t^b}\right)^2\right) + (1-\phi)\omega_t^b\left(\left(\upsilon_t^a\right)^2 + \left(\tilde{\sigma_t^b}\right)^2\right)\right]}_{\mathrm{X}} dt +$$
$$+ dR_t^m - \rho dt + \underbrace{\left(1-\pi_t\right)\phi_t\left(1-\chi_t\right)}_{\omega_t^I \psi_t} \omega_t \upsilon_t^2 dt. \quad \text{(A.10)}$$

The X term of (A.10) cane be written as

$$\left(1-\pi_t\right)\left[\phi_t \chi_t \frac{\left(\omega_t^b\right)^2}{\omega_t^a}\left(\upsilon_t^2 + \left(\tilde{\sigma_t^b}\right)^2\right) + (1-\phi)\omega_t^b\left(\left(\upsilon_t^b\right)^2 + \left(\tilde{\sigma_t^b}\right)^2\right)\right] dt,$$

$$(1-\pi_t)\frac{\omega_t^b}{\omega_t^a}\left(\left(\upsilon_t^b\right)^2+\left(\tilde{\sigma_t^b}\right)^2\right)\underbrace{\left[\phi_t\chi_t\omega_t^b+(1-\phi)\omega_t^a\right]}_{\frac{(1-\psi_t)\omega_t^a\omega_t^b}{(1-\pi_t)}}dt, \tag{A.11}$$

and, by substituting (A.11) into (A.10) and rearranging, we have

$$\frac{d\left(K_t\left(p_t+q_t\right)\right)}{K_t\left(p_t+q_t\right)}=r_t^M-\rho dt+\left(\sigma_t^q-\sigma_t^p\right)\left(\sigma_t^M dt+dW_t\right)+$$
$$+\left(1-\eta_t\right)\left(\omega_t^b\right)^2\left(\left(\upsilon_t^b\right)^2+\left(\tilde{\sigma_t^b}\right)^2\right)dt+\eta_t\left(\omega_t\upsilon_t\right)^2dt.$$

Finally, by Itô's Lemma,

$$d\psi_t=\frac{1}{K_t\left(p_t+q_t\right)}dE_t-\frac{E_t}{\left[K_t\left(p_t+q_t\right)\right]^2}d\left(K_t\left(p_t+q_t\right)\right)-$$
$$+\frac{1}{\left[K_t\left(p_t+q_t\right)\right]^2}d\left(K_t\left(p_t+q_t\right)\right)dE_t+\frac{E_t}{\left[K_t\left(p_t+q_t\right)\right]^3}d\left(K_t\left(p_t+q_t\right)\right)^2,$$

$$d\psi_t=\psi_t\left[dR_t^m-\rho dt+\left(\omega_t^I\upsilon_t^b\right)^2dt+\omega_t^I\upsilon_t^b dW_t\right]-$$
$$+\psi_t\left[dR_t^m-\rho dt+\sigma_t^\pi dW_t+(1-\eta_t)\left(\omega_t^b\right)^2\left(\left(\upsilon_t^b\right)^2+\left(\tilde{\sigma_t^b}\right)^2\right)dt+\psi_t\left(\omega_t^I\upsilon_t^b\right)^2dt\right]-$$
$$+\psi_t\left[(...)dt+\omega_t\upsilon_t^b dW_t\right]\left[(...)dt+\sigma_t^\pi dW_t\right]+\psi_t\left(\sigma_t^\pi\right)^2dt,$$

and, rearranging

$$\frac{d\psi_t}{\psi_t}=(1-\psi_t)\left[\left(\omega_t^I\upsilon_t^b\right)^2-\left(\omega_t^b\right)^2\left(\left(\upsilon_t^b\right)^2+\left(\tilde{\sigma_t^b}\right)^2\right)\right]dt+\left(\omega_t^I\upsilon_t+\sigma_t^\pi\right)\left(dW_t+\sigma_t^\pi dt\right).$$

where $\sigma_t^\pi=(1-\pi_t)(\sigma_t^q-\sigma_t^p)=\frac{q_t}{p_t+q_t}(\sigma_t^q-\sigma_t^p)$.

A.7.4 Amplification

By Itô's Lemma,

$$\frac{d\pi_t}{\pi_t}=(...)dt+\underbrace{\left(\frac{q_t}{p_t+q_t}\right)\left(\sigma_t^q-\sigma_t^p\right)}_{\sigma_t^\pi}dW_t,$$

and

$$d\pi_t=\left[\frac{\pi_\psi}{\pi_t}\psi_t\mu_t^\psi+\frac{1}{2}\frac{\pi_{\psi\psi}}{\pi_t}\left(\psi_t\sigma_t^\psi\right)^2\right]dt+\frac{\pi_\psi}{\pi_t}\psi_t\sigma_t^\psi dW_t.$$

Therefore, by (1.120),

$$\sigma_t^\psi=\frac{\omega_t^I\left(\sigma^b-\sigma_t^K\right)}{1+\frac{\pi_\psi}{\pi_t}\left(\frac{\omega_t}{1-\pi_t}-1\right)}.$$

Appendix B

Chapter 2

B.1 Micro-foundation

The micro-foundation structure proposed in this section is the continuous-time equivalent of the one proposed in Ljungqvist and Sargent (2012), Chapter 12.

Output producing firms There exists a continuum of unitary mass of output producing firms (henceforth, type II). Those firms produce output at a rate A. At each instant of time t, the i^{th} productive firm chooses the physical capital k_t^i in order to solve a static problem

$$\max_{k_t^i \geq 0} \left\{ y_t^i - p_t k_t^i \right\},$$

s.t.

$$y_t^i \leq A k_t^i, \tag{B.1}$$

where p_t is the rental rate of physical capital. Given linearity, the above has an interior solution only when the following zero-profit condition is satisfied:

$$p_t = A. \tag{B.2}$$

If (B.2) holds, the size of the i^{th} firm is indeterminate, and it is willing to supply any market demand.

Capital producing firms There exists a continuum of unitary mass of capital producing firms (henceforth, type I). Those firms transform output into capital, store capital, and earn revenues by renting capital to type II firms at the equilibrium rate $p_t = A$. At each instant of time t, the i^{th} productive firm chooses how much value of capital $k_t^i q_t$ to store in order to earn stochastic returns dR_t^i

per unitary capital, and how much numéraire $\iota_t^i k_t^i$ to purchase to generate new capital $\Phi(\iota_t^i)k_t^i$. Firm i finances itself by issuing state-contingent debt to the agent who supplies the capital stock. Thus, between t and s, the i^{th} firm solves the following problem

$$\max_{\{k_t^i,\iota_t^i\}} \left\{ \mathbb{E}_t^{\mathbb{Q}^i} \underbrace{\left[v_s e^{-\int_t^s r_s du} \right]}_{\text{Discounted "Net" Revenues}} - \underbrace{k_t^i q_t}_{\text{Cost of Capital}} \right\},$$

s.t.

$$T^i: \quad \frac{d\left(k_t^i q_t\right)}{k_t^i q_t} = \left(\Phi(\iota_t^i) - \delta + \mu_t^q - \sigma_t^q \sigma\right) dt + \left(\sigma - \sigma_t^q\right) dW_t + \tilde{\sigma} d\tilde{W}_t^i, \qquad \text{(B.3)}$$

where $\mathbb{Q}^i$ is the risk neutral measure. The revenues v_s are "net" the cost of purchasing the "input", which in returns equals $e^{-\int_t^s \frac{\iota_u^i}{q_u} du}$ for unit of capital. By Equation (B.3), we know that

$$v_s = \underbrace{k_t^i q_t e^{\int_t^s \left(\Phi(\iota_u^i) - \delta + \mu_u^q - \sigma_u^q \sigma\right) du - \frac{1}{2}\|\Sigma_t^i\|^2 du + \int_t^s \Sigma_t^i d\mathbf{W}_t}}_{k_s q_s} e^{\int_t^s \frac{p_u - \iota_u^i}{q_u} du},$$

where $\Sigma_t^i = \begin{bmatrix} \sigma_t & [\mathbb{I}_{i=p}]\,\tilde{\sigma} \end{bmatrix}$ and $d\mathbf{W}_t = \begin{bmatrix} dW_t \\ d\tilde{W}_t \end{bmatrix}$. The FOC on ι_t^i requires that

$$\Phi'(\iota_u^i) = \frac{1}{q_u}, \quad \forall u \in (t, s).$$

By Type II firms optimality condition in (B.2), the FOC on k_t^i implies the zero-profit condition

$$\mathbb{E}_t^{\mathbb{Q}^i} \left[e^{\int_t^s \left(\mu_u - \frac{1}{2}\|\Sigma_t^i\|^2 - r_u\right) du + \int_t^s \Sigma_u^i d\mathbf{W}_u} \right] = 1. \quad \forall i, \qquad \text{(B.4)}$$

Note that the zero profit condition is consistent with the equilibrium return on the i^{th} risky claim dR_t^i where

$$\mu_t := \frac{\mathbb{E}_t\left[dR_t^i\right]}{dt} = \frac{A - \iota_t^i}{q_t} + \Phi(\iota_t^i) + \mu_t^q - \delta - \sigma_t^q \sigma,$$

$$\| \Sigma_t \|^2 = (\sigma - \sigma_t^q)^2 + \tilde{\sigma}^2 = \frac{\mathbb{V}ar_t\left[dR_t^i\right]}{dt} \Longrightarrow \sigma_t := \sigma - \sigma_t^q.$$

Condition (B.4) is equivalent to a non-arbitrage condition so that the return on risky claims issued by type I firms (equity) is such that their present discounted value equals the current value of physical capital stock $k_t^i q_t$ supplied by the agents. If such condition holds, the firm breaks even for each k_t^i, its size is indeterminate, and it is willing to supply each market demand.

To grant the existence (and uniqueness) of the competitive equilibrium, condition (B.4) must be consistent with the no-arbitrage condition for the aggregate portfolio held by the financial sector. The result is summarised in the following proposition:

Proposition 5. ***Risk Neutral Measure***
Given the zero-profit condition in (B.4) and the no arbitrage condition for the aggregate portfolio, the market price of systematic risk equals $\xi_t = \frac{\mu_t^f - r_t}{\sigma_t}$. *The latter implies that there exists a unique* $\mathbb{Q}^i$ *such that the price kernel is well defined,*[1] *and the price of idiosyncratic risk* $\tilde{\xi}_t$ *satisfies*

$$\tilde{\xi}_t = \frac{\mu_t^h - \mu_t^f}{\tilde{\sigma}} \geq 0 \Longleftrightarrow \eta \geq 0.$$

Proof. Given the zero-profit condition in (B.4), by *Girsanov Theorem III* (see **?**), the correspondent Radon-Nykodym derivative equals

$$\frac{d\mathbb{Q}^i}{d\mathbb{P}} = \exp\left\{-\int_t^s \xi_u dW_u - \int_t^s \tilde{\xi}_u d\tilde{W}_u - \frac{1}{2}\int_t^s \left(\xi_u^2 + \tilde{\xi}_u^2\right) du\right\}.$$

where $\mathbb{P}$ is the real probability measure, while ξ_t and $\tilde{\xi}_t$ represent the market prices of systematic and idiosyncratic risk respectively. Given the no-arbitrage condition for the aggregate portfolio:

$$\mathbb{E}_t^{\mathbb{Q}^f}\left[e^{\int_t^s \left(\mu_u^f - \frac{1}{2}\sigma_u^2 - r_u\right)du + \int_t^s \sigma_u^2 dW_u}\right] = 1,$$

it follows that

$$\frac{d\mathbb{Q}^f}{d\mathbb{P}} = \exp\left\{-\int_t^s \xi_t du - \frac{1}{2}\int_t^s \xi_t^2 dW_u\right\} \Longleftrightarrow \xi_t = \frac{\mu_t^f - r_t}{\sigma_t}.$$

The latter implies that the martingale measure for the $i^t h$ firm $\mathbb{Q}^i$ satisfies

$$d\mathbf{W}_t^{\mathbb{Q}^i} = \begin{bmatrix} \xi_t \\ \tilde{\xi}_t \end{bmatrix} dt + d\mathbf{W}_t.$$

where $\tilde{\xi}_t = \frac{\mu_t^h - \mu_t^f}{\tilde{\sigma}} = \frac{1}{\tilde{\sigma}}\frac{\eta}{q_t}$ and, thus

$$k_s^i q_s e^{-\int_t^s \left(r_u - \frac{A - \iota_u}{q_u}\right)du} = k_t^i q_t e^{-\int_t^t \left(r_u - \frac{A - \iota_u}{q_u}\right)du} + \int_t^s \Sigma_t' d\mathbf{W}_t^{\mathbb{Q}^i}.$$

[1] When the intermediation costs are null $\eta = 0$, it follows that $\tilde{\xi}_t = 0$ and, in turn, $\mathbb{Q}^i = \mathbb{Q}^f$. This case is consistent with the benchmark where markets are complete.

By taking the expected value under the probability measure $\mathbb{Q}^i$, it follows that

$$\mathbb{E}_t^{\mathbb{Q}^i}\left[k_s^i q_s e^{-\int_t^s\left(r_u-\frac{A-\iota_u}{q_u}\right)du}\right] = k_t^i q_t + \underbrace{\mathbb{E}_t^{\mathbb{Q}^i}\left[\int_t^s \Sigma_t d\mathbf{W}_t^{\mathbb{Q}^i}\right]}_{0},$$

is a martingale under $\mathbb{Q}^i$. □

B.1.1 Restricted Participation and Transaction Costs

In this appendix we consider the generalisation of the competitive equilibrium in Section 2.3 where both classes of agents, households and financial intermediaries respectively, have full access to risk-free bonds and pooled (p) as well as un-pooled (n) risky claims. In particular we assume that, in order to pool risky claims from different firms, households have to pay a transaction cost ε. We show that *restricted market participation* arises naturally when the transaction cost is big enough with respect to the financial intermediation cost η.

Given problem (2.3), the optimal pooled and un-pooled portfolio choices of both classes of agents satisfy the following:

$$\omega_t^{i,n} = \frac{\mu_t - r_t}{\sigma_t^2 + \tilde{\sigma}^2}, \quad i := \{h, f\}; \tag{B.5}$$

$$\omega_t^{f,p} = \frac{\mu_t - \frac{\eta}{q_t} - r_t}{\sigma_t^2}, \quad \omega_t^{h,p} = \frac{\mu_t - \frac{\varepsilon}{q_t} - r_t}{\sigma_t^2}. \tag{B.6}$$

In equilibrium, the whole amount of wealth invested in risky claims, whether it is pooled or not, must equal the aggregate amount of physical capital, whereas the risk-free bonds must be in zero net supply. By market clearing conditions

$$\left(\omega_t^{h,n} + \omega_t^{h,p}\right)(1-\psi_t) + \left(\omega_t^{f,p} + \omega_t^{f,n}\right)\psi_t = 1, \tag{B.7}$$

$$\left(1 - \omega_t^{h,n} - \omega_t^{h,p}\right)(1-\psi_t) + \left(1 - \omega_t^{f,p} - \omega_t^{f,n}\right)\psi_t = 0. \tag{B.8}$$

By matching equations (B.5) and (B.6), the market clearing conditions (B.7) and (B.8), it follows that

$$\omega_t^f = \omega_t^{f,n} + \omega_t^{f,p} = 1 + \frac{\varepsilon - \eta}{q_t}\frac{1 + \psi_t \frac{\tilde{\sigma}^2}{\sigma_t^2}}{2\sigma_t^2 + \tilde{\sigma}^2}, \tag{B.9}$$

and

$$\omega_t^h = \omega_t^{h,p} + \omega_t^{h,n} = 1 - \frac{\frac{\varepsilon-\eta}{q_t}}{\sigma_t^2}\psi_t. \tag{B.10}$$

We now look for those parametric conditions such that there exists *restricted* *market participation*, i.e. the financial sector always leverages its balance sheet by issuing risk-free bonds in every state. On the contrary, households smooth consumption by allocating their wealth into both risky and risk-free claims in positive amounts whatever share of total wealth. The aforementioned conditions are satisfies if the following holds:

$$\begin{cases} \omega_t^f = \omega_t^{f,p} + \omega_t^{f,n} > 1 \\ \omega_t^h = \omega_t^{h,p} + \omega_t^{h,n} > 0, \\ \omega_t^h = \omega_t^{h,p} + \omega_t^{h,n} < 1. \end{cases} \tag{B.11}$$

By matching equations (B.9) and (B.10) with system (B.11), we find that the following conditions must hold

$$\varepsilon > \eta \Rightarrow \omega_t^h < 1, \omega_t^f > 1,$$

while

$$\varepsilon < \eta + \sigma_t^2 \frac{q_t}{\psi_t} \Rightarrow \omega_t^h > 0.$$

Is summary, the transition cost for h/entrepreneurs ε is required to be bounded:

$$\eta < \varepsilon < \eta + \min_{\psi_t} \left\{ \sigma_t^2(\psi_t) \frac{q_t(\psi_t)}{\psi_t} \right\}.$$

The lower bound grants a comparative advantage to the financial sector at pooling risk, whereas the upper bound prevents the households to short the un-pooled security in equilibrium.

B.2 Competitive Equilibrium: Definition

The formal definition of the competitive equilibrium reads as follows:

Definition 2. ***Competitive Equilibrium: Definition***

Conditional on an initial allocation of capital among the agents, an equilibrium is an adapted stochastic process that maps histories of systematic shocks $\{dW_t\}$ *to prices* $\{q_t\}$*, returns on risky claims* $\left\{dR_t^h, dR_t^f; h \in \mathbb{H}\right\}$*, risk free rates* $\{r_t\}$*, production choices* $\left\{k_t^i, \iota_t^i; i \in [0,1]\right\}$*, consumption choices* $\left\{C_t^h, C_t^f; h \in \mathbb{H}\right\}$*, and asset allocations* $\left\{\omega_t^h, \omega_t^f; h \in \mathbb{H}\right\}$ *such that:*

1. *Firms maximise their profits:*

(a) *Firms of type I*

$$\left\{k_t^i, \iota_t^i\right\} \in \arg \max_{\left\{k_t^i, \iota_t^i\right\} \in T^i} \left\{\mathbb{E}_t^{\mathbb{Q}^i}\left[k_s^i q_s e^{\int_t^s \frac{p_u - \iota_u^i}{q_u} - r_u du}\right] - k_t^i q_t\right\}, \forall i \in [0,1]; \tag{B.12}$$

(b) *Firms of type II*

$$k_t^i \in \arg \max_{k_t^i \geq 0} \left\{(A - p_t)\, k_t^i\right\}, \forall i \in [0,1]. \tag{B.13}$$

2. *Agents maximise their utility:*

$$\left\{c_t^i, \omega_t^i\right\} \in \arg \max_{\left\{c_t^i, \omega_t^i\right\} \in B^i} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \ln c_t^i dt\right], \forall i \in \mathbb{H} \cup \mathbb{F}.$$

3. *All markets clear:*

(a) Risky asset

$$\int_{\mathbb{F}} \omega_t^f e_t^f df + \int_{\mathbb{H}} \omega_t^h e_t^h dh = K_t q_t; \tag{B.14}$$

(b) Bond

$$\int_{\mathbb{F}} (1 - \omega_t^f) e_t^f df + \int_{\mathbb{H}} \left(1 - \omega_t^h\right) e_t^h dh = 0; \tag{B.15}$$

(c) Consumption

$$\int_{\mathbb{F}} \left(A - \iota_t^f - \eta\right) k_t^f df + \int_{\mathbb{H}} \left(A - \iota_t^h\right) k_t^h dh = C_t^f + C_t^h; \tag{B.16}$$

(d) Capital

$$\int_{\mathbb{F}} k_t^f df + \int_{\mathbb{H}} k_t^h dh = K_t. \tag{B.17}$$

B.3 Asset Pricing

To study how financial leverage relates to asset pricing in our theoretical framework, Figure B.1 plots the financial sector risky assets expected returns $\frac{1}{dt}\mathbb{E}\left[dR^f\right] = \mu^f$ (bottom, left) and volatility $\frac{1}{dt}\sqrt{\mathbb{V}ar\left[dR^f\right]} = \sigma^f$ (bottom, right) as a function of ω^f. In the same Figure (top, right) we plot the Sharpe ratios of the financial sector ξ^f (blue, solid) and of the households' ξ^h (blue, dashed) as a function of ω^f. What stands out is that Sharpe ratios are increasing with financial leverage. Accordingly to what we discussed in Section 2.4.1, that there exists a negative relationship between the financial sector relative wealth share and its leverage, since financial leverage is counter-cyclical, so is the corresponding Sharpe ratio. This is because, as long as the financial sector is free to adjust

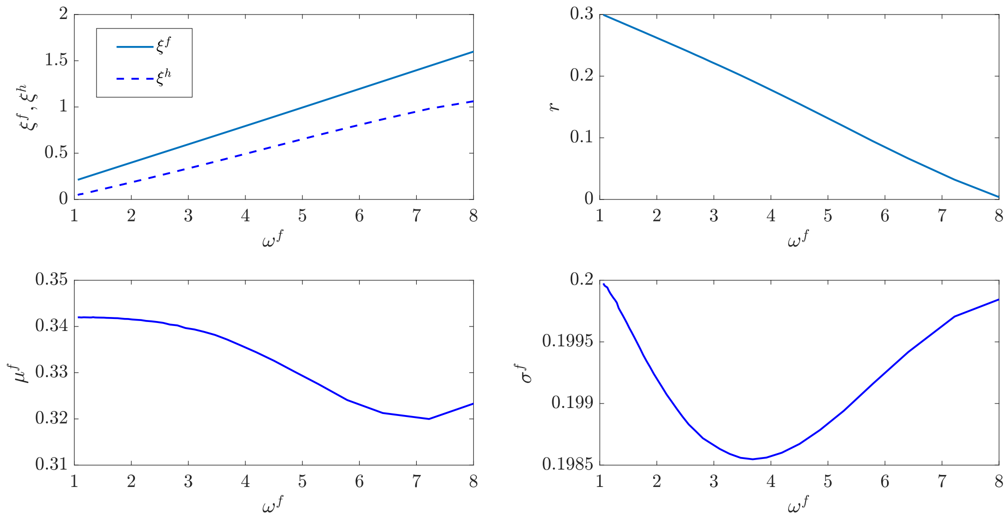

Figure B.1: Top left: Sharpe ratios (left) financial sector ξ^f (blue, solid) and of the households' ξ^h (blue, dashed). Top right: Risk-free interest rate (right) as a function of financial leverage ω^f. Bottom: Financial risky assets expected return μ^f (left) and diffusion σ^f (right) as a function of financial leverage ω^f. Baseline parameters: $A = 0.5$, $\delta = 0.05$, $\tilde{\sigma} = 0.6$, $\sigma = 0.2$, $\eta = 0.05$, $\theta = 2$, and $\rho = 0.05$.

its leverage, its assets covary with leverage, they are riskier, and thus earn a larger risk premium. The plot also clarifies that Sharpe ratios faced by the household (and including idiosyncratic risk) are lower than those faced by the financial sector, consistently with the opposite position that they have in the bond market. Accordingly, risk-free interest rates being decreasing in ω^f (Figure B.1, top, right), are pro-cyclical. This is because, in our model, high leverage corresponds to scarce financial capitalization, and so a scarce supply of risk-free bonds. In this term, the link between financial leverage Sharpe ratios, and interest rates strictly relates to the pooling capacity of the financial sector, and can be decomposed into two different components. First, higher financial leverage corresponds to lower (even negative, depending on the parameters) interest rates. Second, higher leverage corresponds to higher aggregate marginal productivity, decreasing expected price level, and thus lower risky assets returns for the financial sector. Note that the size of idiosyncratic risks contribute also to the financial sector risk premiums despite the fact they can be pooled, and therefore eliminated via diversification. This is due to the assumption of *restricted market participation* as well as to agents' risk aversion. In fact, the households' exposure to idiosyncratic risk, jointly with their share of the aggregate wealth, determines the aggregate demand of risk-free bonds, and so the equilibrium financial leverage. As long as there exists residual (un-pooled) idiosyncratic risk, this is accounted for in the equilibrium risk-free rates. A further interesting implication of our model is that, unlike He and Krishnamurthy (2013), there is no need of binding constraints to link financial leverage to Sharpe ratios: in this terms, it is an inherent effect of financial intermediation.

B.4 Mathematical Appendix - Proofs

B.4.1 The Agents' Problem

Given the agents' problem, the Hamiltonian reads as (we omit the up-scripts for sake of clear notation)

$$\rho V_t = \max_{\{\omega,c\}} \left\{ \log c_t + \frac{1}{dt}\mathbb{E}_t\left[dV_t\right] \right\},$$

subjected to the terminal condition $\lim_{t\to\infty} e^{-\rho t}V(e_t^i) = 0$. Given the generic motion of wealth stock of agent i,

$$\frac{de_t^i}{e_t^i} = \left[r_t + \omega_t\left(\mu_t^i - r_t\right) - \frac{c_t}{e_t}\right]dt + \omega_t\Sigma_t^{\prime}\mathbf{dW}_t,$$

when dynamics of a generic state vector ψ is described by a diffusion as

$$\frac{d\psi_t}{\psi_t} = \mu_t^{\psi}dt + \Omega_t^{\psi}d\mathbf{W}_t,$$

we have that

$$\rho V_t = \log \rho e_t + \partial_\psi V \psi_t \mu_t^{\psi} + \partial_e V e_t \mu_t^e + \partial_{\psi e} V e_t \psi \Sigma_t^{\prime}\Sigma_t^{\psi} + \frac{1}{2}\partial_{\psi\psi}V_t\psi_t \left\|\Sigma_t^{\psi}\right\| + \frac{1}{2}\partial_{ee}V e_t^2 \left\|\Sigma_t\right\|.$$

By considering an ansatz of the value function in the form

$$V_t := H_t(\psi) + \frac{1}{\rho}\log e_t,$$

then, the FOCs on $\{\omega, c\}$ imply

$$\frac{c_t}{e_t} = \rho, \tag{B.18}$$

$$\omega_t = \frac{\mu_t - r_t}{\sqrt{\|\Sigma\|}}. \tag{B.19}$$

and, under the optimal strategy $\{\omega, c\}$, the HJB holds as

$$\rho H(\psi_t) = \log \rho + \frac{1}{\rho}\left(\mu_t^e - \frac{1}{2}\left\|\Sigma_t\right\|\right) + H_\psi \psi \mu^{\psi} + \frac{1}{2}H_{\psi\psi}\psi\left\|\Sigma_t^{\psi}\right\|. \tag{B.20}$$

the By Feynman-Kač Theorem (see Huyên, 2009), the solution to (B.20) equals

$$H(\psi_0) = \frac{1}{\rho}\mathbb{E}_0 \int_0^\infty e^{-\rho t} \left(\mu_t^e - \frac{1}{2}\left\|\Sigma_t\right\|\right) dt - \frac{\log \rho}{\rho}.$$

B.4.2 Equilibrium Portfolios, Leverage, and Prices

According to Definition 1, the market clearing conditions for physical capital and risk-free bonds in Equations (B.14) and (B.15) can be written in terms of relative wealth share as:

$$\omega_t^f E_t^f + \omega_t^h E_t^h = K_t q_t \Longleftrightarrow \omega_t^h (1-\psi_t) + \omega_t^f \psi_t = 1,$$

$$\frac{E_t^f (1-\omega_t^f) + E_t^h (1-\omega_t^h)}{K_t q_t} = 0 \Longleftrightarrow \left(1-\omega_t^h\right)(1-\psi_t) = (\omega_t^f - 1)\psi_t. \quad \text{(B.21)}$$

Accordingly, the market clearing for consumption good (B.16) equals:

$$(A - \iota_t)\left(K_t^h + K_t^f\right) - \eta K_t^f = \rho\left(E_t^h + E_t^f\right). \quad \text{(B.22)}$$

By matching the market clearing condition on capital (B.21) and the optimal portfolios policy, we obtain

$$\omega_t^f = \frac{1}{\psi_t} - \frac{\mu_t - r_t}{\sigma_t^2}\frac{(1-\psi_t)}{\psi_t}$$

which, by Lemma **??**, can be written as

$$\omega_t^f = \frac{1}{\psi_t} - \frac{\frac{\eta}{q_t}}{\sigma_t^2 + \tilde{\sigma}^2}\frac{(1-\psi_t)}{\psi_t} - \frac{(1-\psi_t)}{\psi_t}\underbrace{\frac{\left(\mu_t^f - r_t\right)}{\sigma_t^2}}_{\omega_t^f}\frac{\sigma_t^2}{\sigma_t^2 + \tilde{\sigma}^2}.$$

Rearranging we find that

$$\omega_t^f = \frac{\sigma_t^2 + \tilde{\sigma}^2 - \frac{\eta}{q_t}(1-\psi_t)}{\psi_t \tilde{\sigma}^2 + \sigma_t^2}. \quad \text{(B.23)}$$

By substituting (B.23) into the market clearing for the risk-free bond (B.21), it is straightforward to find ω_t^h. Similarly, the equilibrium interest rate r_t can be obtained from the optimal portfolio policy. The results are summarized in the following Proposition:

Proposition 6. ***Equilibrium Portfolios and Interest Rate***
Equilibrium portfolio shares ω_t^f, ω_t^h and the interest rate r_t depend on relative

wealth share ψ_t only:

$$\omega_t^f = \frac{\tilde{\sigma}^2 + \sigma_t^2 - \frac{\eta}{q_t}(1-\psi_t)}{\psi_t \tilde{\sigma}^2 + \sigma_t^2}, \quad \omega_t^h = 1 - \left(\omega_t^f - 1\right)\frac{\psi_t}{1-\psi_t}.$$

$$r_t = \rho + \left(\psi_t \omega_t^f - 1\right)\frac{\eta}{q_t} + \Phi(\iota_t) + \mu_t^q - \sigma\sigma_t^q - \delta - \sigma_t^2 \omega_t^f.$$

We assumed both classes of agents have the same preferences. It follows that the portfolio share of the financial sector must be greater than or equal to 1. This is because, since the risk-free bond is in zero net supply, a positive portfolio share in bonds by the financial sector must be supplied by households. In equilibrium, this is not possible due to households assets exposure to idiosyncratic risk. Under the assumption that the idiosyncratic volatility $\tilde{\sigma}^2$ is greater then the intermediation cost rate $\frac{\eta}{q_t}$, the financial sector portfolio share ω_t^f is strictly greater then 1. The result is summarised in the following Corollary of Proposition 6 :

Corollary 2. ***Financial Leverage***
When the idiosyncratic volatility is greater then the intermediation cost rate, the financial sector holds a leveraged position, while the households hold positive portfolio shares in both risky and risk-free claims:

$$\tilde{\sigma}^2 > \frac{\eta}{q_t}, \quad \Rightarrow \omega_t^f > 1, \quad \omega_t^h \in (0,1),$$

Proof. The result comes after solving $\omega_t^f > 1$. □

Under the assumption of log investment function $\Phi(\iota_t) = \frac{\ln(\theta\iota_t + 1)}{\theta}$, where the parameter θ represents the cost of technical illiquidity between physical capital and consumption good, rate of re-investment ι_t is an affine transform of the state ψ_t, whereas $q(\psi_t)$ is affine in the equilibrium physical capital holdings of the financial sector $\omega_t^f \psi_t$. In fact, by matching the consumption market clearing condition in (B.22), it follows that:

$$q(\psi_t) = \frac{1 + \theta\left(A - \eta\psi_t\omega_t^f\right)}{1 + \theta\rho}, \quad \iota(\psi_t) = \frac{q(\psi_t) - 1}{\theta}. \tag{B.24}$$

B.4.3 Proof of Theorem 1, points 1 and 2

Given the state

$$\psi_t := \frac{E_t^f}{K_t q_t},$$

by Itô's lemma,

$$d\psi_t = \frac{\partial \psi_t}{\partial E_t^f} dE_t^f + \frac{\partial \psi_t}{\partial K_t q_t} dK_t q_t + \frac{1}{2} \frac{\partial^2 \psi_t}{\partial \left(E_t^f\right)^2} \left(dE_t^f\right)^2 + \\ + \frac{1}{2} \frac{\partial^2 \psi_t}{\partial \left(K_t q_t\right)^2} d\left(K_t q_t\right)^2 + \frac{\partial^2 \psi_t}{\partial \left(K_t q_t\right) \partial E_t^f} dK_t q_t dE_t^f.$$

By substituting the optimal portfolio in the budget constraint of the financial sector we have

$$\frac{dE_t^f}{E_t^f} = \left(1 - \omega_t^f\right) r_t dt - \rho dt + \mu_t^f \omega_t^f dt + \omega_t^f \sigma_t dW_t, \tag{B.25}$$

while the aggregate wealth evolves as

$$\frac{dK_t q_t}{K_t q_t} = \mu_t^f dt + \omega_t^f \psi_t \frac{\eta}{q_t} dt + \sigma_t dW_t - \rho dt. \tag{B.26}$$

Given Equations (B.25) and (B.26) it follows that

$$d\psi_t = \psi_t \frac{dE_t^f}{E_t^f} - \psi_t \frac{dK_t q_t}{K_t q_t} + \psi_t \sigma_t^2 dt - \psi_t \sigma_t^2 \omega_t^f dt.$$

By considering Proposition 6 and rearranging,

$$\frac{d\psi_t}{\psi_t} = \underbrace{\sigma_t^2 \left[1 + \omega_t^f \left(\omega_t^f - 2\right) - \omega_t^f \frac{\psi_t}{\sigma_t^2} \frac{\eta}{q_t}\right]}_{\mu_t^\psi} dt + \underbrace{\sigma_t \left(\omega_t^f - 1\right)}_{\sigma_t^\psi} dW_t.$$

Point 2 can be proved by looking for a Markov equilibrium in the state variable ψ_t. Similarly to Haven et al. (2016), if such an equilibrium exists, one must be able to express both drifts and diffusion in Equation (2.6) as a function of ψ_t only. By Itô's lemma,

$$dq_t = \partial_\psi q(\psi_t) \psi_t \mu_t^\psi dt + \frac{1}{2} \partial_{\psi\psi}^2 q(\psi_t) \psi_t^2 \left(\sigma_t^\psi\right)^2 dt - \partial_\psi q(\psi_t) \psi_t \sigma_t^\psi dW_t. \tag{B.27}$$

By matching drifts and diffusions of the dynamic Equations (B.27) and (2.6) we obtain the system in (2.9).

B.4.4 Proof of Theorem 1, points 3 and 4

Persistent heterogeneity In the neighbourhood of the right-hand side boundary, $\lim_{\psi \to 1^-} \sigma_t^q = 0$ implies, by continuity, that

$$\lim_{\psi \to 1^-} \omega_t^f = 1 \Rightarrow \lim_{\psi \to 1^-} \sigma_t^\psi = 0.$$

By the latter,

$$\lim_{\psi \to 1^-} \mu_t^{\psi} = -\left(\frac{\eta\,(1+\theta\rho)}{1+\theta(A-\eta)}\right) < 0 \Longleftrightarrow \eta > 0. \tag{B.28}$$

Similarly, in the neighbourhood of the left-hand side boundary, $\lim_{\psi_t \to 0^+} \sigma_t^q = 0$. The latter implies that

$$\lim_{\psi \to 0^+} \mu_t^{\psi} = \Delta^2, \quad \lim_{\psi \to 0^+} \sigma_t^{\psi} = \Delta,$$

where, by (B.24),

$$\Delta = \tilde{\sigma}^2 - \underbrace{\left(\frac{\eta\,(1+\theta\rho)}{1+\theta(A-\eta)}\right)}_{\frac{\eta}{q}}$$

is a positive constant. It follows that, in the surroundings of the left-hand side boundary, the dynamics of ψ_t behaves as a geometric Brownian motion with positive drift:

$$\psi_t^{\epsilon} = \epsilon \exp\left\{\left(\frac{1}{2}\Delta^2\right)t + \Delta W_t\right\}, \tag{B.29}$$

where ϵ is a positive number arbitrary close to 0. Hence, the process never reaches the absorbing state $\psi = 0$.

Given the Markov equilibrium in Theorem 1, and conditions (B.28) and (B.29), we know that state drift μ_t^{ψ} has positive sign at the left-hand side boundary whereas it is negative sign at the right-hand side one. It suffices to prove its derivative negative along the whole domain to grant a unique $\hat{\psi} \in (0,1)$ such that $\mu_t^{\psi}(\hat{\psi}) = 0$. In this fashion

$$\frac{\partial}{\partial \psi_t}\mu_t^{\psi} < 0, \forall \psi_t \in (0,1), \tag{B.30}$$

which leads to,

$$\underbrace{2\left(\omega_t^f\right)'(\sigma-\sigma_t^q)^2\left(\omega_t^f-1\right)}_{A} - 2\left(\omega_t^f\right)^2(\sigma-\sigma_t^q)\,(\sigma_t^q)' +$$

$$\underbrace{-\frac{\eta\left(\rho+\frac{1}{\theta}\right)}{\frac{1}{\theta}+\eta\psi_t+A} - \psi_t\frac{\eta^2\left(\rho+\frac{1}{\theta}\right)}{\left[\frac{1}{\theta}+\eta\psi_t+A\right]^2}}_{B} +$$

$$-\,2\,(\sigma-\sigma_t^q)\,(\sigma_t^q)' + 4\,(\sigma-\sigma_t^q)\,(\sigma_t^q)'\,\omega_t^f < 0.$$

and, after some algebra,

$$(\sigma_t^q)'\,(\sigma-\sigma_t^q)\left(\omega_t^f-1\right)^2 > \frac{A+B}{2}. \tag{B.31}$$

Provided that we assume (and numerically check) $(\omega_t^f)' < 0$, $\sigma > \sigma_t^q > 0$ and $(\sigma_t^q)' \geq 0$, condition (B.31) is always satisfied, since $A, B < 0$ and $\omega_t^f > 1$. Moreover, by Theorem 1 $\sigma_t^\psi \propto \sigma_t^q$, it follows that $\sigma_t^q > 0 \Rightarrow \sigma_t^\psi > 0$.

By considering the dynamics of $d\psi_t$ in Theorem 1, a unique stationary distribution $\pi(\psi)$ exists as long as the first two moments of ψ_t exist and are finite. A rigorous discussion of the sufficient conditions of existence of the stationary for Ito's Processes is in Zhenzhong and Chen (2013). Although we cannot derive closed-form solution for ψ_t, its first moment can be determined as

$$d\left(e^{-\int_0^t \mu_s^\psi ds}\psi_t\right) = -e^{-\int_0^t \mu_s^\psi ds}\mu_s^\psi \psi_t dt + e^{-\int_0^t \mu_s^\psi ds} d\psi_t = e^{-\int_0^t \mu_s^\psi ds}\psi_t \sigma_t^\psi dW_t.$$

If we integrate both sides and take expected value, we have

$$\mathbb{E}_0\left[\psi_t\right] = \psi_0 \mathbb{E}_0\left[e^{\int_0^t \mu_s^\psi ds}\right] + \mathbb{E}_0\left[e^{\int_0^t \mu_s^\psi ds}\int_0^t e^{-\int_0^s \mu_u^\psi du}\psi_s \sigma_s^\psi dW_s\right].$$

Since the term in dW_s is an Itô integral, it has expected value equals zero and thus

$$\mathbb{E}_0\left[\psi_t\right] = \psi_0 \mathbb{E}_0\left[e^{\int_0^t \mu_s^\psi ds}\right] \tag{B.32}$$

where ψ_0 is an arbitrary starting point. Thus, the first moment of the distribution is defined as long as $\mathbb{E}_0\left[e^{\int_0^t \mu_s^\psi ds}\right] < \infty$. We prove it numerically by simulation. Similarly we can derive the variance as

$$\mathbb{V}ar_0\left[\psi_t\right] = \mathbb{E}_0\left[\psi_t^2\right] - \mathbb{E}_0\left[\psi_t\right]^2. \tag{B.33}$$

The first term of (B.33) we can be found by solving

$$d(x^2) = 2xdx + 2dx^2$$

where $x = e^{-\int_0^t \mu_s^\psi ds}\psi_t$, which leads to

$$d\left(e^{-2\int_0^t \mu_s^\psi ds}\psi_t^2\right) = 2e^{-\int_0^t \mu_s^\psi ds}\psi_t e^{-\int_0^t \mu_s^\psi ds}\psi_t \sigma_t^\psi dW_t + e^{-2\int_0^t \mu_s^\psi ds}\left(\psi_t \sigma_t^\psi\right)^2 dt.$$

It follows that

$$\mathbb{E}_0\left[\psi_t^2\right] = \psi_0^2 \mathbb{E}_0\left[e^{2\int_0^t \mu_s^\psi ds}\right] + \mathbb{E}_0 E\left[e^{2\int_0^t \mu_s^\psi ds}\int_0^t e^{-2\int_0^t \mu_s^\psi ds}\left(\psi_s \sigma_s^\psi\right)^2 ds\right]$$

and thus

$$\mathbb{V}ar_0\left[\psi_t\right] = 2\mathbb{E}_0\left[e^{2\int_0^t \mu_s^\psi ds}\int_0^t e^{-2\int_0^s \mu_u^\psi du}\left(\psi_s \sigma_s^\psi\right)^2 ds\right].$$

Thus, the second (central) moment of the distribution is defined as long as $\mathbb{V}ar_0\left[\psi_t\right] < \infty$. We prove it numerically by simulation.

The Stationary Density The Fokker-Plank equation for the stationary density satisfies

$$\frac{\partial}{\partial t}\pi(\psi,t) = -\frac{\partial}{\partial\psi}\left\{\psi\mu^{\psi}\pi(\psi,t) - \frac{1}{2}\frac{\partial}{\partial\psi}\left[\psi^2\left(\sigma_t^{\psi}\right)^2\pi(\psi,t)\right]\right\} = 0. \tag{B.34}$$

By integrating over $(0,\psi)$ and rearranging, we can write (B.34) as the following ODE

$$d\ln h(\psi) = 2\frac{\mu^{\psi}}{\psi\left(\sigma^{\psi}\right)^2},$$

where

$$h(\psi) = \pi(\psi)\,\psi^2\left(\sigma^{\psi}\right)^2.$$

By integrating one more time, given a boundary condition $h(0) = h_0$, we obtain the density function of ψ_t as

$$\pi(\psi) = \frac{h_0 e^{\int_0^{\psi}\frac{2\mu^{\psi}(s)}{s\left(\sigma^{\psi}(s)\right)^2}ds}}{\psi^2\left(\sigma^{\psi}\right)^2},$$

where h_0 is such that $\int_0^1 \pi(\psi)d\psi = 1$.

B.4.5 Proof of Proposition 1

By the results summarised in Appendix (B.4.1), it is straightforward that, for a unitary aggregate capital $K_t = 1$, considering agents of the class h

$$W^h(\psi_t) = \frac{\ln\rho q_t(1-\psi_t)}{\rho} + \frac{1}{\rho}H^h(\psi_t).$$

where

$$H(\psi_t)^h = \frac{1}{\rho}\mathbb{E}_0\left[\int_0^{\infty} e^{-\rho s}\underbrace{\mu_s^{e,h} - \frac{1}{2}\left(\omega_s^h\right)^2\left(\sigma_s^2 + \tilde{\sigma}^2\right)}_{f(\psi_s)}ds\right]. \tag{B.35}$$

We compute the value of (B.35) conditional on the state ψ_0 by numerical simulation.

B.4.6 Constrained Portfolios

By considering the constrained version of the problem in Appendix B.4.1, by standard dynamic programming the HJB satisfies

$$\rho V_t = \max_{\{\omega_t, c_t\}}\left\{\ln c_t + \frac{1}{dt}\mathbb{E}_t\left[dV_t\right] - \lambda_t(\omega_t - LC)\right\}$$

where λ_t is the Lagrangian multiplier associated to the constraint

$$\omega_t \leq LC.$$

By taking FOCs and considering complementary slackness, given the dynamics of V_t, the optimal portfolio share ω_t^C satisfies the following system:

$$\begin{cases} \omega_t^U - \omega_t^C = \frac{\lambda_t}{\rho\sigma_t^2}, \\ \lambda_t \left(\omega_t^C - LC\right) = 0, \\ \lambda_t \geq 0, \\ \omega_t^C - LC \leq 0, \end{cases} \tag{B.36}$$

where ω_t^U is the unconstrained solution. The possible couples $\left\{\omega_t^C, \lambda_t\right\}$ that satisfy (B.36) are:

$$\begin{cases} \omega_t^C = \omega_t^U, \lambda_t = 0 & \omega_t^U < LC \\ \omega_t^C = LC, \lambda_t = \rho\sigma_t^2 \left(\omega_t^U - LC\right) & \omega_t^U \geq LC. \end{cases}$$

B.4.7 Macro-dynamics

The aggregate consumption equals $C_t = (A - \iota_t) K_t - \eta K_t^f$. To characterize the dynamics of aggregate consumption, it may be useful to define an auxiliary variable that summarises the fraction of total capital allocated to the financial sector K_t^f. Let κ be

$$\kappa(\psi_t) := \frac{K_t^f}{K_t},$$

with dynamics

$$\frac{d\kappa_t}{\kappa_t} = \mu_t^\kappa dt + \sigma_t^\kappa dW_t,$$

whose drift and diffusion might be pinned down as

$$\mu_t^\kappa \kappa_t = \partial_\psi \kappa_t \psi_t \mu_t^\psi + \frac{1}{2} \partial_{\psi\psi} \kappa_t \left(\psi_t \sigma_t^\psi\right)^2,$$

and

$$\sigma_t^\kappa \kappa_t = \partial_\psi \kappa_t \psi_t \sigma_t^\psi.$$

Therefore, it follows that

$$C_t = (A - \iota_t - \eta\kappa_t) K,$$

and, by Itô's lemma,

$$dC_t = (A - \iota_t - \eta\kappa_t) dK - K d\iota_t - K\eta d\kappa_t - \mathbb{C}ov_t [dK_t, d\iota_t] - \mathbb{C}ov_t [dK_t, d\kappa_t].$$

By considering the stochastic processes dK_t and $d\psi_t$ and

$$d\iota_t = \frac{1}{\theta}\left(q_t\mu_t^q - q_t\sigma_t^q dW_t\right),$$

we obtain, by substitution and rearranging

$$\frac{dC_t}{C_t} = \left[\Phi(\iota_t) - \delta\right]dt - \frac{\frac{q_t}{\theta}\mu_t^q + \eta\kappa_t - \frac{\sigma}{\theta}\sigma_t^q + \kappa_t\sigma_t^\kappa\sigma}{A - \iota_t - \eta\kappa_t}dt + \\ + \sigma\left[1 + \frac{\sigma_t^q}{\sigma}\frac{q_t}{\theta}\frac{1}{A - \iota_t - \eta\kappa_t} - \frac{1}{\sigma}\frac{\eta\kappa_t\sigma_t^\kappa}{A - \iota_t - \eta\kappa_t}\right]dW_t.$$

By Itô's lemma, the dynamics of aggregate investment $I_t = \iota_t K_t$ is given by

$$dI_t = d\left(\iota_t K_t\right) = K_t d\iota_t + \iota_t dK_t + \mathbb{C}ov\left[d\iota_t dK_t\right],$$

and, after substituting and rearranging,

$$\frac{dI_t}{I_t} = \left[\Phi(\iota_t) - \delta + \frac{q_t}{\theta\iota_t}\left(\mu_t^q - \sigma_t^q\sigma\right)\right] + \sigma\left(1 - \frac{1}{\theta}\frac{\sigma_t^q}{\sigma}\frac{q_t}{\iota_t}\right)dW_t.$$

Similarly, the dynamics of aggregate intermediation costs $G_t = \eta\kappa_t K_t$ is given by

$$\frac{dG_t}{G_t} = \left[\Phi(\iota_t) - \delta + \mu_t^\kappa dt + \sigma_t^\kappa\sigma\right]dt + \left(\sigma + \sigma_t^\kappa\right)dW_t.$$

Accordingly,

$$\frac{d\tilde{Y}_t}{\tilde{Y}_t} = \left[\Phi(\iota_t) - \delta - \eta\kappa_t\frac{\mu_t^\kappa + \sigma\sigma_t^\kappa}{A - \eta\kappa_t}\right]dt + \sigma\left(1 - \frac{\eta}{\sigma}\frac{\kappa_t\sigma_t^\kappa}{A - \eta\kappa_t}\right)dW_t.$$

B.4.8 Redistributive Taxation

In this appendix, we describe the equilibrium dynamics of the relative financial capitalization ψ when an exogenous taxation evenly redistributes resources at a rate τ from the financial sector to the households. In this setting, we consider the case where the taxation is constant and equals τ for every value of the state $\psi \in (0,1)$.

Since all the agents have log preferences and the tax transfer is proportional to their whole stock of wealth, it does not directly affect their portfolio and consumption choices. It does instead affect their conditional and unconditional welfare.

Let the dynamic budget constraint of the households' and of the financial sector evolve as

$$dE_t^h = E_t^h\left(\mu_t^{e,h}dt + \sigma_t^{e,h}dW_t\right) + \underbrace{\tau E_t^f dt}_{\text{(Positive) Tax}}, \tag{B.37}$$

$$dE_t^f = E_t^f \left(\mu_t^{e,f} dt + \sigma_t^{e,f} dW_t \right) - \underbrace{\tau E_t^f dt}_{\text{(Negative) Tax}} , \tag{B.38}$$

respectively, where the drift and diffusion terms $\mu_t^{e,i}, \sigma_t^{e,i}$ $i \in \{h, f\}$ are defined in Equation (2.4). The *tax* terms in Equations (B.37) and (B.38) represent the redistribution effect of wealth between sectors by mean of the taxation policy. Note that the absolute value of the tax, τE_t^f, is directly proportional to the financial sector stock of wealth E_t^f; as such it proportionally enters the households' dynamic budget constraint.

By Itô's Lemma, the level of relative financial capitalization evolves as

$$\begin{aligned} d\left(\frac{E_t^b}{E_t^b + E_t^h}\right) = &\frac{E_t^h}{\left(E_t^b + E_t^h\right)^2} dE_t^b - \frac{E_t^b}{\left(E_t^b + E_t^h\right)^2} dE_t^h + \\ &+ \frac{\partial \psi}{\partial E^h \partial E^b} dE_t^h dE_t^b + \frac{1}{2} \frac{\partial^2 \psi}{\partial^2 E^h} \left(dE_t^h\right)^2 + \frac{1}{2} \frac{\partial^2 \psi}{\partial^2 E^b} \left(dE_t^b\right)^2 , \end{aligned}$$

where the dynamics of wealth follow the processes in (B.37) and (B.38). By substituting, rearranging, and considering: $\frac{E_t^h}{E_t^f} := \frac{1}{\psi_t} - 1$:

$$\frac{d\psi_t^\tau}{\psi_t^\tau} = \frac{d\psi_t}{\psi_t} - \tau \frac{\psi_t}{1 - \psi_t} dt,$$

where the process $\frac{d\psi_t}{\psi_t}$ is defined as in (2.8).

B.5 Numerical Solution

Consider the equilibrium outcomes summarised in Theorem 1:

$$q_t \sigma_t^q = \partial_\psi q_t \psi_t \sigma_t^\psi , \tag{B.39}$$

$$\sigma_t^\psi = \omega_t^f (\sigma - \sigma_t^q), \tag{B.40}$$

jointly with the equilibrium price of physical capital as given in Equation (B.24)

$$q(\psi_t) = \frac{1 + \theta \left(A - \eta \psi_t \omega_t^f \right)}{1 + \theta \rho}. \tag{B.41}$$

Let the auxiliary function $\kappa(\psi_t)$ denote the capital holdings of the financial sector $\omega_t^f \psi_t$. As we shall see, the auxiliary function κ is useful to solve the model for its competitive equilibrium as it has a compact support and a well defined boundary condition when $\psi = 0$.

By matching (B.39) and (B.40), it is straightforward to pin down the volatility of physical capital price (see also Brunnermeier and Sannikov, 2014) as

$$\sigma^q(\psi_t) = -\sigma \frac{\partial_\psi q_t \frac{\psi_t}{q_t}(\kappa_t - \psi_t)}{1 - \partial_\psi q \frac{\psi_t}{q_t}(\kappa_t - \psi_t)},$$

that we can substitute into (B.23) to obtain the following bi-variate ODE:

$$\frac{\kappa_t}{\psi_t}\sigma^2 \left(1 - \frac{\partial_\psi q \frac{\kappa_t - \psi_t}{q_t}}{1 + \partial_\psi q \frac{\kappa_t - \psi_t}{q_t}}\right) + \frac{\eta}{q_t} = \left[\sigma^2 \left(1 - \frac{\partial_\psi q \frac{\kappa_t - \psi_t}{q_t}}{1 + \partial_\psi q \frac{\kappa_t - \psi_t}{q_t}}\right)^2 + \tilde{\sigma}^2\right] \frac{1-\kappa_t}{1-\psi_t}. \tag{B.42}$$

By taking the first derivative of (B.41), matching it to (B.42) and rearranging, we obtain the following - fully implicit - system of ODEs

$$\begin{cases} \left(\frac{\kappa_t}{\psi_t} - \frac{1-\kappa_t}{1-\psi_t}\right)\left[\sigma^2\left(1 + \partial_\psi q_t \frac{\kappa_t - \psi_t}{q_t}\right)^{-2}\right] + \frac{\eta}{q_t} - \tilde{\sigma}^2 \frac{1-\kappa_t}{1-\psi_t} = 0; \\ \partial q_\psi + \frac{\theta\eta}{1+\theta\rho}\partial\kappa_\psi = 0, \end{cases}$$

that can be solved numerically given suitable boundary conditions. We solve it by Matlab *ode15i* by setting $q(0) = \frac{1+\theta A}{1+\theta\rho}$ and $\kappa(0) = 0$.

Once that we have the solution vector $\{q, \kappa\}_{\psi \in (0,1)}$, we compute all the equilibrium quantities so that $\omega_t^f = \frac{\kappa_t}{\psi_t}$, where we approximate the first and second derivatives of physical capital prices as $\partial_\psi q(\psi) \approx \frac{q(\psi+\Delta)-q(\psi)}{\Delta}$ and $\partial_{\psi\psi} q(\psi) \approx \frac{q(\psi+\Delta)+q(\psi-\Delta)-2q(\psi)}{2\Delta^2}$, respectively, over the solution grid. Similarly, $\partial_\psi \kappa(\psi) \approx \frac{\kappa(\psi+\Delta)-\kappa(\psi)}{\Delta}$ and $\partial_{\psi\psi}\kappa(\psi) \approx \frac{\kappa(\psi+\Delta)+\kappa(\psi-\Delta)-2\kappa(\psi)}{2\Delta^2}$.

B.6 The Benchmarks

In this section, we introduce the two extreme cases that act as the benchmarks of our analysis. The former is the *no-risk-pooling* economy, where the households hold all the capital and restricted market participation plays a big role. The latter is the *full-risk-pooling* economy, where the financial sector holds the whole stock of physical capital and restricted market participation plays no role.

No-risk-pooling The equilibrium at the left-hand side boundary ($\psi = 0$) implies a constant price of physical capital $q(0)$ (in fact $\mu^q(0) = \sigma^q(0) = 0$), investment $\iota(0)$, risk-free interest rates $r(0)$, risky claim return $\mu^h(0)$, and their Sharpe ratio $\xi^h(0)$. In particular,

$$q(0) = \frac{1+\theta A}{1+\theta\rho}, \quad \iota(0) = \frac{q(0)-1}{\theta} = \frac{A-\rho}{1+\theta\rho}, \quad r(0) = \rho + \Phi(\iota(0)) - \delta - \sigma^2 - \tilde{\sigma}^2,$$

$$\mu^h(0) = \frac{A - \iota(0)}{q(0)} + \Phi(\iota(0)) - \delta, \quad \xi^h(0) = \frac{\frac{A-\iota(0)}{q(0)} + \sigma^2 + \tilde{\sigma}^2 - \rho}{\sqrt{\sigma^2 + \tilde{\sigma}^2}}.$$

In this economy markets are utterly *incomplete.* Financial intermediaries do not supply any risk-mitigation instrument to the economy and each entrepreneur has full exposure to its idiosyncratic shocks. The equilibrium interest rate is lower than how it would be with a financial sector, and it is such that, in absence of risk mitigation assets, agents are happy to invest their wealth in risky claims only. High value of q decrease the dividend yield (but increase the capital gain due to higher investment) and decrease also the Sharpe ratio. The latter depends also on systematic and idiosyncratic risk. Although both increase risk, they also decrease the risk-free rate and thus, overall, increase the equilibrium Sharpe ratio. In this benchmark, the capital stock K_t follows a Geometric Brownian Motion (GBM). The same holds for aggregate output (due to the linearity of type II technology) and for aggregate consumption:

$$\left.\frac{dK_t}{K_t}\right|_{\psi=0} = \left.\frac{dC_t}{C_t}\right|_{\psi=0} = \left.\frac{dY_t}{Y_t}\right|_{\psi=0} = \left[\Phi(\iota(0)) - \delta\right] dt + \sigma dW_t.$$

Although aggregate output and consumption are moved only by the systematic shocks, each entrepreneur individual consumption bears its uninsured idiosyncratic risk leading to a low welfare.

Full-risk-pooling The *full-risk-pooling* economy is reachable when the cost of intermediation equals zero, unless the obvious case when the financial sector is endowed with the whole aggregate wealth at $t = 0$ so that $\psi_0 = 1$. Also this equilibrium implies a constant price of physical capital $q\,(1)$, investment $\iota(1)$, risk-free interest rates $r(1)$, risky claim return $\mu^h(1)$, and their Sharpe ratio $\xi^h(1)$.[2] In particular, we have:

$$q(1) = \frac{1 + \theta\,(A - \eta)}{1 + \theta\rho}, \quad \iota(1) = \frac{A - \rho - \eta}{1 + \theta\rho}, \quad r\,(1) = \rho + \Phi(\iota(1)) - \delta - \sigma^2,$$

$$\mu^f(1) = \frac{A - \iota(1) - \eta}{q(1)} + \Phi(\iota(1)) - \delta, \quad \xi^f(1) = \frac{\frac{A-\iota(1)-\eta}{q(1)} + \sigma^2 - \rho}{\sigma}.$$

Note that when $\eta > 0$ capital prices and investment are lower: $q(1) < q(0)$ implies $\iota(1) < \iota(0)$. Interest rates are higher, $r(1) > r(0)$, due to the fact that the financial sector can diversify all the idiosyncratic risk and thus has a zero demand/supply of risk mitigation for higher rates than when households are alone. Positive intermediation costs imply instead that capital is less productive (some

[2]Also this equilibrium is a special case of John Cox and Ross (1985).

resources are lost by the intermediation process) and its equilibrium prices is lower. Lower prices imply also lower investment and thus lower drift, a *pecuniary externality* of the high intermediation costs. Risk premiums, and so Sharpe ratios, are also a function of capital prices. A low capital price implies a higher dividend yield and a lower capital gain (lower investment). The Sharpe ratio is also lower due to higher interest rates. Also in this benchmark the capita stock follows a GBM, the same process followed by total consumption and output:

$$\left.\frac{dK_t}{K_t}\right|_{\psi=1} = \left.\frac{dC_t}{C_t}\right|_{\psi=1} = \left.\frac{dY_t}{Y_t}\right|_{\psi=1} = [\Phi(\iota(1)) - \delta]\, dt + \sigma dW_t.$$

With positive intermediation cost, $\eta > 0$, the growth rate of output, capital, and consumption is lower in the *full-risk-pooling* economy than in the *no-risk-pooling* case. Nevertheless, in both cases the aggregate volatility is state independent and equals σ. The same process is followed also by the disposable output $\tilde{Y}$, defined as the output net of intermediation costs: $\tilde{Y}_t = Y_t - \eta K_t = (A - \eta)K_t$.

B.7 Comparative Statics

In this Appendix, we discuss the changes of equilibrium dynamics with respect to the key parameters in the model, namely the size of systematic and idiosyncratic risk as well as intermediation costs.

Figure B.2 shows the drift (left) and diffusion (right) of the process $d\psi_t$ as a function of the state $\psi \in (0, 1)$ for different values of systematic diffusion σ. In Figure B.3, we perform the same comparative statics for equilibrium portfolio shares ω^f and ω^h.

With reference to Figure B.2, when the financial sector is arbitrary well capitalised (ψ is high), decreasing systematic risk σ has the effect of reducing σ^ψ: the lower the risk, the lower both state drift and diffusion. When instead ψ approaches the left side boundary $\psi = 0$, a lower σ is associated to higher leverage and reduced risky asset in households' portfolio (Figure B.3). Indeed, higher leverage is associated to a sharper drift μ^ψ. This phenomenon is associated to the so-called *volatility paradox* (Adrian and Boyarchenko, 2012; Brunnermeier and Sannikov, 2014; Phelan, 2016).

Figure B.4 displays a similar exercise by plotting equilibrium portfolio choices over $\psi \in (0, 1)$ with respect to different values of idiosyncratic diffusion $\tilde{\sigma}$. What stands out is that the lower the idiosyncratic risk the lower the equilibrium leverage of the financial sector. This pattern is the consequence of a reduced advantage of the financial sector due to pooling: when idiosyncratic risk is relatively lower, the demand for mitigation is also reduced, household keep a wider

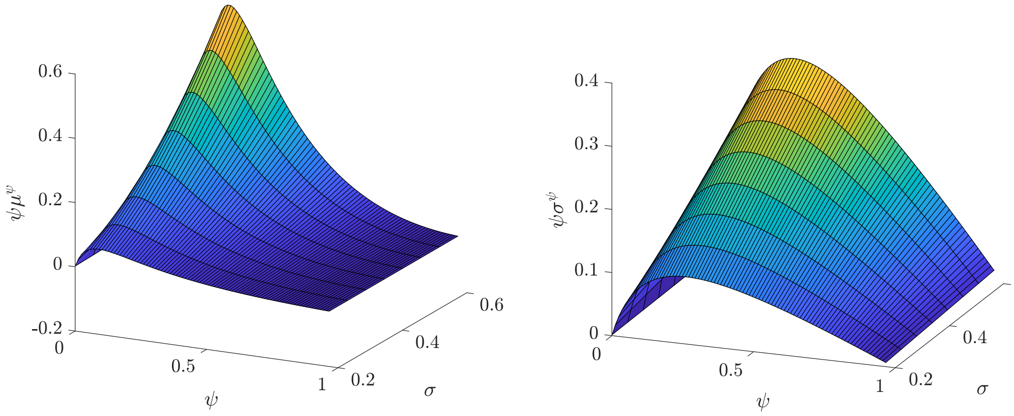

Figure B.2: Drift (left) and diffusion (right) of the process $d\psi_t$ for different values of systematic volatility σ. Baseline parameters: $A = 0.5$, $\delta = 0.05$, $\tilde{\sigma} = 0.6$, $\eta = 0.05$, $\theta = 2$, and $\rho = 0.05$.

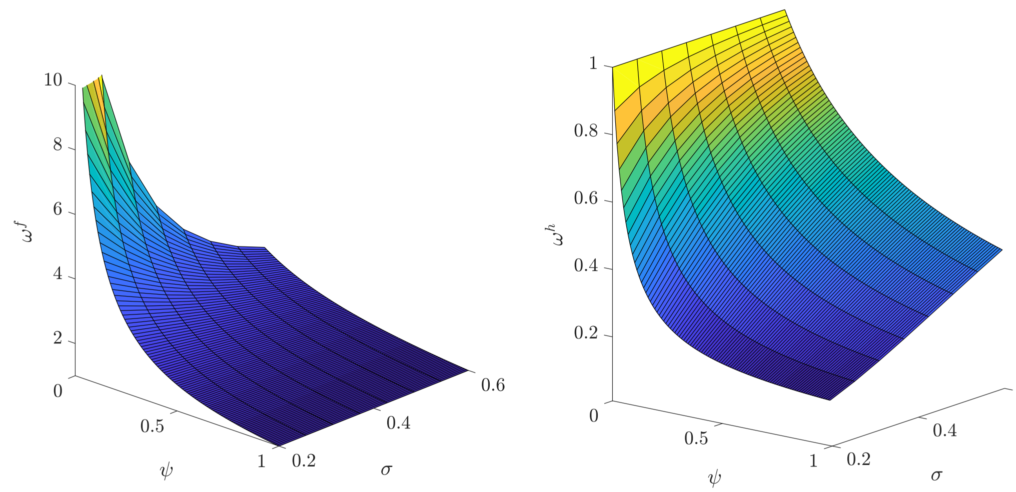

Figure B.3: Equilibrium portfolio shares ω^f (left) and ω^h (right) for different values of systematic diffusion σ. $A = 0.5$, $\delta = 0.05$, $\tilde{\sigma} = 0.6$, $\eta = 0.05$, $\theta = 2$, and $\rho = 0.05$.

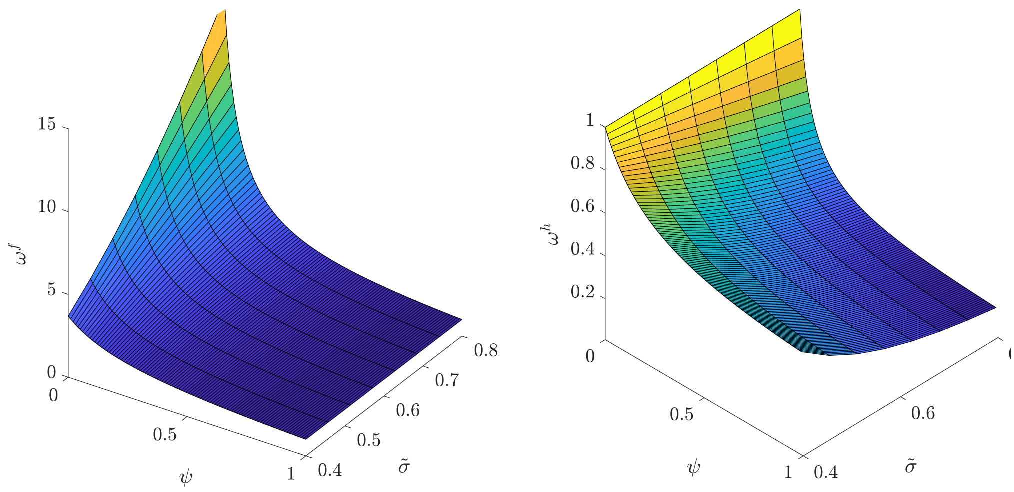

Figure B.4: Equilibrium portfolio shares ω^f (left) and ω^h (right) for different values of idiosyncratic diffusion $\tilde{\sigma}$. Baseline parameters: $A = 0.5$, $\delta = 0.05$, $\sigma = 0.4$, $\eta = 0.1$, $\theta = 2$, and $\rho = 0.05$.

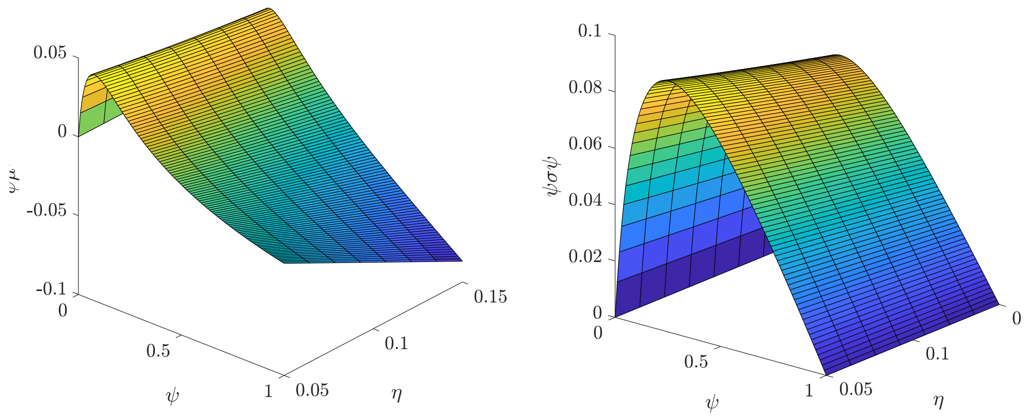

Figure B.5: Drift (left) and diffusion (right) of the process $d\psi_t$ for different intermediation costs η. Baseline parameters: $A = 0.5$, $\delta = 0.05$, $\tilde{\sigma} = 0.6$, $\sigma = 0.2$, $\theta = 2$, and $\rho = 0.05$.

fraction of their wealth allocated in risky claims, and equilibrium risk-free rate is higher.

Finally, in Figure B.5 (top) we repeat the same analysis for different values of intermediation costs η. In the bottom graphs, we consider two sections of the upper ones for null (blue) and positive (green) intermediation costs η. From Figure B.5 we notice that, when there are no intermediation costs, the drift μ_t^ψ is positive for each ψ. In the long-run the financial sector dominates and thus its drains the whole wealth in the economy.[3] Moreover, positive intermediation costs (green) mainly affect the right-hand side of the state space, when ψ approaches 1. Higher costs progressively sharpen the negative drift, when the financial sector is relatively well capitalised, making faster the recovery of households' relative wealth.

[3]This case is equivalent to the equilibrium where markets are complete for both classes agents.

Appendix C

Chapter 3

C.1 The economy with no banks

The natural benchmark case of our analysis is the economy with no banks, where $\psi = \mu^{\psi} = \sigma^{\psi} = 0$, and there are is no recapitalization of the banking sector.

In the economy with no banks, the price $q_t = \bar{q}$ and the re-investment rate $\iota_t = \bar{\iota}$ are constant and equal

$$\bar{\iota} = \frac{\bar{q} - 1}{\theta}, \tag{C.1}$$

$$\bar{q} = \frac{1 + A\theta}{1 + \rho\theta}. \tag{C.2}$$

It follows that the aggregate consumption equals $C_t = \rho K_t \bar{q}$, and its dynamics follows a GBM as well as the one of physical capital

$$\frac{dC_t}{C_t} = \frac{dK_t}{K_t} = \Phi(\bar{\iota})dt + \sigma dW_t, \tag{C.3}$$

while the equilibrium risk-free interest rate on the deposits is constant and equals

$$r = \rho + \Phi(\bar{\iota}) - \sigma^2. \tag{C.4}$$

C.2 Comparative statics

In Figure C.1 we plot the equilibrium market-to-book value of the banking sector v with respect to low (blue) and high (green) parametric values for the banking premium $\bar{\eta}$ (top, left), the recapitalization friction λ (top, right), the exogenous volatility component σ (bottom, left), and the friction on physical capital θ (bottom, right) over the interval $\psi_t \in \left(\underline{\psi}, \bar{\psi}\right)$.

In general, a higher banking premium $\bar{\eta}$ reduces the equilibrium upper threshold above which dividends are paid by the banking sector. This is because a

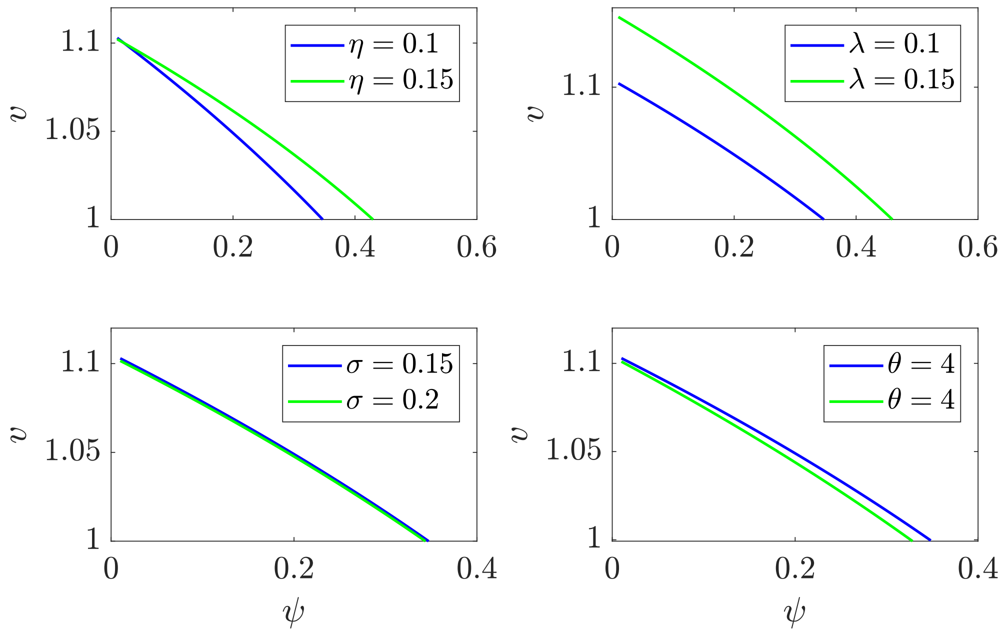

Figure C.1: Comparative statics of the banking sector *market-to-book value* as a function of the key parameters of the model as a function of the state ψ: banking premium η (top, left), recapitalization friction λ (top, right), exogenous systematic volatility σ (bottom, left), and physical capital friction θ (bottom, right). Baseline parameters: $A = 0.1$, $\eta = 0.1$, $\sigma = 0.15$, $\theta = 4$, $\lambda = 0.1$ and $\rho = 0.05$.

higher premium corresponds to "more profitable" monitoring services supplied by the banking sector with respect to the households'. In equilibrium, this implies the marginal value of banking equity being lower for every state ψ_t. Correspondingly, the banks reach the threshold wealth share such that v has unitary value more frequently, they require lower capital buffers, and so pay dividends flows to the households more often.

For similar reasons, the same pattern holds for both increases in aggregate exogenous systematic volatility σ and physical capital friction θ. While the former effect relates to the capacity of the bank to supply the households with risk-free deposits, the latter implies the banks doing more investments, since more output is required to produce new capital.

Conversely, the market-to-book value shifts to the right for higher value of the recapitalization friction (Figure C.1, top right), and so the banks require higher capitalization before they pay out dividends. This effect is due to a "precautionary motif"; since for higher λ default contingencies are costlier, the banks react by increasing their *equity buffers*. As a result, the banking sector is more careful and decides to retain dividends to grow its equity to a higher level before it is optimal to pay out dividends.

C.2.1 The Economy in the Long-run

Aiming to understand the features of the equilibrium dynamics in the long-run, Figure C.2 (left) depicts two realized paths of the stochastic process that drives the relative wealth share ψ simulated over 10,000 periods. The red lines depict the upper (dotted) and lower (dashed) thresholds that determine the payment of dividends and recapitalization flows, respectively. Due to their cost advantage at monitoring the capital producing firm, the banks benefit from the higher expected return on their assets, and thus from a higher growth rate of equity with respect to households' wealth. The cost advantage is directly proportional to the banking premium η, inversely to the price level q, that is decreasing for lower values of ψ. This means that when the banking sector is already well capitalized, it grows at a lower rate.

The second component is due to households' consumption; it takes place through a price *externality*, and it affects the cost advantage due to the presence of the banking premium. As banks do not consume, the higher their wealth share (equity), the higher the equilibrium price level q, and so the firms re-investment rate ι (note that, since $\iota \propto q$, higher financial leverage associates to lower prices q and channels a stronger benefit of the banking premium over banks' wealth dynamics). Moreover, due to the assumption of log utility, the states where the prices are low are also those where households' consumption is high. Therefore, ψ grows at a higher rate (note that, due to log utility, the households consume a fixed fraction ρ of their wealth E_t^h that, for a unitary aggregate capital $K = 1$, is proportional to $q_t(1 - \psi)$).

Another aspect that is relevant to stress is that, due to financial leverage, the state dynamics is much more volatility for lower than for higher values of ψ, when the banks' are "poorly" capitalized.

This pattern emerges even clearer if we look at the lung-run (stationary) distribution of ψ. In Figure C.3, we plot the stationary density $\pi(\psi)$ (left) jointly with the drift (top, right) and diffusion (bottom, right) of the banks' relative capitalization a function of the state (see Klimenko et al., 2017, for the details of the computational methods). According to what we observe in the simulated paths, the economy drifts, and spends most of the time, to states where the banking sector is well capitalized. Accordingly, those are the state where banks persistently pay out dividends to their shareholders. Nevertheless, there is a small but positive probability that ψ floats through states of low capitalization. Conditional on those states, being the volatility term σ^ψ much higher as $\psi \to \underline{\psi}$, its much likelier that the banking sector defaults.

Perhaps the most compelling element of the states featuring low banks capitalization is that they may be arbitrarily persistent. Conditional on the banks' optimal recapitalization policy (that supply to the banks just enough equity to remain at the lower threshold $\underline{\psi}$), in the neighbourhood of the lower bound the volatility component dominates the corresponding drift. Thus, being the source of noise dW_t an i.i.d. process, a negative stream of adverse exogenous shocks may dump the system around $\underline{\psi}$, where frequent equity issuances are required, and aggregate investments is low.

C.3 Proofs

C.3.1 HJB and the households' problem

For sake of clear notation, we omit all time subscripts. By standard continuous-time stochastic control methods, the households' HJB reads as follows:

$$\rho H = \max_{\{c,\omega^h\}\in G} \left\{ \log c + \frac{1}{dt}\mathbb{E}\left[\frac{\partial H}{\partial e}de + \frac{1}{2}\frac{\partial^2 H}{\partial e^2}de^2\right] + \right.$$
$$\left. + \frac{1}{dt}\mathbb{E}\left[\frac{\partial H}{\partial \psi}d\psi + \frac{1}{2}\frac{\partial^2 H}{\partial \psi^2}d\psi^2 + \frac{\partial^2 H}{\partial \psi \partial e}d\psi de\right]\right\}, \quad \text{(C.5)}$$

where ψ_t is the aggregate banking sector relative share of wealth. We postulate that the dynamics of ψ follows a diffusion process as

$$\frac{d\psi}{\psi} = \mu^\psi dt + \sigma^\psi dW + d\Xi, \quad \text{(C.6)}$$

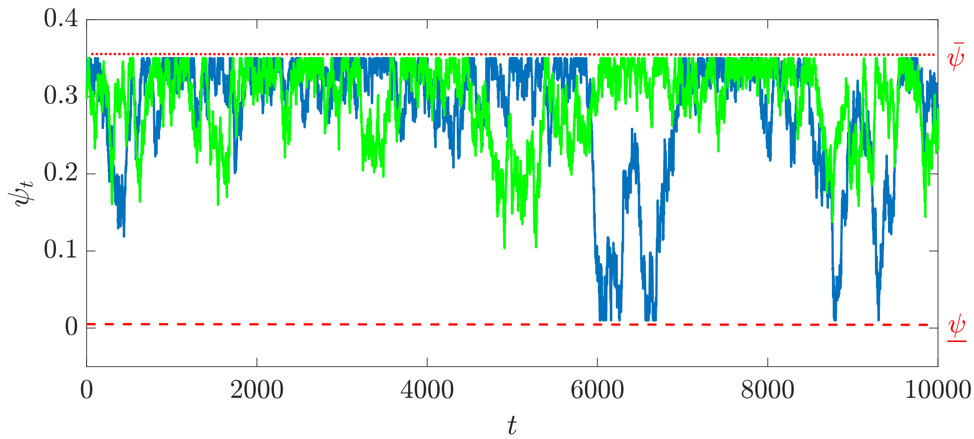

Figure C.2: Two realizations of the equilibrium relative wealth share stochastic process in blue and green, respectively, over a time interval $T = 10,000$. In red, the upper (dotted) and lower (dashed) thresholds above and below which the banking sector either pays out dividends or asks for recapitalization. Baseline parameters: $A = 0.1$, $\eta = 0.1$, $\sigma = 0.15$, $\theta = 4$, $\lambda = 0.1$, and $\rho = 0.05$.

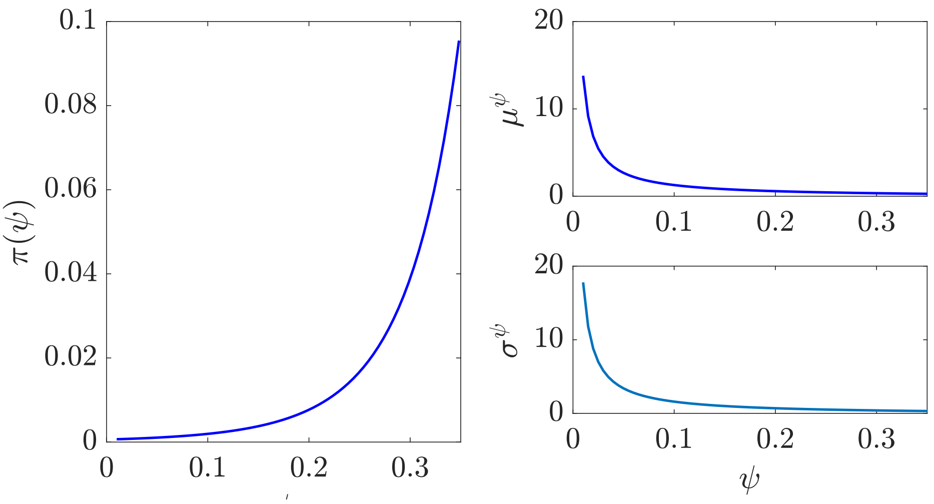

Figure C.3: Left: long-run (stationary) density function. Right: drift (top) and diffusion (bottom) of the state dynamics $\frac{d\psi}{\psi}$ as a function of ψ. Baseline parameters: $A = 0.1$, $\eta = 0.1$, $\sigma = 0.15$, $\theta = 4$, $\lambda = 0.1$, and $\rho = 0.05$.

where $d\Xi$ is an impulse term that adjusts the dynamics of banks' relative capitalization for dividends payouts and equity issuance flows, and is consistent with banks' optimal policy.

Given the ansatz for the value function $H := h(\psi) + \bar{b}\log e_t^h$, jointly with the dynamics of the households' wealth in (3.6) and the process (C.6), it follows that

$$\rho H = \max_{\{c,\omega^h\}\in G}\left\{\log c + b\left[r + \omega\left(\mu^h - r\right) + \frac{d\delta}{e} - (1+\lambda)\frac{d\pi}{e} - \frac{c}{e}\right] + \right. \\ \left. - b(\sigma\omega)^2 + h_\psi\psi\left(\mu^\psi + d\Xi\right) + \frac{1}{2}h_{\psi\psi}\left(\psi\sigma^\psi\right)^2\right\} \quad \text{(C.7)}$$

and the FOCs read as

$$\frac{c}{e} = \frac{1}{b}, \qquad \omega = \frac{\mu^h - r}{\sigma^2}; \quad \text{(C.8)}$$

and h satisfies the following ODE:

$$\rho h = \ln\rho - 1 + \frac{1}{\rho}\left[r + \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2} + \frac{d\delta}{e} - (1+\lambda)\frac{d\pi}{e}\right] + \\ + h_\psi\psi\left(\mu^\psi + d\Xi\right) + \frac{1}{2}h_{\psi\psi}\left(\psi\sigma^\psi\right)^2, \quad \text{(C.9)}$$

and, accordingly,

$$\bar{b} = \frac{1}{\rho}. \quad \text{(C.10)}$$

With reference to the results of Proposition 2, it is relevant to stress that the h^{th} household takes the dividends and recapitalization flows from and to its own bank as given. Moreover, the effect on welfare of the perspective of paying or receiving those capital flows is accounted for in the h_t component of the value function only, and is conditional on the state ψ. As we shall see, the h_t component of the households' welfare function H_t is fundamentally affected by banks' cash flows $d\delta^b$ and $d\pi^b$.

C.3.2 Equilibrium

The formal statement of the competitive equilibrium reads as follows:

Definition 2. ***Competitive Equilibrium***
Conditional on an initial allocation of capital between the aggregate banking sector equity and the households wealth, an equilibrium is an adapted stochastic process that maps histories of systematic shocks $\{dW_t\}$ *to prices* $\{q_t\}$*, return on risky claims* $\{dR_t\}$*, risk-free interest rate on short-term bank liabilities* $\{r_t\}$*, production choices* $\{K_t, \iota_t\}$*, consumption choices* $\left\{c_t^h : h \in \mathbb{H}\right\}$*, asset allo-*

cations $\left\{\omega_t^h, \omega_t^b : h \in \mathbb{H}, b \in \mathbb{B}\right\}$*, as well as dividend and recapitalization strategies* $\left\{d\delta_t^b, d\pi_t^b : b \in \mathbb{B}\right\}$ *such that:*

1. *The firms maximise their profits:*

   (a) *Capital producing firm:*

$$\{K_t, \iota_t\} = \arg \max_{\{K_t, \iota_t\} \in T} \left\{\mathbb{E}_t^{\mathbb{Q}} \left[V_s e^{-\int_t^s r_u du}\right] - K_t q_t\right\}, \tag{C.11}$$

   where V_s *are the firms revenues at between* t *and* $s = t + dt$ *at time* s*;*

   (b) *Output producing firm:*

$$K_t = \arg \max_{K_t \geq 0} K_t \left(A - p_t\right). \tag{C.12}$$

2. *The households* $h \in \mathbb{H}$ *maximise their utility:*

$$\left\{c_t^h, \omega_t^h\right\} \in \arg \max_{\left\{c_t, \omega_t^h\right\} \in G_t^h} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \ln c_t^h dt. \tag{C.13}$$

3. *The banks* $b \in \mathbb{B}$ *maximise their market value:*

$$\left\{d\delta_t^b, d\pi_t^b, \omega_t^b\right\} \in \arg \max_{\left\{d\delta_t^b, d\pi_t^b, \omega_t^b\right\} \in B_t^b} \mathbb{E}_0^{\mathbb{Q}} \int_0^\infty \Lambda_t \left[d\delta_t^b - (1+\lambda) d\pi_t^b\right] dt. \tag{C.14}$$

4. *All markets clear:*

   (a) *Risky assets:*

$$\int_{\mathbb{H}} e_t^h \omega_t^h dh + \int_{\mathbb{B}} e_t^b \omega_t^b db = K_t q_t; \tag{C.15}$$

   (b) *Deposits:*

$$\int_{\mathbb{H}} e_t^h \left(1 - \omega_t^h\right) dh + \int_{\mathbb{B}} e_t^b \left(1 - \omega_t^b\right) db = 0; \tag{C.16}$$

   (c) *Consumption:*

$$\int_{\mathbb{H}} \left(A - \iota_t - \eta\right) k_t^h dh + \int_{\mathbb{B}} \left(A - \iota_t\right) k_t^b db = C_t^h; \tag{C.17}$$

   (d) *Physical capital:*

$$\int_{\mathbb{H}} k_t^h dh + \int_{\mathbb{B}} k_t^b db = K_t. \tag{C.18}$$

C.3.3 HJB and the banks' problem

For sake of clear notation, I omit all the time subscripts. By standard continuous-time stochastic control methods, the HJB of the banking sector satisfies the following:

$$rJdt = \max_{\{d\delta, d\pi, \omega\} \in B} \Big\{ d\delta - (1+\lambda)d\pi + \mathbb{E}^{\mathbb{Q}} \left(\partial_e J de + \partial_v J dv_t + \partial_v \partial_e J de dv \right) + \\ + \frac{1}{2} \mathbb{E}^{\mathbb{Q}} \left[\partial_e^2 J de^2 + \partial_v^2 J dv^2 \right] \Big\}, \quad \text{(C.19)}$$

Given the ansatz for the value function, $J := ve$, jointly with the processes (3.4), (3.15), and (3.18) the HJB is separable in all the controls

$$- \mu^v - \sigma^v \xi = \max_{\{d\delta\} \in B} \left[\frac{d\delta}{ve} - \frac{d\delta}{e} \right] + \max_{\{d\pi\} \in B} \left[\frac{d\pi}{e} - (1+\lambda) \frac{d\pi}{ve} \right] + \\ + \max_{\{d\omega\} \in B} \left[\omega^b (\mu - r) - \omega^b \sigma (\sigma^v + \xi) \right]. \quad \text{(C.20)}$$

The optimality conditions on dividends, (3.19), recapitalization (3.20), and the banking sector risk premia (3.21) follow by FOCs. Moreover, when $1 < v_t < 1+\lambda$,

$$-\mu^v - \sigma^v \xi = \omega^b \left[(\mu - r) - \sigma (\sigma^v + \xi) \right], \quad \text{(C.21)}$$

C.3.4 State Variable

Given that $\psi_t := \frac{E_t^b}{K_t q_t}$, and $K_t q_t = E_t^b + E_t^h$, it follows that

$$\frac{d\psi_t}{\psi_t} = \frac{dE_t^b}{E_t^b} - \frac{d(K_t q_t)}{K_t q_t} - \frac{dE_t^b}{E_t^b} \frac{d(K_t q_t)}{K_t q_t} + \frac{d(K_t q_t)^2}{(K_t q_t)^2}, \quad \text{(C.22)}$$

and, given $d(K_t q_t) = dE_t^b + dE_t^h$, (3.6), and (3.15)

$$\frac{d(K_t q_t)}{K_t q_t} = \left[r_t - \rho (1 - \psi_t) + \psi_t \omega_t^b (\mu_t - r) + (1 - \psi_t) \omega_t^h \left(\mu_t^h - r \right) \right] dt + \\ - \frac{d\Pi_t}{K_t q_t} \lambda + \left[\psi_t \omega_t^b \sigma_t + (1 - \psi_t) \omega_t^h \sigma_t \right] dW_t, \quad \text{(C.23)}$$

and thus, given (3.4), (C.17), and (C.15),

$$\frac{d\psi_t}{\psi_t} = \underbrace{(1-\psi_t)\left[\rho + \omega_t^b \frac{\eta}{q_t} + \omega_t^b \sigma_t^2 \omega_t^h - \left(\omega_t^h \sigma_t\right)^2\right] dt - \sigma_t^2 \left(\omega_t^b - 1\right) dt +}_{\mu_t^\psi}$$

$$+ \underbrace{\frac{d\Pi_t}{K_t q_t}\left(\frac{1-\psi_t\lambda}{\psi_t}\right) - \frac{d\Delta_t}{K_t q_t}\frac{1}{\psi_t}}_{d\Xi_t} + \underbrace{\sigma_t \left(\omega_t^b - 1\right)}_{\sigma_t^\psi} dW_t. \quad \text{(C.24)}$$

C.3.5 Equilibrium (Unconstrained)

To the aim of solve the model we turn the key equilibrium SDEs, whose drift and diffusion components are unknown, into a system of ODEs. By Itô's lemma:

$$dq_t = \left(q_\psi \psi_t \mu_t^\psi + \frac{1}{2} q_{\psi\psi} \left(\psi_t \sigma_t^\psi\right)^2\right) dt + q_\psi \psi_t \sigma_t^\psi dW_t, \quad \text{(C.25)}$$

$$dv_t = \left(v_\psi \psi_t \mu_t^\psi + \frac{1}{2} v_{\psi\psi} \left(\psi_t \sigma_t^\psi\right)^2\right) dt + v_\psi \psi_t \sigma_t^\psi dW_t, \quad \text{(C.26)}$$

that imply, by matching (C.25) and (C.26) to the conjectured processes (3.23) and (3.18), that

$$\begin{cases} \mu_t^q q_t = q_\psi \psi_t \mu_t^\psi + \frac{1}{2} q_{\psi\psi} \left(\psi_t \sigma_t^\psi\right)^2 \\ \mu_t^v v_t = v_\psi \psi_t \mu_t^\psi + \frac{1}{2} v_{\psi\psi} \left(\psi_t \sigma_t^\psi\right)^2 \\ \sigma_t^q q_t = q_\psi \psi_t \sigma_t^\psi \\ \sigma_t^v v_t = -v_\psi \psi_t \sigma_t^\psi. \end{cases} \quad \text{(C.27)}$$

System (C.27), together with the conditions of Definition 2, implies that the following system of first order DAEs holds in equilibrium:

$$\begin{cases} g_t \psi_t \mu_t^\psi + \frac{1}{2} g_\psi \left(\psi_t \sigma_t^\psi\right)^2 = q_t \mu_t^q \\ g_t = q_\psi \\ q_\psi \psi_t \sigma_t^\psi = q_t \sigma_t^q \\ w_t \eta_t \mu_t^\psi + \frac{1}{2} w_\psi \left(\psi_t \sigma_t^\psi\right)^2 = \mu_t^v v_t \\ w_t = v_\psi \\ -v_\psi \psi_t \sigma_t^\psi = v_t \sigma_t^v \\ A - \iota_t = \rho q_t (1-\psi_t) \\ 1 + \theta \iota_t = q_t. \end{cases} \quad \text{(C.28)}$$

In particular, when (3.21) holds with equality, then $\omega_t^b = \frac{1}{\psi_t}$ and $\omega_t^h = 0$, and

(3.22) reduces to

$$-\mu_t^v = \frac{1-(\sigma+\sigma_t^q)}{\psi_t}\sigma_t^v, \tag{C.29}$$

and thus,

$$\sigma_t^v = \frac{\overline{\eta}}{q_t}. \tag{C.30}$$

Moreover, $q(\psi_t)$ can be expressed as a function of ψ explicitly

$$q_t = \frac{1+\theta A}{1+\theta\rho\,(1-\psi_t)}, \tag{C.31}$$

and thus,

$$\mu_t^q = \frac{\theta\rho}{1+\theta\rho\,(1-\psi_t)}\psi_t\mu_t^\psi + \frac{(\theta\rho)^2}{[1+\theta\rho\,(1-\psi_t)]^2}\left(\psi_t\sigma_t^\psi\right)^2, \tag{C.32}$$

$$\sigma_t^q = \frac{\theta\rho}{1+\theta\rho\,(1-\psi_t)}\psi_t\sigma_t^\psi. \tag{C.33}$$

If follows that the drift and diffusion terms of (3.27) reduces to (3.34) and (3.35), respectively.

Finally, by matching the third to last equations of system (C.28) to (C.30) we obtain the following ODE

$$-\frac{\partial v_t}{\partial \psi_t}\frac{\psi\sigma_t^\psi}{v_t} = \frac{\eta}{q_t}, \tag{C.34}$$

which has a unique solution, given the boundary condition $v\left(\underline{\psi}\right) = 1+\lambda$.

It follows that

$$v(\overline{\psi}) = v(\underline{\psi})e^{\int_{\underline{\psi}}^{\overline{\psi}} \frac{1}{\sigma[1+\sigma\theta\rho(1-\psi_t)]}\left[\frac{\eta\rho}{1+\theta A}+\frac{\eta}{(1+\theta A)(1-\psi_t)}\right]d\psi_t}, \tag{C.35}$$

The elementary integral at the exponent of (C.35) can be solved by parts and leads to the uniquely determined upper bound $\overline{\psi}$

$$\overline{\psi} = 1-\left(1-\underline{\psi}\right)\left(\frac{1}{1+\lambda}\right)^\chi, \tag{C.36}$$

where $\chi = \sigma\frac{1+A\theta}{\eta}$, and it is below one as long as $\left(\frac{1}{1+\lambda}\right)^\chi > 0$.

C.4 Welfare

In the unconstrained equilibrium, $\mu^h = r$. Thus,

$$\rho h = \log \rho - 1 + \frac{r}{\rho} + \frac{1}{\rho}\left(\frac{d\Delta}{E^b} - (1+\lambda)\frac{d\Pi}{E^b}\right)\frac{\psi_t}{1-\psi_t} + h_\psi \psi \mu^\psi + \frac{1}{2} h_{\psi\psi}\left(\psi\sigma^\psi\right)^2, \tag{C.37}$$

As long as neither dividends nor recapitalization flows are paid or asked by the banks, (C.37) reduces to

$$\rho h = \log \rho - 1 + \frac{r}{\rho} + h_\psi \psi \mu^\psi + \frac{1}{2} h_{\psi\psi}\left(\psi\sigma^\psi\right)^2. \tag{C.38}$$

Estratto per riassunto della tesi di dottorato

L'estratto (max. 1000 battute) deve essere redatto sia in lingua italiana che in lingua inglese e nella lingua straniera eventualmente indicata dal Collegio dei docenti.

L'estratto va firmato e rilegato come ultimo foglio della tesi.

Studente: Andrea Modena Matricola: 956312

Dottorato: Economics

Ciclo: XXXII

Titolo della tesi : Three Essays in Continuous-time Macro-Finance

Abstract: Questo lavoro consiste di tre articoli che studiano modelli di macrofinanza in tempo continuo in un contesto di equilibrio economico generale. Il capitolo 1 esamina alcuni importanti temi metodologici in relazione ai meccanismi fondamentali prominenti nella neonata letteratura di macrofinanza. I contenuti fungono da base per gli ulteriori sviluppi nei capitoli secondo e terzo. Il capitolo 2 studia l'interdipendenza tra l'attività di pooling del rischio degli intermediari finanziari, la macro-dinamica economica ed il benessere delle famiglie. Secondo il modello, le famiglie beneficiano di più quando il settore finanziario non è né troppo piccolo né troppo grande. Il capitolo 3 sviluppa un modello teorico per studiare in che modo i regimi di risoluzione delle crisi bancarie possano influire sul benessere delle famiglie nel breve e nel lungo periodo. Il modello implica che una risoluzione che complementa le politiche individualmente ottimali di ricapitalizzazione delle banche può migliorare il benessere a lungo termine, anche quando tutti gli agenti sono omogenei all'interno della propria classe e non esiste alcun rischio idiosicratico.

Abstract (Eng): This work consists of three papers on continuous-time general equilibrium models in macro-finance. Chapter 1, reviews some important topics of continuous-time methods as they relate to the core mechanisms prominent in the new-born macro-finance literature. The contents act as a baseline for further developments in the second and third chapters. Chapter 2 studies the inter-dependence between financial intermediaries' risk-pooling activity, economic macro-dynamics and, in turn, households' welfare. According to the model, the households benefit the most when the financial sector is neither too small nor too big. Chapter 3 develops a theoretical model to study how banks resolution regimes may affect households' welfare in the short and in the long-run. We show that a bailout resolution that tops up banks' individual optimal recapitalization policies may improve long-run welfare, even when all actors of the same type are homogeneous, and there does not exist idiosyncratic insurable risk.

Firma dello studente

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