DOCTORAL THESIS

UNIVERSITÉ PARIS 1
PANTHÉON SORBONNE

Università
Ca'Foscari
Venezia

Fixed point, Game and Selection Theory: From the Hairy Ball Theorem to a Non Hair-Pulling Conversation.

Author:
Nadia MAAGLI DRISSA

This thesis has been written within the European Doctorate in Economics - Erasmus Mundus (EDEEM), with the purpose to obtain a joint doctorate degree in economics at the Department of Applied Mathematics, Centre d'Économie de la Sorbonne (CES), Université Paris 1 Panthéon–Sorbonne at the Department of Economics of Ca' Foscari University of Venice.

June 27, 2016

Declaration of Authorship

I, Nadia MAAGLI DRISSA, declare that this thesis titled, “Fixed point, Game and Selection Theory: From the Hairy Ball Theorem to a Non Hair-Pulling Conversation.” and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.

Signed:

Date:

PhD Committee

Advisors:

Prof. **Pascal GOURDEL**, Université Paris 1–Panthéon Sorbonne

Prof. **Marco LICALZI**, Ca' Foscari University of Venice.

Other members:

Prof. **Hichem BEN-EL-MECHAIEKH**, Brock University, Referee.

Prof. **Bernard DE MEYER**, Université Paris 1 Panthéon–Sorbonne.

Prof. **Roberto LUCCHETTI**, Politecnico di Milano, Referee.

Prof. **Yannick VIOSSAT**, Ceremade-Université Paris-Dauphine.

ليس العقل ما يميز الانسان عن الحيوان ولكن الضمير.
توفيق الحكيم

"ما دام يوجد خطأ فلا بد أن يوجد صواب. وإذا وجد الصواب مرة فيمكن أن يوجد مرة أخرى. وإذا كان قد انتكس بعد وجوده فيمكن أن نضمن له حياة لا تعرف الانتكاسة."
نجيب محفوظ

Abstract

This thesis has three main areas of concern namely fixed point theory, selection theory and an application of game theory to cognitive science.

The first step of our work is dedicated to the hairy ball theorem. We mainly introduced an equivalent version to the latest theorem in the form of a fixed point theorem. In this way, ensuring the proof of this equivalent version gives a new insight on the hairy ball theorem. In order to achieve our result, we used different tools as for instance approximation methods, homotopy, topological degree as well as connected components.

In a second phase, we manipulated lower semicontinuous correspondences and we established a selection theorem related to one of Michael's selection theorems. Beyond the existence result, we introduced a new geometric concept of convex analysis, namely "the peeling concept" whose independence from the main result could be explored in other different issues.

In the final chapter, the starting point is an innovative idea of Warglien and Gärdenfors who rely on the theory of fixed points to argue for the plausibility of two individuals achieving a "meeting of minds". But their approach is merely existential. To describe the structure of a possible outcome, we model the process as a simple non cooperative game using conceptual spaces as a new tool of modelization in cognitive science.

Acknowledgements

Firstly, I would like to express my sincere gratitude to my advisors. Many thanks to **Pascal Gourdel**. Thank you for your support, your regular presence and your pedagogical contribution over these past years that I cannot count any more.

Many thanks also to **Marco LiCalzi**. I enjoyed working in a unique environment of hard working, stimulating original thinking with an incredible view in a breathtaking city.

Besides my advisors, I would like to thank the rest of my thesis committee: **Hichem Ben-El-Mechaiekh**, **Bernard De Meyer**, **Roberto Lucchetti** and **Yannick Viossat** for their insightful comments, but also for taking time to review my work.

I would also like to show gratitude to **Philippe Bich**. His teaching style and enthusiasm made a strong impression on me. I have also benefited from constructive comments during our discussions.

I am also grateful to **Jean-Marc Bonnisseau** for not missing a beat helping me and helping the students most in need.

I would also express my warm thanks to the department UFR 27 of Paris 1 especially Brigitte Augarde and Corinne Aboulker. Brigitte thank you for keeping always your office door open for me with a maternal smile and affection and Corinne I appreciate your dynamism and professionalism.

I also thank the Axe Math & Jeux, in particular **Agnieszka Rusinowska** for trying everyday to improve our research environment, Elda André, Joel Faron, Marie-Lou Margaria, Michela Siciliano and Maria Varela for their welcome and their extreme kindness.
My sincere thanks also goes to all EDEEM staff. In particular, I am grateful to **Bertrand Wigniolle**, Caroline Dutry, Lisa Negrello and Guillaume Glade.

I place on record, my sincere thank you to **Rim Lahmandi Ayed** who gave me a taste for applied mathematics at l'Ecole Polytechnique de Tunisie and to all my professors at l'Ecole Normale Supérieure de Tunis.

I am also grateful to Abdennebi Achour. I am indebted to him for attention support and the thrill of a brief taste of humour and freedom since my childhood.

Getting through my phd thesis required more than academic support, and I have few people to thank for listening to all my stories over the past years. I cannot express my appreciation for their friendship: Armagan and her little princesses, Carla, Lalaina and Thais. For many memorable days, lunch breaks, lots of laughs (Thais's voice will resound forever as a message of peace!). Lalaina thank you for being always a faithful guardian angel for me and for all of us: the GBA members!

I do not forget our incredible team at the MSE: Abhi, Aurélien, Clément and Mathiew (I will never forget our Venice experience together), Cuong, Fatma, Federica, Florent and our common sense of humour, Gäetan you are still my best Frenshy (after Atef), Hyejin, Katya and her beautiful Russian smile, Lorenzo, Mustapha, Nikos my best Greek (unless one day Atef will decide to become Greek), Okay, Paulo, Peter (we miss your smile, since your are getting married maybe we are going to miss it more!), Salah, Sang, Seba, Simon, Sinda, Stéphane "Qui aime bien, châtie bien...ou pas", Veronica, Vincenzo, Xavier (I would have loved to know you sooner) , Yacoub and Zeineb.

I also owe a lot to my favourite twins during all those years, Amine and Anis, I am lucky to have you. Many thanks to my closed friends Hajer, Maha (and Solla), Meriem (and Dadouma), Nesrine (and Titou), evidence that distance takes away but never separates.

Most importantly, I would also like to thank my family.
It would be an understatement to say that none of this could have happened without my father. I am forever grateful. I cannot imagine having my life without you.
My mother who offered her encouragement through calls every day, almost every hour. Even my Italian friends found this too much!
I am also grateful to have a second mother who dedicated her time and life to take care of us. Thank you Foufou, I can not even imagine the supreme sacrifices that you made for us.

Many thanks to my two brothers, my grand-mother, my in-laws, all my uncles, aunts and my cousins. Do not be surprised because I did not tell your names, you are so numerous that the earth would have to be much larger in order to hold you. You are my light and I love you all.

Finally, none of this could have happened without the man of my life. You have been my backbone for so many years in Tunis, Paris, Venice and everywhere. Thank you for always pushing me to bring out the best in me. Thank you for putting up with my mood swings, bad temper and my “always Mrs right” attitude. Above all, thank you for Camélia Frida, who looks just like you with her fresh eyes looking at the world. No matter what happens, I will be here for you and I will love you both forever (remember forever is a relative concept!).

Nadia MAAGLI DRISSA
Paris, June 2016.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

Fixed point theory is at the heart of nonlinear analysis. Actually, it provides many tools in order to get existence results in many nonlinear problems. A fixed point theorem is a result asserting that under some conditions a function f has at least one fixed point, that is, there exists x such that $f(x) = x$. One of the first examples easy to be understood by a student with basic knowledge in calculus and analysis is the following (you may consider Figure 1.1): Any continuous function f from the closed unit interval to itself has at least one fixed point. Indeed, suppose by contradiction that f has no fixed point in $[0, 1]$, then $f(0) > 0$. Then, it must be $f(x) \geq x$ for all x, otherwise a fixed point occurs. This implies that $f(1) > 1$, which establishes a contradiction.

From this basic example, one might wonder what kind of func-

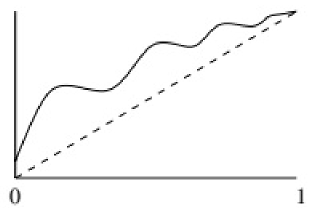


FIGURE 1.1: Fixed point on $[0, 1]$

tions on what type of sets can generate fixed points. Since then, several results have been emerged and fixed point theorems are considered amongst the most useful in mathematics with several types of applications. Here, we can distinguish two main approaches: a non constructive and a constructive approach. Namely, Brouwer's fixed point theorem and Banach fixed point theorem. The result provided by Brouwer (1910) stated that any continuous function from the closed unit ball of $\mathbb{R}^n$, denoted by B^n, into itself admits a fixed point. It is worth noting that the result of Brouwer holds true not only for B^n but also for any compact convex set of $\mathbb{R}^n$. However, the fixed point is not necessarily unique. We can ask the following: How

can we strengthen the hypothesis to get the uniqueness?
To get an idea, we consider the following example nearly identical to our simple first example. Let $f : [-1, 1] \to [-1, 1]$ be a continuous function. Brouwer's theorem for the case $n = 1$ stated that f has at least a fixed point x^*. Suppose that there exists a second solution $\tilde{x} \neq x^*$, then we have $\frac{f(\tilde{x})-f(x^*)}{\tilde{x}-x^*} = 1$. So, if for instance, f is differentiable satisfying $f'(x) < 1$, for any $x \in [-1, 1]$, then by the mean value theorem, we can guarantee the uniqueness of the fixed point.
In n dimension, a natural generalization is to look to the quantity $\frac{\|f(\tilde{x})-f(x^*)\|}{\|\tilde{x}-x^*\|}$, for $\tilde{x} \neq x^*$, potentially in more complicated spaces than $\mathbb{R}^n$. A related finding is given by the Banach fixed point theorem stated below (1922). Unlike Brouwer's theorem, its approach is constructive. Indeed, under specific conditions, the procedure of iterating a function yields a unique fixed point.
Before working in depth on Brouwer's fixed point theorem, we will recall the Banach fixed point theorem and some applications.

1.0.1 The Banach fixed point theorem

First, we recall the Banach fixed point theorem.

Theorem 1.0.1 (Banach, (1922)) [4] Let (E, d) be a complete non empty metric space and let $T : E \to E$ be a contraction (i.e there exists $k \in (0, 1)$ and $(x, y) \in E \times E$ such that $d(T(x), T(y)) \leq kd(x, y)$), then T has a unique fixed point x_0 in E. Besides, the fixed point x_0 can be calculated as the limit of $T^n(z)$, for any $z \in E$.

Note that in addition to the capital contribution ensured by the uniqueness, the error of the distance between x_n and the fixed point is explicitly given by the contraction constant k and the initial displacement $d(x_0, x_1)$.
In order to gain some intuition about the importance of the Banach fixed point theorem, we will start by exploring the example of Rousseau [2]. We take the famous box of the laughing cow (see Figure 1.2). The right earring of the cow is also a laughing cow. We consider the function from the cover to itself such that to each point of the cover, we associate the corresponding point on the right earring. An interesting question is the following: is there a fixed point of this function? Remark that this function has a special property. Since the images are much smaller than the domain, then it is a contraction in $\mathbb{R}^2$, which is a complete metric space. Therefore, the above Banach fixed point theorem states that our process has a unique fixed point.

FIGURE 1.2: The laughing cow cover

The joy of this theorem is that the uniqueness of the fixed point is not visible to the naked eye.

Applications

Despite the numerous applications in analysis of the Banach fixed point theorem, the existence and uniqueness of solutions of some class of differential equations remain among the most known. For instance, we consider the existence of solutions for the Volterra integral equation of the second kind [8] given by the following.

Theorem 1.0.2 Let $K : [0,T] \times [0,T] \times \mathbb{R} \to \mathbb{R}$ be continuous and suppose it satisfies a Lipschitz condition

$$|K(t,s,x) - K(t,s,y)| \leq L|x-y|$$

for all $(s,t) \in [0,T] \times [0,T]$, $x, y \in \mathbb{R}$ and $L \geq 0$. Then for any $v \in C([0,T])$ the equation

$$u(t) = v(t) + \int_0^t K(t,s,u(s))ds, \quad (0 \leq t \leq T)$$

has a unique solution $u \in C([0,T])$.

The above result is also connected to the next differential equation problem formulated by the next theorem.

Theorem 1.0.3 Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be continuous and suppose it satisfies the Lipschitz condition

$$|f(s,x) - f(s,y)| \leq L|x-y|$$

for $s \in [0,T]\,, x, y \in \mathbb{R}$, then the Cauchy problem

$$\frac{du}{ds} = f(s,u), \quad u(0) = 0$$

has exactly one solution u defined on the entire interval $[0, T]$.

It is worth noting that some further facts are also important. For instance, a second application of the Banach fixed point theorem is also to prove the implicit function theorem [13]. For more recent applications, one can consult for example the application studied by Rousseau [2] about the image compression.
Now, let us return to the matter at hand, the Brouwer's fixed point theorem.

1.0.2 Brouwer's fixed point theorem

Theorem 1.0.4 (Brouwer, (1912)**)** Let B^n be the closed n-ball, and suppose that f is continuous from B^n to itself. Then, there exists some $x^* \in B^n$, such that $f(x^*) = x^*$.

Before approaching the historical aspect of Brouwer's fixed point theorem, note that we can easily prove that the regular version of Brouwer's fixed point theorem (C^1 version) is equivalent to the continuous version. In 1909, Brouwer met a very important circle of french mathematicians: Borel, Hadamard and Poincaré. The fruit of this encounter brought Brouwer to exchange his ideas with Hadamard. At this time, Brouwer was able to prove his theorem only for the two dimensional case and it was Hadamard who in 1910 gave the first proof for any arbitrary finite dimension. Yet, in 1912, Brouwer found an alternative proof independent of Hadamard's proof. It is worth noting that in 1886, Poincaré [7] published his results about the homotopy invariance of the index. Poincaré's result was implicitly rediscovered by Brouwer in 1911. However, we waited until 1940, the year in which Miranda proved that Poincaré's result is equivalent to Brouwer's theorem and since then has been known as the Poincaré-Miranda theorem. For more details, see for example Kulpa [30].
All this made Brouwer's fixed point theorem the cradle of algebraic topology. Thus, Brouwer's proof are essentially topological. For a deeper survey, the reader can consult Stuckless [27]. Among the full range of proofs, we can cite the proof of Knaster-Kuratowski-Mazurkiewiez (1929) using the lemma of Sperner. This gave birth to the KKM theory but we will not follow this direction of study. However, our interest is focused on equivalent results to Brouwer's fixed point theorem. Here, we decided to present the proof of a relatively early equivalence between Brouwer's fixed point theorem and the

non retraction theorem. Then, our attention is devoted to underline the tight association between the works of Poincaré and Brouwer.

From the non retraction theorem to Brouwer's theorem

We recall that a retraction from a topological space to a subset of this space is a continuous application from this space with values in the subset whose restriction to this subset is the identity. Formally, we have

Definition 1 If Y is a subset of $\mathbb{R}^n$, and $X \subset Y$, then a retraction $r : Y \to X$ is a continuous mapping such that $r(x) = x$ for all $x \in X$.

Here, we announce the C^1 version of the non retraction theorem. Indeed, one of the motivations behind it is that usually we adopt a proof using differential computations where the C^1 character is needed.

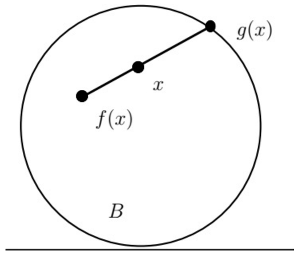
Theorem 1.0.5 (The non retraction Theorem) There exists no C^1 retraction from the unit ball of $\mathbb{R}^n$, $B^n$to the unit sphere, S^{n-1}.

Proof of the equivalence

$$\text{Brouwer's fixed point theorem } (A) \Leftrightarrow \text{the non retraction theorem } (B)$$

We will prove equivalently that non (B) is equivalent to non (A). We start by proving that non (B) implies non (A). Suppose that the non retraction theorem is not verified, then there exists $r : B^n \to S^{n-1}, C^1$ such that $r(x) = x$, for any $x \in S^{n-1}$. Let $f : B^n \to B^n$, $x \to -r(x)$. The function f is clearly continuous and for any $x \in S^{n-1}$, $f(x) = -x \neq x$. Therefore, f has no fixed point, which establishes the result. Conversely, suppose that Brouwer's fixed point theorem doesn't hold true. This implies, that there exists $f : B^n \to B^n$, such that $f(x) \neq x, \forall x \in B^n$. Let $u(x) : B^n \to S^{n-1}$ defined by $u(x) = \frac{f(x)-x}{\|f(x)-x\|}$. Then, we consider the line passing through x and directed by $u(x)$. We will denote by $g(x)$ the intersection point of this line and the sphere on x side (see the following figure).

By construction, g belongs to S^{n-1} and if $x \in S^{n-1}$, then $g(x) = x$. It remains to check that g is C^1. Without loss of generality, we can suppose that f is C^1. Therefore, the function u is also C^1. Now, by definition, there exists $t(x) \geq 0$, such that $g(x) = x + t(x)u(x)$. Solving $\|g(x)\|^2 = 1$, leads to the value of $t(x)$ given by $t(x) = -x.u(x) + \sqrt{1 - \|x\|^2 + (x.u(x))^2}$. It suffices to remark that the quantity under the root square is positive and conclude that t is C^1.



That is g is C^1 and then the non retraction theorem doesn't hold true, which proves the result.
The next section is devoted to the connection between some works of Poincaré and Brouwer.

Poincaré-Miranda theorem

Mentioning again the Brouwer's fixed point theorem for $n = 1$. It is easy to see that the proof is trivial since it is an immediate consequence of the intermediate value theorem which says that a continuous function $f : [a, b] \to \mathbb{R}$ such that $f(a).f(b) \leq 0$ must have a zero at some point in $[a, b]$. As noticed in Browder [6], Poincaré, in 1884, announced the following without proof:
" Let $\xi_1, \xi_2, \cdots \xi_n$ be n continuous functions of n variables $x_1, x_2, \cdots x_n$: the variable x_i is subject to vary between the limits a_i and $-a_i$. Let us suppose that for $x_i = a_i$, ξ_i is constantly positive, and that for $x_i = -a_i$, ξ_i is constantly negative; I say there will exist a system of values of x for which all the $\xi's$ vanish".
Poincaré proved this result two years after. However, in 1940, the result became interlinked with Miranda who proves that it's equivalent to Brouwer's fixed point theorem.

Theorem 1.0.6 (The Poincaré-Miranda theorem, (1940)) Let $\Omega = \{x \in \mathbb{R}^n : |x_i| \leq a_i, i = 1, \cdots, n\}$ and $f : \Omega \to \mathbb{R}^n$ be continuous satisfying

$$f_i(x_1, x_2, \cdots, x_{i-1}, -a_i, x_{i+1}, \cdots, x_n) \geq 0, \text{for } 1 \leq i \leq n$$
$$f_i(x_1, x_2, \cdots, x_{i-1}, a_i, x_{i+1}, \cdots, x_n) \leq 0, \text{for } 1 \leq i \leq n$$

Then, $f(x) = 0$ has a solution in Ω.

It is important to notice that we can reformulate the Poincaré-Miranda theorem as a fixed point theorem (see for example [28]), where the assumptions are unchanged. This is given by the following theorem.

Theorem 1.0.7 Let $\Omega = \{x \in \mathbb{R}^n : |x_i| \leq a_i, i = 1, \cdots, n\}$ and $f : \Omega \to \mathbb{R}^n$ be continuous satisfying

$$f_i(x_1, x_2, \cdots, x_{i-1}, -a_i, x_{i+1}, \cdots, x_n) \geq 0, \text{for } 1 \leq i \leq n$$
$$f_i(x_1, x_2, \cdots, x_{i-1}, a_i, x_{i+1}, \cdots, x_n) \leq 0, \text{for } 1 \leq i \leq n$$

Then, $f(x) = x$ has a solution in Ω.

A key point of this theorem is that the function f is not self mapping as it is quite frequent in fixed point theorems.
Now, once more, a common feature between Brouwer and Poincaré is given by the following result.

Theorem 1.0.8 (The hairy ball theorem, (1912)) Let $f : S^{2n} \to \mathbb{R}^{2n+1}$ be continuous such that for any $x \in S^{2n}$, the scalar product $f(x).x = 0$, then there exists $\overline{x}$ such that $f(\overline{x}) = 0$.

The above theorem can be seen as a corollary of the Poincaré-Hopf index theorem. Note that the case $n = 1$ was independently proved by Poincaré in 1885. In 1926, Hopf proved the full result (for more details see for example [11, 18]). However, in 1912, Brouwer provided an independent proof of the hairy ball theorem. We dedicate the next section to the hairy ball theorem which is our second chapter of this thesis.

1.1 Chapter 2: The hairy ball theorem

First, consider Figure 1.3, remark that it is not difficult to see that S^1 has a non vanishing continuous tangent vector field. Now, for S^2

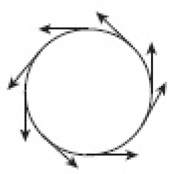


FIGURE 1.3: A non vanishing continuous tangent vector field on S^1.

imagine brushing a ball perfectly covered with fine hair all over. After brushing, there is no way that all the hair lies flat on the sphere. One hair will stand up perpendicular to the surface. An another intuition is also to say that winds moves in continuous stokes over the surface of the earth. Since the earth is assimilated to a sphere, then there is one point at which there is no wind (See Figure 1.4). However, despite the inspiration for curious people wishing to chase

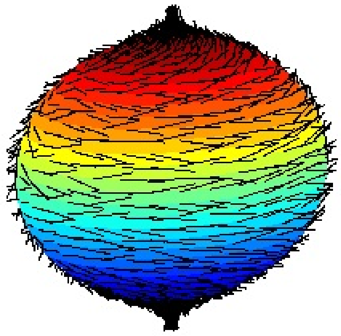


FIGURE 1.4: A continuous tangent vector field on S^2.

a point where the wind is not blowing off, the hairy ball theorem presents a lot of applications not only limited to mathematics but also to physics, (see for example [29, 21]).
There are several ways in order to prove the hairy ball theorem. Brouwer's proof is based on homotopy and the degree of a mapping. Brouwer's and the hairy ball theorem were proved using essentially algebraic topology. In 1978, Milnor gave an analytic proof for both theorems from a differential point of view and did not make effective use of algebraic topology. Doubtless, there exists plenty of proofs of the hairy ball theorem that are analytic. We think that Milnor's proof is worth quoting. Indeed, in his paper [10], Milnor wrote "This note will present strange but quite elementary proofs of two classical theorems of topology based on a volume computation in Euclidean space and the observation that the function $(1+t^2)^{\frac{n}{2}}$ is not a polynomial when n is odd". We dedicate the next section to a sketch of Milnor's proof.

1.1.1 Milnor's proof

Theorem 1.1.1 An even dimensional sphere does not admit any continuous field of non-zero tangent vectors.

As explained by Milnor, the proof of the above theorem is based on the following Lemmas (whose proofs are omitted).

Theorem 1.1.2 An even-dimensional sphere does not possess any continuously differentiable field of unit tangent vectors.

Let $A \subset \mathbb{R}^{n+1}$ be a compact set and $x \to v(x)$ be a continuously differentiable vector field which is defined throughout a neighborhood of A. Consider for any $t \in \mathbb{R}$, $f_t : A \to \mathbb{R}^{n+1}$, $x \to x + tv(x)$.

Lemma 1.1.1 If the parameter t is sufficiently small, then the function f_t is one to one and transforms the region A onto a nearby region $f_t(A)$ whose volume can be expressed as a polynomial function of t.

Lemma 1.1.2 If t is sufficiently small, then $f_t : S^{2k} \to \sqrt{1+t^2}S^{2k}$ is onto.

Before stating Milnor's proof, note that a curious fact related to the Banach fixed point theorem is that Milnor gave two proofs for Lemma 1.1.2, the first one is based on the Banach fixed point theorem and the second one on the inverse function theorem.
Proof of Theorem 1.1.1
By contradiction, suppose that the sphere possesses a continuous field of non zero tangent vectors $v(x)$. Let us consider $m = \min\left\{\|v(x)\|, x \in S^{2k}\right\}$. First, note that $m > 0$. Second, by the Weierstrass approximation theorem, there exists a polynomial $p : S^{2k} \to \mathbb{R}^{n+1}$ satisfying $\|p(u) - v(u)\| < \frac{m}{2}$, for any $u \in S^{2k}$. Define $w(u) = p(u) - (p(u).u)u$, for any $u \in S^{2k}$. In order to conclude, it suffices to remark that $w(u)$ is tangent to S^{2k} at u. Besides, the above inequality and the triangle inequality show that $w \neq 0$, then applying Theorem 1.1.2 to $\frac{w(u)}{\|w(u)\|}$ yields to a contradiction.

1.1.2 Main results

The purpose of the first chapter is essentially to introduce an equivalent version of the hairy ball theorem in the form of a 'fixed point theorem'. In this way, ensuring the proof of this equivalent theorem gives a new insight on the hairy ball theorem. The result is given by the following.

Theorem 1.1.3 Let $f : S^{2n} \to S^{2n}$ be a continuous function such that for any $x \in S^{2n}$, $f(x).x \geq \frac{1}{2}$, then it possesses a fixed point.

In the sequel, to abbreviate notations, we will denote by $S := S^{2n}$. In order to prove our main theorem, we built an explicit function α such that α possesses the north pole x_0 and the south pole $-x_0$ as unique fixed points. Then, we construct an homotopy F between f and α. As it is classical, we can think of F as a time indexed family of continuous maps such that when time t varies from 0 to 1, F continuously deforms α into f. The main result stated that there exists

a connected component connecting one fixed point of α and at least one fixed point of f. Formally, our result is summarized by the following.
Let $F : [0,1] \times S \to S$ be a continuous function, we denote by $C_F := \{(t,x) \in [0,1] \times S : F(t,x) = x\}$ and $\mathscr{H}$ the set of continuous functions $F : [0,1] \times S \to S$ such that $C_F \cap (\{0\} \times S) = \{x_0, -x_0\}$.

Proposition 1.1.1 If $f : S \to S$ is continuous such that for any $x \in S$, $f(x).x \geq \frac{1}{2}$, then there exists $F \in \mathscr{H}$ such that $F(1,.) = f$.

Theorem 1.1.4 Let F be the function given by Proposition 2.4.1, then there exists a connected component Γ subset of C_F such that $\Gamma \cap (\{0\} \times S) \neq \emptyset$ and $\Gamma \cap (\{1\} \times S) \neq \emptyset$. Consequently, $F(1,.)$ has a fixed point.

Now, it is clear that once we proved Proposition 1.1.1 and Theorem 1.1.4, then Theorem 1.1.3 is deduced immediately.
On the other hand, it is important to notice that implementable methods are desirable for computing fixed points but only available for following smooth paths. Unfortunately, we have no guarantee that the connected component Γ, obtained in Theorem 1.1.4, is smooth (see Figure 1.5).

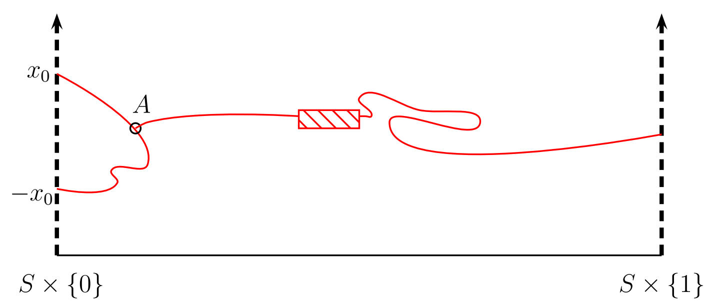


FIGURE 1.5: Example of a non smooth path Γ

At the point A, a bifurcation occurs creating a loss of the direction to follow and the shaded area makes the tracking even harder. In the sequel, the purpose is to present the method to find the desirable smooth case. But, before giving our formal results, we present here the main idea of the process.
Indeed, the key intuition is to approximate F by a sequence of smooth mappings G_p. This allows us to obtain a sequence of smooth connected components Γ_p. Then, applying a result of Mas-Colell, we claim that every Γ_p such that Γ_p intersect $\Omega \times \{0\}$ is diffeomorphic to

a segment, where Ω is the open set given by $\Omega = B(S, 1/2)$.
Moreover, recycling the same argument used for F, we prove that for p large enough, there exists a pull through connected component intersecting $\Omega \times \{1\}$ (see Figure 1.6).

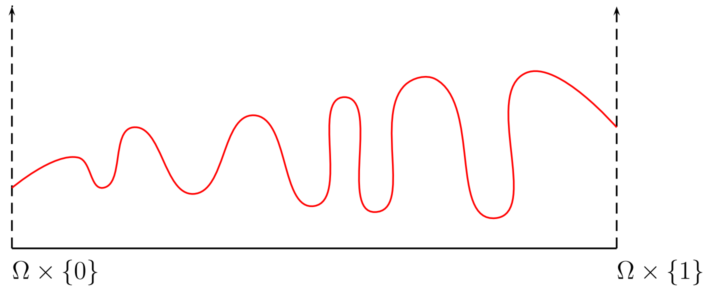


FIGURE 1.6: Example of a smooth path Γ_p

Let Ω be an open subset of $\mathbb{R}^n$. Let $A \subset \Omega$ be open and such that $\overline{A} \subset \Omega$ and let $\mathbb{F}$ be the set of twice continuously differentiable functions: $F : [0,1] \times \Omega \to A$.

Formally, our result is given by the following proposition.

Proposition 1.1.2 There exists an open dense set $\mathbb{F}' \subset \mathbb{F}$ and a function $\widetilde{G}_p : [0,1] \times \Omega \to A$, where A is the neighbourhood of S given by $A = B(S, \frac{3}{8})$ such that

1. For any $p \in \mathbb{N}$, $\widetilde{G}_p \in \mathbb{F}'$ and for any $(t, x) \in [0,1] \times \Omega$, we have $\left\| \widetilde{G}_p(t,x) - F(t, \frac{x}{\|x\|}) \right\| \leq \frac{1}{p}$.
2. For p large enough, there exists a connected component $\Gamma_p \subset C_{\widetilde{G}_p}$ such that $\Gamma_p \cap (\{0\} \times \Omega) \neq \emptyset$ is diffeomorphic to a segment.
3. For p large enough, there exists a connected component $\Gamma_p \subset C_{\widetilde{G}_p}$ such that $\Gamma_p \cap (\{0\} \times \Omega) \neq \emptyset$ and $\Gamma_p \cap (\{1\} \times \Omega) \neq \emptyset$.

Finally, to recover the continuous case, we partially followed a useful hint given by Milnor [11]. Indeed, while proving that the regular version C^1 of Brouwer's fixed point theorem is equivalent to the continuous version, Milnor highlighted a very important point. Since the proof is based on Weierstrass approximation theorem, he wrote the following: " to prove a proposition about continuous mappings, we first establish the result for smooth mappings and then try to use an approximation theorem to pass to the continuous case".
In our case, based on the result on $\widetilde{G}_p$ and in the spirit of Kuratowski's limit, we construct $Z \subset C_F$ such that Z is connected and pull through.

This concludes our second chapter of the thesis. In the next section, we examine an another branch related to Brouwer's fixed point theorem.

1.2 Brouwer's theorem extensions

After reviewing some equivalent results to Brouwer's fixed point theorem in $\mathbb{R}^n$, we consider now Brouwer's fixed point theorem extensions. Many paths could be taken since we can extend the convexity, compactness, continuity, singlevaluedness and the finite dimension. A first direction to take is the following: What about normed linear spaces? Can we extend the result to more general sets?

1.2.1 Topological extension

Before announcing the next result, we recall that a map $f : X \to Y$ is called compact if $f(X) \subset K$, for some compact set K subset of Y. One of the most famous results taking this path are the following.

Theorem 1.2.1 (Schauder's fixed point theorem, 1927) Let C be a convex subset of a normed linear space E . Then each compact map $f : C \to C$ has at least a fixed point.

In 1930, Mazur showed that the convex closure of a compact set in a Banach space is compact. This yields a more general version of the Schauder theorem (See for example [8]).

Theorem 1.2.2 Let C be a convex closed subset of a Banach space E and $f : C \to C$ continuous with compact values. Then, f has a fixed point.

We can weaken the hypothesis on f and strength those on C. Clearly, a continuous mapping on a compact set is compact. The result is given by the following.

Theorem 1.2.3 Let C be a convex compact subset of a Banach space E and $f : C \to C$ continuous. Then, f has a fixed point.

Another perspective is to weaken the hypothesis on the regularity of the space. Then, another result is derived for locally convex topological vector spaces and known as the Tychonoff's fixed point theorem.

Theorem 1.2.4 (Tychonoff's fixed point theorem, 1934) Let C be a convex compact subset of E a locally convex topological vector space and $f : C \to C$ continuous. Then, f has a fixed point.

The next result is due to Hukuhara [17] who unified the two last results and the theorem is known as the Schauder-Tychonoff theorem.

Theorem 1.2.5 (Schauder-Tychonoff, (1950)) Let C be a non empty convex subset of a locally convex linear topological space E, and let $f : C \to C$ be a compact map. Then, f has a fixed point.

Now, we will present an another approach in order to generalize the Brouwer's fixed point theorem. Namely, staying on $\mathbb{R}^n$, we extend the theorem to multifunctions.

1.2.2 Multimaps extension

By a multifunction, we mean a map $\varphi : X \to 2^Y$ which assigns to each point $x \in X$ a subset $\varphi(x)$ of Y. Remark that a single-valued map $f : X \to Y$ can be identified with a multivalued mapping $\varphi : X \to 2^Y$ by setting $\varphi(x) = \{f(x)\}$. In the sequel, the word multifunction is substituted by the word correspondence. Given that Brouwer's theorem is based on the continuity of the function being considered, it is possible to ask what types of continuity can be generalized to multifunctions. For single valued functions, we say that $f : X \to Y$ is continuous if given any open set of Y, the preimage by f is open in X. However, the notion of preimage for correspondences differs from the classical one. Here, we have the generalization of the concept of continuity to set valued mappings.

Definition 2 Let $\varphi : X \to 2^Y$ be a correspondence, $B \subset Y$. We define by

$$\varphi^+(B) = \{x \in X \mid \varphi(x) \subset B\}, \qquad \varphi^-(B) = \{x \in X \mid \varphi(x) \cap B \neq \emptyset\}.$$

Definition 3 Let X and Y be topological spaces and $\varphi : X \to 2^Y$ be a correspondence.

1. φ is called upper semicontinuous on X if
   - For all open set $V \subset Y$, $\varphi^+(V)$ is open.
   - For all closed set $V \subset Y$, $\varphi^-(V)$ is closed.
2. φ is called lower semicontinuous on X if

- For all open set $V \subset Y$, $\varphi^{-}(V)$ is open.
- For all closed set $V \subset Y$, $\varphi^{+}(V)$ is closed.

3. φ is called continuous if it is both lower semicontinuous and upper semicontinuous.

Note that if φ is identified with a single valued map f, then the two notions of semicontinuity coincide with the classical continuity of f. Kakutani's fixed point theorem was the first result about correspondences. This section is devoted to this theorem and some related results.

Kakutani's fixed point theorem

The result is stated as follows.

Theorem 1.2.6 (Kakutani's theorem (1941)) Let K be a non empty compact convex subset of $\mathbb{R}^n$. Let $\varphi : K \to 2^K$ an upper semicontinuous correspondence with nonempty closed convex values. Then φ has a fixed point.

In order to understand the notion of fixed points for correspondences, Franklin [9] used a simple and homely metaphor: "Let X be the set of all men. If x is a man, let $F(x)$ equal the subset of men whose manners are known to the man x. Kakutani's theorem talks about the relationship $x \in F(x)$, which says that x is a member of the subset $F(x)$. In our example a man x lies in $F(x)$ if he knows his own name. In other words, $x \in F(x)$ unless x has amnesia".
In order to draw an analogy with the first part, like Brouwer's theorem, Kakutani's theorem has been generalized also to topological vector spaces. The result is known as Kakutani-Fan-Glicksberg theorem.

Theorem 1.2.7 Let X be a locally convex topological vector space and let $K \subset X$ be non empty compact and convex. Let $\varphi : K \to 2^K$ an upper semicontinuous correspondence with nonempty closed and convex values. Then, φ has a fixed point.

This is only the beginning of hundreds of results concerned with the class of upper semicontinuous correspondences with closed convex values, (commonly called Kakutani maps). A mix of many methods were being used to prove Kakutani's theorem, while the initial proof (see for example [9, 25]) is based on the Brouwer's fixed point theorem and geometric techniques on simplexes.

In the same perspective of Brouwer's fixed point theorem, we are stating one of the valuable results which is equivalent to Kakutani's fixed point theorem, namely Von Neumann intersection lemma given by the following.

Proposition 1.2.1 (Von Neumann's intersection lemma (1937)) Let $X \subset \mathbb{R}^m$, $Y \subset \mathbb{R}^n$ be compact convex sets and M, N closed subsets of $X \times Y$ satisfying

$$\forall x \in X, M_x = \{y \in Y : (x,y) \in M\},$$

$$\forall y \in Y, N_y = \{x \in X : (x,y) \in N\},$$

are non empty and convex. Then $M \cap N$ is a non empty compact set.

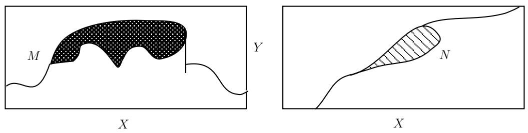


FIGURE 1.7: Von Neumann's intersection lemma

This lemma is deduced directly from Kakutani's fixed point theorem. However, the equivalence is due to Nikaido [22].
Like Poincaré and Brouwer, Kakutani and Von Neumann works were frequently described together. For instance, Kakutani's theorem was developed in many ways. As part of this development, Kakutani's theorem can be seen as a direct consequence of Cellina theorem [15] which can be proved using Von Neumann's approximation lemma (1937). Both results that have just been announced are given by the following.

Proposition 1.2.2 (Von Neumann's approximation lemma, (1937)) Let $\Gamma : E \to F$ be an upper semicontinuous correspondence with compact convex values, $E \subset \mathbb{R}^m$ compact and $F \subset \mathbb{R}^k$ convex. Then, for any $\varepsilon > 0$, there exists f continuous such that $Gr(f) \subset N_\varepsilon(Gr(\Gamma))$, where $N_\varepsilon(Gr(\Gamma)) = \cup_{x \in Gr(\Gamma)} B(x, \varepsilon)$.

Theorem 1.2.8 (Cellina, 1969) Let $K \in \mathbb{R}^m$ be nonempty, compact and convex, and let $\mu : K \to K$. Suppose that there is a closed correspondence $\gamma : K \to F$ with nonempty compact convex values, where $F \subset \mathbb{R}^k$ is compact and convex, and a continuous $f : K \times F \to K$ such that for each $x \in K$, $\mu(x) = \{f(x,y), y \in \gamma(x)\}$. Then, μ has a fixed point.

Remark that, in Von Neumann's approximation lemma, the problem is translated to functions. Here, a function approximate the correspondence. It can be also a selection of the correspondence. This branch is known as the selection theory. We decided to propose another intuitive and basic way to look at the Kakutani's theorem. If the essential details of the proof are not known, in our opinion, a natural question will arise: what about the class of lower semicontinuous correspondences? What class of fixed points theorems is available? Actually, there is relatively not many fixed points related to the above class. Yet, in 1956, Michael's interest to this class gave rise to a new theory, called Selection theory. Chapter 3 of this thesis is devoted mainly to this branch of study.

1.3 Chapter 3: Selection theory

1.3.1 Michael's selection theorems

During many years, the class of lower semicontinous multifunctions (commonly called Michael maps) has been the subject of several generalizations. First, recall that a single valued mapping $f : X \to Y$ is called a selection of $\varphi : X \to 2^Y$ if for any $x \in X$, $f(x) \in \varphi(x)$.
The most interesting part is to prove the existence of continuous selections. The first result of Michael is given by the following.

Theorem 1.3.1 Let X be a paracompact space, Y a Banach space and $\varphi : X \to 2^Y$ a lower semicontinuous correspondence with nonempty closed convex values. Then φ admits a continuous single-valued selection.

Not surprisingly, note that the above theorem was generalized in multiple ways by modifying the assumptions on both sets X and Y and also by introducing many generalizations of the convexity. However, all theorems were obtained for closed-valued correspondences. One of the selection theorems obtained by Michael in order to relax the closeness restriction is the following.

Theorem 1.3.2 Let X be a perfectly normal space[1], Y a separable Banach space and $\varphi : X \to Y$ a lower semicontinuous correspondence

[1]A topological space is called perfectly normal if it is normal and every closed subset is a G_δ subset, where a subset is G_δ if it is expressible as a countable intersection of open subsets.

with nonempty convex values. If for any $x \in X$, $\varphi(x)$ is either finite dimensional, or closed, or has an interior point, then φ admits a continuous single-valued selection.

An interesting question may arise: Is it possible to relax the separability of Y? To answer this question, Michael provided in his paper a counter example showing that the separability of Y can not be omitted. Even though, the correspondence has open values, Michael established an overall conclusion. Therefore, another interesting question might be asked: Under which conditions, can we omit the separability of Y when the correspondence has either finite dimensional or closed values. Chapter 3 aims to prove that for the metric case the answer to this question is affirmative.

1.3.2 Main Results

We start by recalling that if X is a metric space, then it is both paracompact and perfectly normal. In many applications, both paracompactness and perfect normality aspects are ensured by the metric character. Besides, perfectly normality does not imply paracompactness nor the converse. Namely, the metric case unify the two properties (for more details see the following diagram (Michael [5])).

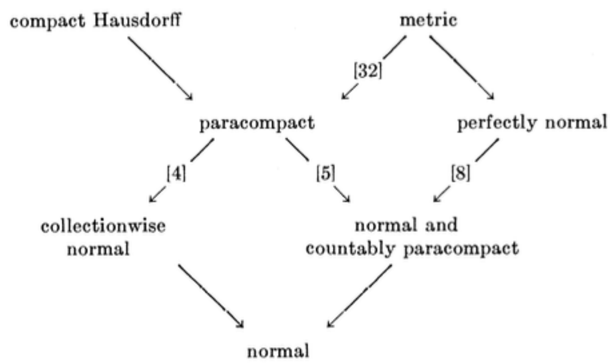


FIGURE 1.8: Connection between some topological spaces

Therefore, we assume that (X,d) is a metric space and $(Y,\|.\|)$ is a Banach space. We denote by $\dim_a(C) = \dim \operatorname{aff}(C)$ the dimension of C. Moreover, we recall that the relative interior of a convex set C is a convex set of same dimension and that $\operatorname{ri}(\overline{C}) = \operatorname{ri}(C)$ and $\overline{\operatorname{ri}(C)} = \overline{C}$. One important tool of our proof is the introduction of the concept of peeling of a finite dimensional convex set.

As a first step, this geometric tool enables us to extend by induction the result when the dimension of the image is finite and constant and then to establish the result on X. Therefore, the main result is given by the following.

Theorem 1.3.3 Let $D_i := \{x \in X, \dim_a \varphi(x) = i\}$ and $\varphi : X \to 2^Y$ be a lower semicontinuous correspondence with nonempty convex values. Then, for any $i \in \mathbb{N}$, the restriction of $\mathrm{ri}(\varphi)$ to D_i admits a continuous single-valued selection $h_i : D_i \to Y$. In addition, if $i > 0$, then there exists a continuous function $\beta_i : D_i \to]0, +\infty[$ such that for any $x \in D_i$, we have $\overline{B}_{\varphi(x)}(h_i(x), \beta_i(x)) \subset \mathrm{ri}(\varphi(x))$.

Once we have the above theorem, we were able to prove our main result given by

Theorem 1.3.4 Let X be a metric space and Y a Banach space. Let $\varphi : X \to 2^Y$ be a lower semicontinuous correspondence with nonempty convex values. If for any $x \in X$, $\varphi(x)$ is either finite dimensional or closed, then φ admits a continuous single-valued selection.

Note that Theorem 1.3.4 does not imply Theorem 1.3.2 neither the converse. This concludes this section and mainly Chapter 3 of this thesis. The last section is dedicated to adopt some applications of both Brouwer's and Kakutani's fixed point theorems.

1.4 Chapter 4: Applications of Brouwer's and Kakutani's fixed point theorems

1.4.1 Brouwer's fixed point theorem applications

Brouwer's theorem and many of its extensions were used in order to prove the existence of solutions for many problems in nonlinear analysis.
For more uncommon applications, Park [26], wrote: " One interesting application of the Brouwer theorem is due to Zeeman, who described a model of brain". Here, we will present another unusual application of the Brouwer's fixed point theorem to cognitive science. Cognitive science is the scientific study of the mind and its processes. One branch of cognitive science is semantics. Traditionally, it is associated to a mapping between a language and the world. However, it ruled out the language role in meaning negotiation. Another possible way is to focus on how linguistic expressions are linked to its mental representation. But, cognitive science fails to explain how social interactions can affect semantics.

Warglien and Gärdenfors [20] introduced an innovative view of semantics described by "a meeting of minds". They modelise the communication as a mapping between individual mental spaces. They embedded mental spaces with topological and geometrical structures. Namely, they argued that a mental space is a collection of convex compact regions called concepts. Besides, if communication is enough smooth, then it can be viewed as an implicit unknown continuous function. Therefore, by Brouwer's fixed point theorem a meeting of minds can be reached.
This point is discussed in details in the latest Chapter of the thesis.

1.4.2 Kakutani's fixed point theorem applications

Joint applications with selection theory: the equilibrium theory

In 1975, Gale and Mas-Colell [3] stated the following result using both Kakutani's theorem and Theorem 1.3.2 of Michael.

Theorem 1.4.1 Given $X = \prod_{i=1}^{n} X_i$, where X_i is a non-empty compact convex subset of $\mathbb{R}^n$, let $\varphi_i : X \to 2^{X_i}$ be n convex (possibly empty) valued mappings whose graphs are open in $X \times X_i$. Then there exist $x \in X$ such that for each i either $x_i \in \varphi_i(x)$ or $\varphi_i(x) = \emptyset$.

From this result, Gale and Mas-Colell obtained their equilibrium theorem [3] for a game without ordered preferences. A trivial corollary is derived from Theorem 1.4.1.

Corollary 1.4.1 For each i, the correspondence $U_i : X \to 2^{X_i}$, has an open graph and satisfies for each x, $x_i \notin \operatorname{co} U_i(x)$, then there exists $x \in X$ with $U_i(x) = \emptyset$, for all i.

This result was also exploited by Shafer and Sonnenschein (for more details, see for example Border [15], Florenzano [4]).

Game theory: the Nash equilibrium

A game $G((X_i)_{i\in N}, (U_i)_{i\in N})$ is described by a finite set $N = \{1, \cdots, n\}$ of players such that for any $i \in N$, we associate a set of strategies X_i and a payoff function $U_i : X = \prod_{i=1}^{n} X_i \to \mathbb{R}$. We denote by $BR_i : X_{-i} = \prod_{j\neq i} X_j \to X_i$, $x_{-i} \to \operatorname{argmax}_{x_i\in X_i} U_i(x_{-i}, x_i)$, the best reply correspondence of player i consisting of the strategies leading to greatest payoff of player i against $x_{-i} = (x_j)_{j\neq i}$.

Definition 4 (Nash equilibrium) We call $x^* = (x_i^*, x_{-i}^*)$ a Nash equilibrium if and only if

$$\forall i \in \mathbb{N}, x_i^* \in BR_i(x_{-i}^*).$$

Or identically,

$$\forall i \in, U_i(x_{-i}^*, x_i^*) \geq U_i(x_{-i}^*, x_i).$$

Therefore, a Nash equilibrium is a profile of strategies (x_i^*, x_{-i}^*), where no player i has an action yielding to an output preferable to the one given by choosing x_i^*, knowing that every other player j chooses his action x_j^*. That is, no player can profitably deviate given the actions of other players.
An issue to examine is the existence of such equilibrium. Notice that not every strategic game has a Nash equilibrium. For instance, the following game commonly called Matching Pennies game has no Nash equilibrium in pure strategies(see for example [19]).

Example 1.4.1 Each of two people chooses either Head or Tail. If the choices differ, person 1 pays person 2 a dollar; if they are the same, person 2 pays person 1 a dollar. Each person cares only about the amount of money that he receives. A game that models this situation is shown in the following Table 1.1.

TABLE 1.1: Matching Pennies

	Head	Tail
Head	(1-1)	(-1,1)
Tail	(-1,1)	(1,-1)

However, by definition of a Nash equilibrium, in order to prove the existence of such equilibrium for a game, it suffices to prove that there is $x^* = (x_i^*, x_{-i}^*)$ such that $\forall i \in \mathbb{N}, x_i^* \in BR_i(x_{-i}^*)$. Therefore, we define the correspondence BR given by $BR : X \rightarrow X$, $x \rightarrow (BR_1(x_{-1}), \cdots, BR_n(x_{-n}))$. Then, the above condition can be rewritten as $x^* \in BR(x^*)$. Kakutani's fixed point theorem gives conditions under them the relation $x^* \in BR(x^*)$ is fulfilled. The existence result is given by the following. [2]

Theorem 1.4.2 (Glicksberg theorem) The strategic game G has a Nash equilibrium if for all $i \in \mathbb{N}$

[2]Note that this result is slightly different from the original work of Nash [12], which takes place within the context of finite games and mixed strategies.

1. The set X_i is nonempty compact convex subset of a Euclidean space.

2. U_i is continuous and quasi concave on X_i.

The existence of a Nash equilibrium enabled us to achieve our goals in proving the last results of this thesis. The next section is dealing with our last chapter results. As mentioned before, our work borrows from the work of Warglien and Gärdenfors [20]. Their approach, however, is merely existential and thus offers no insight in the structure of the possible outcomes associated with establishing a common conceptual space. Gärdenfors [23] highlighted this point by writing "The result by Warglien and Gärdenfors(2003) only shows that fixpoints always exist, but it says very little about how such fixpoints are achieved. In practice there exist many methods for interlocutors to reach a fixpoint that depend on perceptually or culturally available salient features".

1.4.3 Main results

In introducing this section we quote this following expression by Stalnaker (1979) [24]:
"One may think a nondefective conversation as a game where the common context set is the playing field and the moves are either attempts to reduce the size of the set in certain ways or rejections of such moves by others. The participants have a common interest in reducing the size of the set, but their interest may diverge when it comes to the question of how it should be reduced. The overall point of the game will of course depend on what kind of conversation it is, for example, whether it is an exchange of information, an argument, or a briefing".
We shed a constructive light by framing the problem of how two agents reach a common understanding as the equilibrium outcome of a bargaining procedure. We borrow from the theory of conceptual spaces the assumption that agents' categorisations correspond to a collection of convex categories or, for short, to a convex categorisation. However, the neutrality of this latter term is meant to help the reader keeping in mind that our results are consistent with, but logically independent from, the theory of conceptual spaces.
Each agent has his own binary convex categorisation over the closed unit disk C in $\mathbb{R}^2$. The categorisation of an agent over C corresponds to two regions L (Left) and R (Right) fully characterised by the chord (tb). This may look like the following figure 1.9.

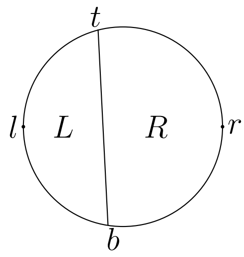


FIGURE 1.9: A binary convex categorisation.

Now, Figure 1.10 provides a pictorial representation for the process: Agent 1 (Primus) and Agent 2 (Secunda) negotiate a common categorisation as a compromise between their own individual systems of categories.

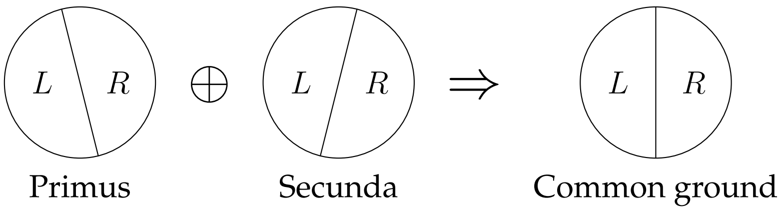


FIGURE 1.10: The search for a common categorisation.

We provide a simple game-theoretic model for their interaction and study the equilibrium outcomes. Each agent evaluates the common categorisation against his own. Superimposing these two spaces, there is one region where the common categorisation and the individual one agree and (possibly) a second region where they disagree. Each agent wants to minimise the disagreement between his own individual and the common categorisation. For simplicity, assume that the payoff for an agent is the opposite of the area of the disagreement region D; that is, $U_i = -\lambda(D_i)$ where λ is the Lebesgue measure. We distinguish two cases depending on whether the disagreement between agents individual spaces is focused or widespread (see Figure 1.11).

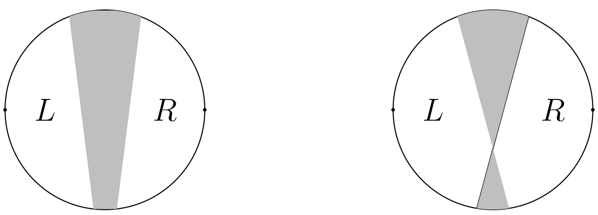


FIGURE 1.11: Focused (left) and widespread disagreement (right).

Under focused disagreement, t_1 precedes t_2 and b_2 precedes b_1 in the clockwise order. The disagreement region is convex and the interaction is a game of conflict: as Primus's choice of t moves clockwise, his disagreement region (with respect to the common categorisation) increases, while Secunda's decreases. In particular, under our simplifying assumption that payoffs are the opposites of the disagreement areas, this is a zero-sum game. Intuitively, players have opposing interests over giving up on their individual categorisations. Therefore, we expect that in equilibrium each player concedes as little as possible. In our model, this leads to the stark result that they make no concessions at all over whatever is under their control. That is, they exhibit maximal stubbornness. This is made precise in the following theorem, that characterises the unique equilibrium.

Theorem 1.4.3 Under focused disagreement, the unique Nash equilibrium is $(t^*, b^*) = (t_1, b_2)$. Moreover, the equilibrium strategies are dominant.

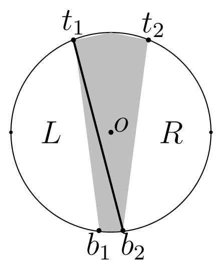


FIGURE 1.12: The unique equilibrium outcome under focused disagreement.

Now, under widespread disagreement, t_1 precedes t_2 and b_1 precedes b_2 in the clockwise order. The disagreement region is not convex and the interaction is no longer a zero-sum game. We simplify the analysis by making the assumption that the two chords characterising the players' categorisations are diameters. Then the two angular distances $\tau = \widehat{t_1 o t_2}$ and $\beta = \widehat{b_1 o b_2}$ are equal, the players have the same strength and the game is symmetric.

Players' stubbornness now has a double-edged effect, leading to a retraction of consensus at the unique equilibrium. Before stating it formally, we illustrate this result with the help of Figure 1.13, drawn for the special case $\tau = \beta = \pi/2$.

The thick line depicts the common categorisation at the unique equilibrium for this situation. Consider Primus. Choosing t very close to t_1 concedes little on the upper circular sector, but exposes him to the

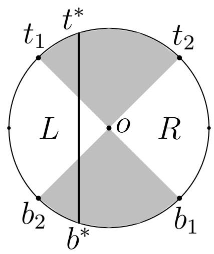


FIGURE 1.13: The unique equilibrium outcome under widespread disagreement.

risk of a substantial loss in the lower sector. This temperates Primus' stubbornness and, in equilibrium, leads him to choose a value of t^* away from t_1. However, as his opponent's choice makes the loss from the lower sector smaller than the advantage gained in the upper sector, the best reply t^* stays closer to t_1 than to t_2. An analogous argument holds for Secunda. A surprising side-effect of these tensions is that, in equilibrium, the common categorisation labels the small white triangle between the thick line and the origin as R, in spite of both agents classifying it as L in their own individual systems of categories. That is, in order to find a common ground, players retract their consensus on a small region and agree to recategorize it. The following theorem characterise the unique equilibrium by means of the two angular distances $\widehat{t^*ot_1}$ and $\widehat{b^*ob_2}$. It is an immediate corollary that the retraction of consensus always occurs, unless $\tau = 0$ and the two agents start off with identical categorisations.

Theorem 1.4.4 Suppose that the individual categorisations are supported by diameters, so that $\tau = \beta$. Under widespread disagreement, there is a unique Nash equilibrium (t^*, b^*) characterised by

$$\widehat{t^*ot_1} = \widehat{b^*ob_2} = \arctan\left(\frac{\sin\tau}{\sqrt{2} + 1 + \cos\tau}\right).$$

Our remaining effort is to prove that since the equilibrium necessitates a retraction of consensus, it should not be surprising that we have an efficiency loss that we call the cost of consensus. The equilibrium strategies lead to payoffs that are Pareto dominated by those obtained under different strategy profiles. The following result exemplifies the existence of such cost using the natural benchmark provided by the Nash bargaining solution (t^s, b^s), with t^s and b^s being the midpoints of the respective arc intervals.

Proposition 1.4.1 Suppose that the individual categorisations are supported by diameters. Under widespread disagreement,

$u_i(t^*, b^*) \leq u_i(t^s, b^s)$ for each player $i = 1, 2$, with the strict inequality holding unless $\tau = 0$.

In concluding this section, notice that as explained before the existence results which proofs are based on Brouwer's or Kakutani's fixed point theorems, are not constructive and do not usually lead to efficient methods for computing explicitly fixed points. Therefore, even computing two players Nash equilibrium may be quite involved. This is one of the reasons why it would be wise to notice that the proofs of our results are essentially based on trigonometry formulas. Once again, we gave an answer to a common question that could be heard: " When am I ever going to make use of my trigo formula?". Many do not appreciate the importance of Trigonometry, at least after the last chapter of this thesis, we deeply do.

Bibliography

[1] A. Mas-Colell (1973), "A note on a theorem of F. Browder", Math. Programming, volume 6, 229–233.

[2] C. Rousseau (2010), Banach fixed point theorem and applications notes, university of Montreal.

[3] D. Gale. and A. Mas-Colell (1975), "An equilibrium existence theorem for a general model without ordered preferences", Journal of Mathematical Economics 2, 9-15.

[4] D.R. Smart (1974), Fixed point theorems, Cambridge tracts in mathematics, Cambridge university Press.

[5] E. Michael (1956), Continuous selections I, Ann. of Math. (2) 63, 361-382.

[6] F. E. Browder (1983), Fixed point theory and nonlinear problem, Bulletin(New Series)of the American mathematical society.

[7] H. Poincaré, Sur les courbes définies par une équation différentielle

[8] J. Dugundji and A. Granas (1982), Fixed Point Theory, Polish Scientific Publishers, Warszawa.

[9] J. Franklin (1980), Methods of Mathematical Economics, Springer-Verlag, New York.

[10] J.W. Milnor (1978), "Analytic proofs of the "hairy ball theorem" and the Brouwer fixed point theorem", Institute of advanced study, Princeton university.

[11] J. W. Milnor (1965), From the differentiable viewpoint, Princeton University.

[12] J. F Nash Jr. (1950), "Equilibrium points in n-person games". Proceedings of the National Academy of Sciences of the United States of America, 36:48–49.

[13] J. M. Ortega, W. C. Rheinboldt (1970), “Iterative solutions of nonlinear equations in several variables”, Academic Press, New York.

[14] K. Fan (1966), “Applications of a theorem concerning sets with convex sections”, Math. Ann. 163 189-203.

[15] K. C. Border (1985), Fixed Point Theorems with Applications to Economics and Game Theory. Cambridge University Press.

[16] M. Florenzano (2003), General Equilibrium Analysis, Springer.

[17] M. Hukuhara (1950), “Sur l’existence des points invariants d’une transformation dans l’espace fonctionnel”, Jap. J. Math. 20 1-4

[18] M. W. Hirsch (1976), Differential topology, Springer verlag New York.

[19] M. Osborne and A. Rubinstein (1994), A Course in Game Theory, The MIT Press Cambridge, Massachusetts London, England.

[20] M. Warglien, Gärdenfors, P. (2013). “Semantics, conceptual spaces, and the meeting of minds”. Synthese,190, 2165-2193.

[21] M.Werner (2007), “A Lefschetz fixed point theorem in gravitational lensing", Institute of Astronomy, University of Cambridge.

[22] Nikaido (1968), Convex structures and economic theory (Academic Press, New York).

[23] P. Gärdenfors (2014a). “Levels of communication and lexical semantics". Synthese, accepted for publication.

[24] P. Portner and B .H. Parte (2002), Formal semantics: the essential reading, Blackwell publishers.

[25] S. Kakutani (1941), “A generalization of Brouwer’s fixed point theorem", Duke Math. J., vol. 8, no 3. p. 457-459.

[26] S. Park (1999), “Ninety years of the Brouwer fixed point theorem”, Vietnam J. Math., 27, pp. 193-232.

[27] T. Stuckless (1999), Brouwer’s fixed point theorem: Methods of proof and generalizations, B.Sc. Memorial university of Newfoundland.

[28] U. Schäfer (2014), From Sperner's lemma to differential equations in Banach spaces.

[29] V. Ovsienko, S. Tabachnikov (2015), Affine Hopf fibrations, Working paper.

[30] W. Kulpa (1997), "The Poincaré-Miranda Theorem", The American Mathematical Monthly, Vol. 104, No. 6

Chapter 2

A new approach of the Hairy ball theorem

Abstract. In this paper, we establish an equivalent version of the hairy ball theorem in the form of a fixed point theorem. In order to prove our main result we use homotopy transformations, the topological degree properties and mainly connected paths. Thereafter, in order to recover the case where smooth paths desirable for implementable methods are obtained, we employ approximation methods for continuous functions by constructing a sequence of smooth regular mappings. Finally, we reconnect with the continuous case and ensure that the transition smooth continuous case is possible. [1]

Keywords: Hairy ball theorem, fixed point theorems, approximation methods, homotopy, topological degree, connected components.

2.1 Introduction

In his paper [2] about the mathematical heritage of H. Poincaré, Browder wrote: " Among the most original and far-reaching of the contributions made by Henri Poincaré to mathematics was his introduction of the use of topological or "qualitative" methods in the study of nonlinear problems in analysis...The ideas introduced by Poincaré include the use of fixed point theorems, the continuation method, and the general concept of global analysis."
This paper is devoted to one of the results stated by H. Poincaré in the late of 19th century given by the following: There is no non vanishing continuous tangent vector field on even dimensional n- spheres.

[1]This Chapter is based on "*A new approach of the Hairy ball theorem* " co-authored with Pascal Gourdel [12].

The theorem was proven for two dimensions by H. Poincaré and generalised by H. Hopf for higher dimensions. In 1910, Brouwer gave another proof for this theorem, better known as the hairy ball theorem. Doubtless, currently there is numerous other proofs of this theorem. Yet, this paper is devoted to present differently this result. Namely, we prove that the hairy ball theorem is equivalent to the following fixed point theorem: If $f : S^{2n} \to S^{2n}$ is continuous and satisfying for any $x \in S^{2n}, f(x).x \geq \frac{1}{2}$, then it possesses a fixed point. Thus, proving the above theorem gives an alternative proof of the hairy ball theorem. A second and very important issue is the techniques used in order to prove our main result. We construct an explicit continuous homotopy F between a function α and the function f. As it is classical, we can think of F as a time indexed family of continuous maps such that when times t varies from 0 to 1, F continuously deform α into f.

We proved that there exists at least one connected component starting from a fixed point of α and intersecting $\{1\} \times S^{2n}$. In other words, this connected component enables us to recover some fixed points of f and proved our main theorem.

On the other hand, it is important to notice that implementable methods are desirable for computing fixed points but only available for following smooth paths. However, we have no guarantee that our pull through connected component is smooth.

Our remaining effort is to approximate F by a sequence of smooth mappings. This allowed us to obtain a sequence of smooth connected components with at least one which is pull through. Besides, in the last section, we prove that it is possible to guarantee the transition smooth-continuous version. However, we may not recover necessarily a connected component but only a connected set leading to some fixed points of f.

The paper is organized as follows. Section 2.2 provides some basic preliminaries and notations. In Section 2.3, we show the equivalence between the hairy ball theorem and our main theorem. The proof of the latter is postponed in Section 2.4. In Section 2.5, we present the smooth case and recover our main result by ensuring the transition smooth-continuous case. Section 2.6 is an Appendix collecting the proofs of some intermediate results.

2.2 Preliminaries and notations

Throughout this paper, we shall use the following notations and definitions. Let $S = S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ be the unit n- sphere. In the sequel, we will suppose that the integer n is even. For any $0 < r < 1$, we denote by $B(S, r) = \{x \in \mathbb{R}^{n+1} : d(x, S) < r\} = \{x \in \mathbb{R}^{n+1} : 1 - r < \|x\| < 1 + r\}$. For a subset $X \subset \mathbb{R}^n$, we denote by $\overline{X}$ the closure of X, by X^c the complement of X, by $\text{int}(X)$ the interior of X, and by $\partial X = \overline{X} \backslash \text{int}(X)$ the boundary of X. We denote by $x_0 = (0, 0, \cdots, 1)$ and $-x_0 = (0, 0, \cdots, -1)$, respectively the north and south pole of S and by 'deg' the classical topological degree.
Let $F : [0, 1] \times S \to S$ be a continuous function, we denote by $C_F := \{(t, x) \in [0, 1] \times S : F(t, x) = x\}$ and $\mathscr{H}$ the set of continuous functions $F : [0, 1] \times S \to S$ such that $C_F \cap (\{0\} \times S) = \{x_0, -x_0\}$. Finally, the translation $\overline{F}$ of F is defined by $\overline{F}(t, x) = F(t, x) - x$.
For getting our main result, we need the following topological degree properties (see for example [13]).

Proposition 2.2.1 Let Ω be a bounded open set of $\mathbb{R}^m$, $f : \overline{\Omega} \longrightarrow \mathbb{R}^m$ be a continuous function and $y \in \mathbb{R}^m$ such that $y \notin f(\partial\Omega)$. Then, we have

1. (**Local constancy**) The $\deg(., \Omega, y)$ is constant in $\{g \in C(\overline{\Omega}) \backslash \|g - f\| < r\}$ where $r = d(y, f(\partial\Omega))$.

2. (**Excision**) Let Ω_1 be an open set of Ω. If $y \notin f(\overline{\Omega} \backslash \Omega_1)$, then $\deg(f, \Omega, y) = \deg(f, \Omega_1, y)$.

3. (**Homotopy invariance**) Let V be an open and bounded set of $[0, 1] \times \mathbb{R}^m$, $V(t) := \{x \in \mathbb{R}^m : (t, x) \in V\}$ and $F : \overline{V} \to \mathbb{R}^n$, with $f_t = F(t, .) \in C^1(\overline{V}(t))$. Suppose that there exists a continuous path $t \to p_t$ such that $p_t \notin f_t(\partial V(t))$, then $t \to \deg(f_t, V(t), p_t)$ is constant on $[0, 1]$.

We need also the following.

Proposition 2.2.2 [5] In a compact set, each connected component is the intersection of all open and closed sets that contain it.

Proof. See the appendix.
The aim of this note is to provide a variant proof of the following theorem.

Theorem 2.2.1 (Hairy ball theorem) An even dimensional sphere does not admit any continuous field of non-zero tangent vectors.

In other terms, if $g : S \to \mathbb{R}^{n+1}$ is continuous and for every $x \in S$, we have $g(x).x = 0$, then there exists $\overline{x}$ such that $g(\overline{x}) = 0$.
In what follows, we establish two equivalent versions of the hairy ball theorem presented as fixed point theorems.

2.3 Equivalent versions

In the following, we state the first equivalent version to the hairy ball theorem.

Theorem 2.3.1 If $f : S \to S$ is continuous, then either f or $-f$ possesses a fixed point.

As a first step, we prove that Theorem 2.3.1 is equivalent to the hairy ball theorem.
Proof. First, we claim that the hairy ball theorem implies Theorem 2.3.1. Indeed, let $f : S \to S$ be a continuous function and consider the tangent component $g : S \to \mathbb{R}^{n+1}$ given by $g(x) = f(x) - (x.f(x))x$. By the hairy ball theorem, g has a zero $\overline{x}$ on S. That is, $f(\overline{x}) = (\overline{x}.f(\overline{x}))\overline{x}$. Now, using that $f(\overline{x})$ is collinear to $\overline{x}$ and that both of them belongs to the sphere, we conclude that either $f(\overline{x}) = \overline{x}$ or $f(\overline{x}) = -\overline{x}$.
Conversely, let $g : S \to \mathbb{R}^{n+1}$ be a continuous function such that for any $x \in S$, $g(x).x = 0$. Then, translating the radial component gives that for any $x \in S$, $g(x) + x \neq 0$. So, we consider the function $f(x) = \frac{x+g(x)}{\|x+g(x)\|}$. By Theorem 2.3.1, there exists $\overline{x}$ such that $f(\overline{x}) = \varepsilon\overline{x}$, where $\varepsilon \in \{-1, 1\}$. This implies that $g(\overline{x}) = 0$. ∎

Now, we state the main result and prove that it is equivalent to the hairy ball theorem.

Theorem 2.3.2 If $f : S \to S$ is continuous and satisfying for any $x \in S$, $f(x).x \geq \frac{1}{2}$, then it possesses a fixed point.

Proof. Obviously, Theorem 2.3.1 implies Theorem 2.3.2. In fact, since $\|x\| = 1$ on S, then it is trivial to see that $f(x) = -x$ is impossible. Conversely, we will prove that Theorem 2.3.2 implies the hairy ball theorem. Indeed, let $g : S \to \mathbb{R}^{n+1}$ be a continuous function such that for any $x \in S$, $g(x).x = 0$. Let $M = \sup_{x\in S} \|g(x)\|$. So, put $\alpha = \frac{\sqrt{3}}{M}$ and consider the function $f_\alpha(x) = \frac{x+\alpha g(x)}{\|x+\alpha g(x)\|}$. We have $f_\alpha(x).x = \frac{1}{\|x+\alpha g(x)\|} > 0$ and by simple calculus, we obtain

$$(f_\alpha(x).x)^2 = \frac{1}{\|x + \alpha g(x)\|^2} = \frac{1}{\|x\|^2 + \alpha^2 \|g(x)\|^2} \geq \frac{1}{1 + \alpha^2 M^2} = \frac{1}{4}.$$

By Theorem 2.3.2, f_α possesses a fixed point $\overline{x}$. Setting $f_\alpha(\overline{x}).\overline{x} = 1$ above implies that $g(\overline{x}) = 0$, and the result follows. ■

Remark 2.3.1 Let us notice that the choice of the real number $1/2$ is arbitrary. The main idea of the theorem is that we can allow a radial component if it is not fully opposite to x. Here we state the general version.
For any real number $\lambda > -1$, we denote by P_λ the following proposition
If $f : S \to S$ is continuous and for any $x \in S$, $f(x).x \geq \lambda$, then f has a fixed point.

It is not difficult to prove that P_λ is equivalent to Theorem 2.3.1.
To sum up, we have Theorem 2.3.2 is equivalent to the hairy ball theorem. Thereafter, providing a proof of Theorem 2.3.2 will enable us to have a new proof of the hairy ball theorem which differs from the classical proofs [10].

2.4 Main results

First, remark that the proof of Theorem 2.3.2 depends on Lemma 2.4.1 (Subsection 2.4.1) and Theorem 2.4.1 (Subsection 2.4.2).

Lemma 2.4.1 If $f : S \to S$ is continuous such that for any $x \in S$, $f(x).x \geq \frac{1}{2}$, then there exists $F \in \mathscr{H}$ such that $F(1,.) = f$.

Theorem 2.4.1 There exists a connected component Γ subset of C_F such that $\Gamma \cap (\{0\} \times S) \neq \emptyset$ and $\Gamma \cap (\{1\} \times S) \neq \emptyset$. Consequently, $F(1,.)$ has a fixed point.

Once we prove Lemma 2.4.1 and Theorem 2.4.1, then Theorem 2.3.2 is deduced immediately.

2.4.1 Proof of Lemma 2.4.1

For any $(t, x) \in [0, 1] \times S$, we will consider the function F given by the normalisation of an homotopy between f and some function α

$$F(t,x) = \frac{tf(x) + (1-t)\alpha(x)}{\|tf(x) + (1-t)\alpha(x)\|}. \tag{2.1}$$

Before introducing precisely the function α, we can easily remark that F satisfies the conclusion of Lemma 2.4.1 provided that for any $(t, x) \in [0, 1] \times S$, the function α met the three following properties

(P_1) $tf(x) + (1 - t)\alpha(x) \neq 0$.

(P_2) $\alpha : S \to S$ is continuous.

(P_3) "$\alpha(x)$ is positively collinear to x" is equivalent to " $x = \pm x_0$".

First, we will start by constructing the function α.

The construction of the function α

First, let us define the function $\beta : S \to S$ by $\beta(x) = y$, where $x = (x_1, \cdots, x_{n+1})$ and $y = (y_1, \cdots, y_{n+1})$ such that

$$\forall i \in \{1, \cdots, n\}\,,\; y_i = x_i\sqrt{x_1^2 + \cdots + x_n^2} = x_i\sqrt{1 - x_{n+1}^2},$$

and

$$y_{n+1} = x_{n+1}\sqrt{2 - x_{n+1}^2}.$$

Second, consider the following function R_θ whose matrix is given by

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & \cdots & 0 & 0 \\ \sin\theta & \cos\theta & \vdots & \vdots & \\ \vdots & & \cos\theta & -\sin\theta & \vdots \\ 0 & & \sin\theta & \cos\theta & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 \end{pmatrix}$$

where $0 < \theta < \frac{\pi}{2}$.
Finally, we define the function $\alpha : S \to S$ by $\alpha(x) = R_\theta(\beta(x))$.
In the following, we will present some of its geometrical properties.

Geometrical properties of the function α: For any $\overline{x}_{n+1} \in [-1, 1]$, let us denote by $P_{\overline{x}_{n+1}}$, the following set $P_{\overline{x}_{n+1}} := \{x \in S \text{ such that } x_{n+1} = \overline{x}_{n+1}\}$. By analogy with the unit sphere of $\mathbb{R}^3$, this set is called a 'parallel' of altitude $\overline{x}_{n+1}$. Remark that except at the poles, it is a sphere of dimension $n - 1$.

- The image by α of a parallel of altitude x_{n+1} is a parallel of altitude $x_{n+1}\sqrt{2 - x_{n+1}^2}$ and closer to the corresponding pole.
- The image by α of a polar cap of altitude x_{n+1} is a polar cap of altitude $x_{n+1}\sqrt{2 - x_{n+1}^2}$ and closer to the corresponding pole.

To gain intuition, this may look like the following figure 2.1.

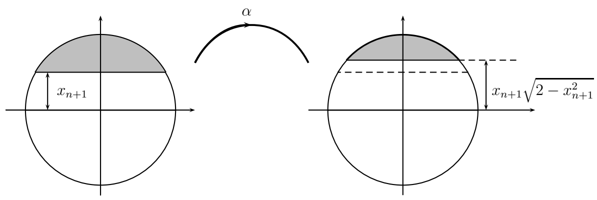


FIGURE 2.1: The image of a polar cap by α

Therefore, the parallel of altitude -1 (reduced to the south pole), of altitude 0 (reduced to the equator) and of altitude 1 (reduced to the north pole) are the only one that are globally invariant. In addition, since $\alpha(x)$ and x are on the sphere and belongs both either to the north semi-sphere or to the south semi-sphere, then positive colinearity means equality. Finally, the following proposition shows that we can construct a set closed to the north pole which is 'strongly' invariant by α.

Proposition 2.4.1 For any $0 < \mu < \frac{1}{2}$, we have

$$\alpha(\overline{B}(x_0, \mu) \cap S) \subset \overline{B}(x_0, \frac{\mu}{2}) \cap S.$$

Proof. See the appendix. ■

Now, in the next section, we prove that the properties (P_1), (P_2) and (P_3) are satisfied.

Principal step

In order to prove (P_1), it suffices to prove that $(tf(x)+(1-t)\alpha(x)).x > 0$. Moreover, since $f(x).x > \frac{1}{2}$, then we just need to prove that $\alpha(x).x > 0$. This follows from the expression of α. Indeed, we have

$$\alpha(x) = R_\theta(\beta(x)) = \begin{pmatrix} \sqrt{1-x_{n+1}^2}(x_1\cos\theta - x_2\sin\theta) \\ \sqrt{1-x_{n+1}^2}(x_1\sin\theta - x_2\cos\theta) \\ \sqrt{1-x_{n+1}^2}(x_3\cos\theta - x_4\sin\theta) \\ \sqrt{1-x_{n+1}^2}(x_3\sin\theta - x_4\cos\theta) \\ \vdots \\ \sqrt{1-x_{n+1}^2}(x_{n-1}\cos\theta - x_n\sin\theta) \\ \sqrt{1-x_{n+1}^2}(x_{n-1}\sin\theta - x_n\cos\theta) \\ x_{n+1}\sqrt{2-x_{n+1}^2} \end{pmatrix}$$

Therefore, we obtain that

$$\alpha(x).x = (1-x_{n+1}^2)^{\frac{3}{2}}\cos\theta + x_{n+1}^2\sqrt{2-x_{n+1}^2}$$
$$\geq (1-x_{n+1}^2)^{\frac{3}{2}}\cos\overline{\theta} + x_{n+1}^2 \geq \min(\frac{2}{3}, \cos\overline{\theta}) > 0,$$

for any $\theta \in]0, \overline{\theta}]$, where $\overline{\theta} < \pi/2$.
On the other hand, since all the components of the function β are continuous, then (P_2) is trivial. This completes the proof of (P_1) and (P_2). In order to prove (P_3), note that the sufficient condition is easy to verify. Now, suppose that α and x are positively colinear. This implies that there exists $\lambda > 0$ such that $\alpha(x) = \lambda x$. Yet, since both α and x belong to the sphere, then we have $\lambda = 1$. Thus, we have $x_{n+1}\sqrt{2-x_{n+1}^2} = x_{n+1}$. That is, $x_{n+1} \in \{-1, 0, 1\}$. In order to finish the proof, it remains to show that $x_{n+1} = 0$ is impossible. By contradiction, if it is the case, then we obtain $\beta(x) = x$. Therefore α coincide with the 'rotation' R_θ. Since $\alpha(x) = x$, then $R_\theta(x) = x$. We can easily check that the only fixed points of R_θ are $\pm x_0$ (a similar argument has been used in [11]), then $x_{n+1} \neq 0$.

□

2.4.2 Proof of Theorem 2.4.1

Let F be the function given by Equation (2.1) and $\Omega = B(S, 1/2)$. Consider the extension $\tilde{F} : [0,1] \times \overline{\Omega} \to S$ defined by $\widetilde{F}(t,x) = F(t, \frac{x}{\|x\|})$.
First, remark that we have both C_F and $C_{\widetilde{F}}$ are included in $[0,1]\times S \subset [0,1] \times \Omega$ and F and $\widetilde{F}$ coincide on $[0,1] \times S$, then $C_{\widetilde{F}} = C_F$. Second, notice that since $[0,1] \times S$ is compact, then $C_{\widetilde{F}}$ is compact.
On the other hand, it is important to notice that we can partition the set of components in $C_{\widetilde{F}}$ as (C_1, C_2), where C_1 denotes the family of components intersecting $\{0\} \times \Omega$ and C_2 the family of components that does not intersect $\{0\} \times \Omega$. We index the family C_i by I_i. To gain some intuition, you can consider the following figure 2.2 .

Now, in order to prove our result suppose that there exists no component in C_1 intersecting $\{1\} \times \Omega$. Then, since for any $i \in I_1$, $\Gamma_i \in C_1$ is compact, then there exists $t_i < 1$ such that $\Gamma_i \subset [0, t_i) \times \Omega$.
By Proposition 2.2.2, we have $\Gamma_i = \cap_{j \in J_i} U_{i,j}$, where for any $j \in J_i$, $U_{i,j}$ is open and closed in $C_{\widetilde{F}}$. Let $D_i = C_{\widetilde{F}} \backslash ([0,t_i) \times \Omega)$, then $(\cap_{j\in J_i} U_{i,j}) \cap D_i = \emptyset$. Remark that since D_i can be rewritten as $C_{\widetilde{F}} \backslash ((-1, t_i) \times \Omega)$, then D_i is compact. Consequently, since $U_{i,j}$ are

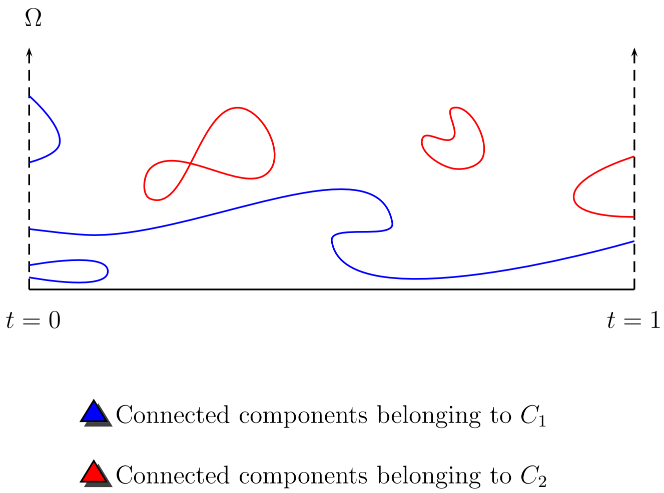


FIGURE 2.2: Example of a partition (C_1, C_2)

closed, then there exists $J_i^1 \subset J_i$ finite such that

$$(\cap_{j \in J_i^1} U_{i,j}) \cap D_i = \emptyset \tag{2.2}$$

Note that $U_i = \cap_{j \in J_i^1} U_{i,j}$ remains an open and closed set containing Γ_i, then without loss of generality, we may assume that U_i can be written as U_{i,j_0} and that $j_0 \in J_i^1$. Using the definition of D_i and Equation 2.2, we conclude that $U_i = U_{i,j_0}$ is included in $[0, t_i) \times \Omega$.
For the second type of components belonging to $C_{\widetilde{F}}$, by a similar argument, there exists $j_0' \in J_i^2$ finite such that $U = U'_{i,j_0'}$ open and closed included in $(r_i, 1] \times \Omega$, for some $r_i > 0$.

The family $\left\{ \{U_{i,j_0}\}_{i \in I_1}, \left\{ U'_{i,j_0'} \right\}_{i \in I_2} \right\}$ forms an open covering of the compact set $C_{\widetilde{F}}$. Therefore, we can extract a finite sub-covering $\{U_1, \cdots, U_k, U'_{k+1}, \cdots, U'_n\}$ such that the first k members are included in $[0, \bar{t}) \times \Omega$ and the remaining $(n-k)$ members are included in $(\bar{r}, 1] \times \Omega$, where $\bar{t} = \max(t_i)_{1 \le i \le k}$ and $\bar{r} = \min(r_i)_{k+1 \le i \le n}$.
The set $E = \cup_{i \le k} U_i$ is open and closed in the compact set $C_{\widetilde{F}}$. So E is compact and contained in $[0, \bar{t}) \times \Omega$. Moreover, $E^c = C_{\widetilde{F}} \backslash E$ is also compact, open and contained in $(\bar{r}, 1] \times \Omega$. Using the separation criteria T_4 in the metric space $[0, 1] \times \Omega$, there exists two disjoint open sets V_1 and V_2 in $[0, 1] \times \Omega$ such that $E \subset V_1$, and $E^c \subset V_2$.

Now, consider the function $\overline{F} : [0, 1] \times \overline{\Omega} \to \mathbb{R}^{n+1}$ defined by $\overline{F}(t, x) = \widetilde{F}(t, x) - x$ and $V_1(t) := \{x \in \Omega : (t, x) \in V_1\}$. Then, we obtain the following.[2]

(1) $\deg(\overline{F}(0,.), \Omega, 0) = -2$.

(2) $\deg(\overline{F}(0,.), \Omega, 0) = \deg(\overline{F}(0,.), V_1(0), 0)$.

(3) $t \to \deg(\overline{F}(t,.), V_1(t), 0)$ is constant on $[0,1]$.

Assuming these computations are established, then we will obtain a contradiction. Indeed, by statement (1), (2) and (3), we have

$$\deg(\overline{F}(t,.), V_1(t), 0) = \deg(\overline{F}(0,.), V_1(0), 0) = -2.$$

Yet, by construction for t close enough to 1, we have $V_1(t) = \emptyset$. Therefore, $\deg(\overline{F}(t,.), V_1(t), 0) = 0$, which establishes a contradiction, as required. In conclusion, we proved that there exists a component $\Gamma \subset C_F$ such that $\Gamma \cap (\{0\} \times S) \neq \emptyset$ and $\Gamma \cap (\{1\} \times S) \neq \emptyset$.

2.5 Transition smooth-continuous version

To sum up, we obtained the existence of a connected component Γ of C_F starting from a fixed point of a function α and enables us to recover some fixed points of f. Besides, it is important to notice that computational methods for finding such fixed points are desirable. Yet, implementable methods are available for following *smooth* path (see for example [3]). However, we have no guarantee that Γ is smooth (see Figure 2.3).

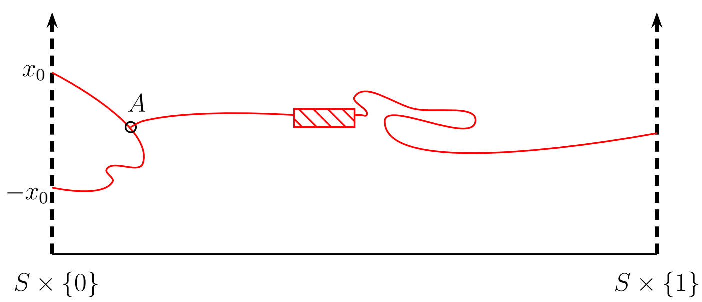


FIGURE 2.3: Example of a non smooth path Γ

[2] The proof of the following result is based on the degree properties and is related in the appendix.

At the point A, a bifurcation occurs creating a loss of the direction to follow and the shaded area makes the tracking even harder. In the sequel, the purpose is to present the method to find the desirable smooth case. But, before giving our formal results and proofs postponed in Section 2.5, we present here the main idea of the process.
Indeed, the key intuition is to approximate F by a sequence of smooth mappings G_p. This allows us to obtain a sequence of smooth connected components Γ_p. Then, applying a result of Mas-Colell, we claim that every Γ_p such that Γ_p intersect $\Omega \times \{0\}$ is diffeomorphic to a segment.
Moreover, recycling the same argument used for F, we prove that for p large enough, there exists a pull through connected component intersecting $\Omega \times \{1\}$ (see Figure 2.4).

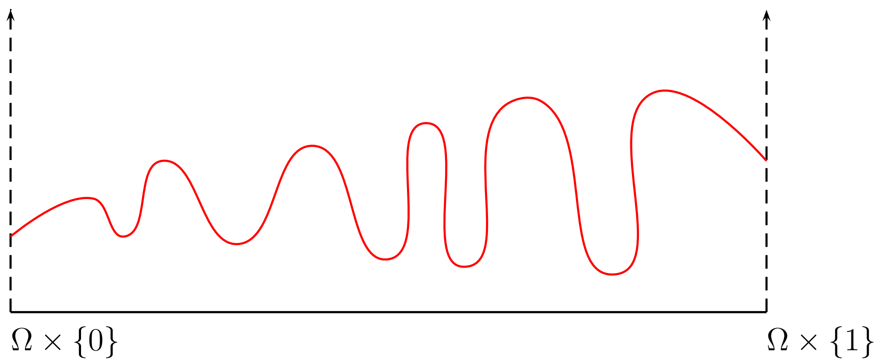


FIGURE 2.4: Example of a smooth path Γ_p

Finally, taking an appropriate limit, we construct a compact set Z such that $Z \subset C_F$satisfying Z is connected, $Z \cap (S \times \{0\}) \neq \emptyset$ and $Z \cap (S \times \{1\}) \neq \emptyset$. Note that Z is not necessarily a connected component.

In the sequel, we will collect formally those results. The main proposition is given by the following.

Proposition 2.5.1 Let Ω be an open subset of $\mathbb{R}^n$. Let $A \subset \Omega$ be open and such that $\overline{A} \subset \Omega$ and let $\mathbb{F}$ be the set of twice continuously differentiable functions: $F : [0,1] \times \Omega \to A$. Then, there exists an open dense set $\mathbb{F}' \subset \mathbb{F}$ and a function $\widetilde{G}_p : [0,1] \times \Omega \to A$, where $A = B(S, \frac{3}{8})$ such that

1. For any $p \in \mathbb{N}$, $\widetilde{G}_p \in \mathbb{F}'$ and for any $(t,x) \in [0,1] \times \Omega$, we have $\left\|\widetilde{G}_p(t,x) - F(t, \frac{x}{\|x\|})\right\| \leq \frac{1}{p}$.

2. For p large enough, there exists a connected component $\Gamma_p \subset C_{\widetilde{G}_p}$ such that $\Gamma_p \cap (\{0\} \times \Omega) \neq \emptyset$ is diffeomorphic to a segment.

3. For p large enough, there exists a connected component $\Gamma_p \subset C_{\widetilde{G}_p}$ such that $\Gamma_p \cap (\{0\} \times \Omega) \neq \emptyset$ and $\Gamma_p \cap (\{1\} \times \Omega) \neq \emptyset$.

2.5.1 Proof of Proposition 2.5.1

The proof of each statement is stored in a different subsection. As explained before, in the sequel, we will use the following theorem of Mas-Colell [8].

Theorem 2.5.1 (Mas-Colell) There is an open and dense set $\mathbb{F}' \subset \mathbb{F}$ such that for every $F \in \mathbb{F}'$, any non empty component Γ of C_F with $\Gamma \cap (\{0\} \times \Omega) \neq \emptyset$ is diffeomorphic to a segment.

Notice that Mas-Colell presented and proved his result on a convex set Ω. However, a careful reading of the proof shows that this assumption was only used in a subsequent part.

The construction of $\widetilde{G}_p$

Let F be the function given by (2.1) and recall the function $\widetilde{F} : [0,1] \times \overline{\Omega} \to S$ defined by $\widetilde{F}(t,x) = F(t, \frac{x}{\|x\|})$. By Stone Weierstrass approximation method, for any integer $p \geq 2$, there exists a C^2 function $\widetilde{F}_p : [0,1] \times \overline{\Omega} \to A$, such that $\left\| \widetilde{F} - \widetilde{F}_p \right\|_\infty \leq \frac{1}{2p}$. Let $\mathbb{F}'$ be an open dense set given by Mas-Colell's theorem. Since $\widetilde{F}_p \in \mathbb{F} = \overline{\mathbb{F}'}$, then there exists $\widetilde{G}_p \in \mathbb{F}'$ such that $\widetilde{G}_p : [0,1] \times \Omega \to A$ and $\left\| \widetilde{G}_p - \widetilde{F}_p \right\|_\infty \leq \frac{1}{2p}$. So, we obtain that

$$\left\| \widetilde{G}_p - \widetilde{F} \right\|_\infty \leq \frac{1}{p}. \tag{2.3}$$

Existence of connected components

For p large enough, we have to show that $C_{\widetilde{G}_p} \neq \emptyset$. This follows from $\deg(\overline{G}_p(0,.), \Omega, 0) = -2$. Indeed, let f and g_p given by the following: for any $x \in \overline{\Omega}$, $f(x) = \overline{F}(0,x) = \widetilde{F}(0,x) - x = \alpha(\frac{x}{\|x\|}) - x$ and $g_p(x) = \overline{G}_p(0,x)$. We claim that $0 \notin f(\partial\Omega)$. Indeed, if there exists $x \in \partial\Omega$ such that $f(x) = 0$, then x is a fixed point of α. That is, $x = \pm x_0 \notin \partial\Omega$, which yields a contradiction. On the other hand, for any $x \in \overline{\Omega}$, we have $\left\| \widetilde{G}_p(0,x) - \alpha(\frac{x}{\|x\|}) \right\| \leq \frac{1}{p}$. That is,

$\|g_p(x) - f(x)\| \leq \frac{1}{p} \leq r$, where $r = d(0, f(\partial\Omega)) > 0$. By Proposition 2.2.1 (i), we have $\deg(\overline{G}_p(0,.), \Omega, 0) = \deg(\overline{F}(0,.), \Omega, 0) = -2$.

Note that using Proposition 2.4.1, we can prove alternatively that $C_{\widetilde{G}_p} \neq \emptyset$. Indeed, it can be shown that there exists a neighborhood of x_0 which is stable by $\widetilde{G}_p$ (see Extra result in the appendix), which allows us to apply Brouwer's fixed point theorem.

A pull through connected component

At this step, applying Theorem 2.5.1 to $\widetilde{G}_p$, we conclude that any component $\Gamma_p \in C_{\widetilde{G}_p}$ intersecting $(\{0\} \times \Omega)$ is diffeomorphic to a segment.
In order to prove the last statement, it remains to show that there exists a component Γ_p such that in addition $\Gamma_p \cap (\{1\} \times \Omega) \neq \emptyset$. The result follows immediately by reproducing the same arguments for G_p then those used for the function F in Section 2.4.2.

2.5.2 Transition smooth-continuous versions

Now, we claim that there exists Z connected subset of C_F such that Z intersect $(\{0\} \times S)$ and $(\{1\} \times S)$. The proof is ruled as follows.

1. Construction of Z.
   We have already established that for p large enough, there exists a connected component $\Gamma_p \subset C_{\widetilde{G}_p}$ such that $\Gamma_p \cap (\{0\} \times \Omega) \neq \emptyset$ and $\Gamma_p \cap (\{1\} \times \Omega) \neq \emptyset$. Let $x_p \in \Gamma_p \cap (\{0\} \times \Omega)$. Since the sequence x_p is bounded, then we may assume that it converges to some $\overline{x}$. Now, we denote by Γ_p^{tr}, the translated component of Γ_p, given by $\Gamma_p^{tr} = \Gamma_p + (\overline{x} - x_p)$. In the spirit of Kuratowski's limit, we introduce Z given by $Z = (\cap_{p\geq 1}\overline{\cup_{k\geq p}\Gamma_k^{tr}})$[3].

2. Connectedness of Z.
   Since the components Γ_k^{tr} are connected, for all k, and contains a common point $(0, \overline{x})$, then $\cup_{k\geq p}\Gamma_k^{tr}$ is connected. Therefore, the set $Z_p = \overline{\cup_{k\geq p}\Gamma_k^{tr}}$ is the closure of a connected set, then it is connected compact. Finally, using that the intersection of a decreasing sequence of connected compact sets is connected, we conclude that Z is connected.

[3]Note that since the translation tends to zero, then the set Z can be written as $\cap_{p\geq 1}\overline{\cup_{k\geq p}\Gamma_k}$ but the formulation used above allows us to conclude about the connectedness of Z.

3. $Z \subset C_F$.
   Let $(t, z) \in Z$, using the alternative definition of Z, we obtain that $\forall p \geq 1$, $(t, z) \in \overline{\cup_{k \geq p} \Gamma_k}$. That is, $\forall p \geq 1$, $B((t, z), \frac{1}{p}) \cap (\cup_{k \geq p} \Gamma_k) \neq \emptyset$. Therefore, there exists $k(p) \geq p$ such that $(t_p, z_p) \in \Gamma_{k(p)} \cap B((t, z), \frac{1}{p})$. Thus, $\widetilde{G}_{k(p)}(t_p, z_p) = z_p$. Passing to the limit yields to $\widetilde{F}(t, z) = z$. Consequently, $(t, z) \in C_{\widetilde{F}} = C_F$.

4. $Z \cap (\{1\} \times S) \neq \emptyset$.
   We have already established that for p large enough, $\Gamma_p \cap (\{1\} \times \Omega) \neq \emptyset$. Therefore, there exists $x'_p \in \Omega$ such that $\widetilde{G}_p(1, x'_p) = x'_p$. Adopting the same argument and notations, passing to the limit yields to $\widetilde{F}(1, \overline{x}') = \overline{x}'$, which implies that $\overline{x}' \in S$. On the other hand, $(1, x'_p) \in \Gamma_p \in \overline{\cup_{k \geq p} \Gamma_k} \subset Z$. By compactness, x'_p converges to $\overline{x}' \in Z$. That is, $Z \cap (\{1\} \times S) \neq \emptyset$.

In the light of the above, we conclude this paper with a series of remarks related to the difference between the limit taken above and the classical Kuratowski limit.

Remark 2.5.1 Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of sets of $\mathbb{R}^p$ and the Kuratowski limit given by $\limsup\limits_{p \longrightarrow +\infty} A_p = \cap_{p \geq 1} \cup_{k \geq p} A_k$.
Then, it is important to notice that trivially $\overline{\limsup\limits_{p \longrightarrow +\infty}} A_p$ is a subset of $Z = \cap_{p \geq 1} \overline{\cup_{k \geq p} A_k}$. However, Z is not necessarily identical to $\overline{\limsup\limits_{p \longrightarrow +\infty}} A_p$. Indeed, let us consider for any $p \in \mathbb{N}$, $f_p(x) : [0, 1] \longrightarrow [0, 1], x \longrightarrow \frac{1}{px+1}$.
Let $A_p = \text{Gr } (f_p) = \{(x, \frac{1}{px+1}), x \in [0, 1]\}$. It is easy to compute that $\overline{\limsup\limits_{p \longrightarrow +\infty}} A_p = \{(0, 1)\}$, while $Z = (\{0\} \times [0, 1]) \cup ([0, 1] \times \{0\})$.

Remark 2.5.2 Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of sets of $\mathbb{R}^p$ and the Kuratowski limit given by $\limsup\limits_{p \longrightarrow +\infty} A_p = \cap_{p \geq 1} \cup_{k \geq p} A_k$. We can notice that even if for any $p \geq 1$, A_p are smooth and diffeomorphic to a segment, then using our concept of limit, we may end up with a 'thick' set Z. Indeed, let us consider for any $p \in \mathbb{N}$, $f_p(x) : [0, 1] \longrightarrow [0, 1], x \longrightarrow \sin^2(px)$. Let $A_p = \text{Gr } (f_p) = \{(x, \sin^2(px)), x \in [0, 1]\}$. Then, we obtain that $Z = [0, 1]^2$.

2.6 Appendix

1. *Proof of Proposition 2.2.2*
   Let C be a connected component of the compact set K and $(K_i(C))_{i\in I}$ be the family of all open and closed sets of K that contains C. We denote by $K(C) = \cap_{i\in I} K_i(C)$. We have, $K(C)$ is closed in the compact set K, then compact. We intend to prove that $C = K(C)$. We have $C \subset K_i(C)$, for any $i \in I$. Indeed, since $K_i(C)^c$ is also open and closed, then if $C \cap K_i(C)^c \neq \emptyset$, this will contradicts that C is connected. Therefore, $C \subset K(C)$. Conversely, it suffices to prove that $K(C)$ is connected. We argue by contradiction. Suppose that $K(C) = F_1 \cup F_2$, where F_1 and F_2 are nonempty, open, closed and disjoints sets. Using the separation criteria T_4, we obtain that there exists U_1, U_2 two disjoints open sets of K such that $F_1 \subset U_1$ and $F_2 \subset U_2$. Since $C \subset K(C)$, then we may assume that $C \subset F_1 \subset U_1$. Let $U = U_1 \cup U_2$, then $K(C) \cap U^c = \emptyset$. That is, $\cap_{i\in I} K_i(C) \cap U^c = \emptyset$. Using the finite intersection property, we obtain that there exists $J \subset I$ finite such that $\cap_{i\in J} K_i(C) \cap U^c = \emptyset$. Let $K_{i_0} = \cap_{i\in J} K_i(C)$, then $K_{i_0} \subset U$. However, $K_{i_0} \cap U_1$ is open and closed in K containing C but not $K(C)$, which establish a contradiction.

   $\square$

2. *Proof of Proposition 2.4.1*
   Let $x \in \overline{B}(x_0, \mu) \cap S$, then $\|x - x_0\|^2 = 2(1 - x_{n+1}) \le \mu^2$.
   On the other hand, we have

$$\begin{aligned}\|\alpha(x) - x_0\|^2 = \|\beta(x) - x_0\|^2 &= 2(1 - x_{n+1}\sqrt{2 - x_{n+1}^2})\\ &= \frac{2(1 - x_{n+1}^2(2 - x_{n+1}^2))}{1 + x_{n+1}\sqrt{2 - x_{n+1}^2}} = \frac{2(1 - x_{n+1}^2)^2}{1 + x_{n+1}\sqrt{2 - x_{n+1}^2}}\\ &\le \frac{2(1 - x_{n+1})^2(1 + x_{n+1})^2}{1 + x_{n+1}} = 2(1 - x_{n+1})^2(1 + x_{n+1})\\ &\le 4(1 - x_{n+1})^2 \le \mu^4 \le \frac{\mu^2}{4}\end{aligned}$$

   , for any $0 < \mu < \frac{1}{2}$.

   $\square$

Proposition 2.6.1 (Extra result)

Let $0 < r < \frac{1}{3}$, then for any $x \in \overline{B}(x_0, r)$, we have $\alpha(\frac{x}{\|x\|}) \in \overline{B}(x_0, \frac{r}{\sqrt{2}\sqrt{1+\sqrt{1-r^2}}})$.

Proof. Let $x \in \overline{B}(x_0, r)$, then $\|x - x_0\|^2 = \|x\|^2 - 2x_{n+1} + 1 \leq r^2$. Let $t = x_1^2 + \cdots + x_n^2$ and $s = x_{n+1}$, then $t + s^2 - 2s + 1 = (1-s)^2 + t \leq r^2$. On the other hand, we have $\left\|\frac{x}{\|x\|} - x_0\right\|^2 = 2(1 - \frac{x_{n+1}}{\|x\|}) = 2(1 - \frac{s}{\sqrt{t+s^2}})$. Put $f(t) = 2(1 - \frac{s}{\sqrt{t+s^2}})$ for any $t \in (0, r^2 - (1-s)^2)$, then f is increasing and $f(t) \leq f(r^2 - (1-s)^2) = 2(1 - \frac{s}{\sqrt{r^2-1+2s}}) = g(s)$. Now, the function g is defined on $(1-r, 1)$ and g reaches its maximum at $(1-r^2)$, then $g(s) \leq g(1-r^2) = 2(1 - \sqrt{1-r^2}) = \frac{2r^2}{1+\sqrt{1-r^2}}$. Hence, we obtain that $\left\|\frac{x}{\|x\|} - x_0\right\| \leq \frac{\sqrt{2}r}{\sqrt{1+\sqrt{1-r^2}}} < \frac{1}{2}$. By Proposition 2.4.1, we have $\alpha(\frac{x}{\|x\|}) \in \overline{B}(x_0, \frac{r}{\sqrt{2}\sqrt{1+\sqrt{1-r^2}}})$, as required. ■

Proof. of degree properties

1. We claim that $\deg(\overline{F}(0,.), \Omega, 0) = -2$. Indeed, recall that

$$\begin{aligned}\deg(\overline{F}(0,.), \Omega, 0) &= \sum_{x \in \overline{F}^{-1}(0,.)(\{0\})} \operatorname{sgn}(\det\ D\overline{F}(0,x)) \\ &= \sum_{x \in \{x_0, -x_0\}} \operatorname{sgn}(\det\ D\overline{F}(0,x)).\end{aligned}$$

Next, we compute the differential of $\overline{F}(0,.)$ at the points x_0 and $-x_0$. Define $h : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{n+1}$ by $h(x) = \beta(\frac{x}{\|x\|})$, then we have for $x \neq 0$, $D\overline{F}(0,.)(x) = R_\theta(Dh(x)) - I_{n+1}$.

Let us denote by $x = (x', x_{n+1})$, we have

$$h(x') = \frac{\|x'\|\, x'}{\|x\|^2},$$

and

$$(h(x))_{n+1} = \frac{x_{n+1}\sqrt{2\,\|x'\|^2 + x_{n+1}^2}}{\|x\|^2}.$$

The function h is differentiable at x_0 and $Dh(x_0) = 0$. Indeed, we have $h(x) - h(x_0) = (\frac{\|x'\|x'}{\|x\|^2}, \frac{x_{n+1}(x_{n+1}^2+2\|x'\|^2)^{\frac{1}{2}}}{\|x\|^2} - 1)$.
It suffices to prove that $h(x) - h(x_0) = o(\|x - x_0\|)$. Since we have $\|x - x_0\|^2 = \|x'\|^2 + (x_{n+1} - 1)^2$, then we have only to prove that every component belongs to $o(\|x'\|)$. It is clear that

$\frac{\|x'\|x'}{\|x\|^2} = o(\|x'\|)$. On the other hand, for the second component of $h(x) - h(x_0)$, we have

$$\frac{x_{n+1}(x_{n+1}^2 + 2\|x'\|^2)^{\frac{1}{2}}}{\|x\|^2} - 1 = \frac{x_{n+1}^2(1 + \frac{2\|x'\|^2}{x_{n+1}^2})^{\frac{1}{2}} - \|x\|^2}{\|x\|^2}$$

$$= \frac{x_{n+1}^2(1 + \frac{\|x'\|^2}{x_{n+1}^2} + o(\frac{\|x'\|^2}{x_{n+1}^2})) - \|x'\|^2 - x_{n+1}^2}{\|x\|^2} = \frac{o(\|x'\|^2)x_{n+1}^2}{\|x\|^2}.$$

Hence, h is differentiable at x_0 and $Dh(x_0) = 0$. A similar calculus leads to $Dh(-x_0) = 0$. Finally, we can conclude that $D\overline{F}(0,.)x_0 = D\overline{F}(0,.)(-x_0) = -I_{n+1}$. Thus, since n is even then, we have

$$\deg(\overline{F}(0,.), X, 0) = (-1)^{n+1} + (-1)^{n+1} = -2,$$

and the result follows.

2. We recall that $V_1(t) = \{x \in \Omega : (t,x) \in V_1\}$ and let $\Omega_1 = V_1(0)$. We have $0 \notin \overline{F}(\overline{\Omega}\backslash V_1(0))$. Indeed, suppose that there exists $x \in \overline{\Omega}\backslash V_1(0)$ such that $\overline{F}(0,x) = 0$. That is, $x \notin V_1(0)$ and $(0,x) \in C_{\overline{F}} \subset V_1 \cup V_2$. This implies that $(0,x) \in V_2$. Though, by construction, V_2 doesn't contain $(0,x)$ which set a contradiction. By Proposition 2.2.1 (ii), we get $\deg(\overline{F}(0,.), \Omega, 0) = \deg(\overline{F}(0,.), V_1(0), 0)$.

3. Let $V(t) = V_1(t)$, $f_t = \overline{F}(t,.)$ and $p_t = 0$. We have $0 \notin f_t(\partial V_1(t))$. Suppose that there exists $x \in \partial V_1(t)$ such that $\overline{F}(t,x) = 0$, then by definition $(t,x) \in \partial V_1 = \overline{V}_1\backslash V_1$ and $(t,x) \in C_{\overline{F}} = E \cup E^c$. Thus, we have two cases. If $(t,x) \in E$, then $(t,x) \in V_1$, an impossibility since $(t,x) \in V_1^c$. Otherwise, we have $(t,x) \in E^c$, then by construction of V_1 and V_2, we have $E^c \cap \partial V_1 = \emptyset$, contradiction. Hence, by By Proposition 2.2.1 (iii), we conclude that $\deg(\overline{F}(t,.), V_1(t), 0)$ is constant in $t \in [0,1]$.

■

Bibliography

[1] F.E. Browder (1960), “On continuity of fixed points under deformations of continuous mappings", Summa Brasiliensis Mathematicae Vol 4.

[2] F.E. Browder (1983), “Fixed point theory and nonlinear problem”, Bulletin(New Series)of the American mathematical society.

[3] S.N. Chow and al. (1978), “Finding Zeroes of Maps: Homotopy Methods that are constructive with probability one”, Math. of computation V 32, 887–899.

[4] B.C. Eaves (1972), “Homotopies for the computation of fixed points”, Math. Programming, v3, 1–22.

[5] R. Engelking. General Topology, Heldermann Verlag, Berlin, 1989. Revised and completed edition, Sigma Series in Pure Mathematics, Vol. 6.

[6] A. Granas and J.Dugundji (2003), Fixed point theory, Springer Press.

[7] M.W. Hirsch (1976), “Differential Topology” Springer-Verlag.

[8] A. Mas-Colell (1973), “A note on a theorem of F.Browder”, Math. Programming, v6, 229–233.

[9] J. Milnor (1965), Topology from the Differentiable Viewpoint, Univ of Virginia Press.

[10] J. Milnor, (1978), “Analytic proofs of the "Hairy ball theorem" and the Brouwer fixed point theorem”, Princeton press.

[11] N. Prabhu (1992), “Fixed points of rotations of n-sphere”, Internat.J.Math. Math.Sci. V22, 221–222.

[12] P. Gourdel, N. Mâagli, (2014), “A new approach of the Hairy ball theorem”.

[13] R. Vandervorst (2008), “Topological methods for nonlinear differential equations: From degree theory to Floer homology”, lecture notes at Vrije university.

Chapter 3

A convex selection theorem with a non separable Banach space

Abstract. As in Michael's convex-valued selection theorem, we consider a nonempty convex valued lower semicontinuous correspondence $\varphi : X \to 2^Y$. We prove that if φ has either closed or finite dimensional images, then there admits a continuous single valued selection, where X is a metric space and Y is a Banach space. We provide a geometric and constructive proof of our main result.

Keywords: barycentric coordinates, continuous selections, lower semicontinuous correspondence, closed valued correspondence, finite dimensional convex values, separable Banach spaces. [1]

3.1 Introduction

The area of continuous selections is closely associated with the publication by Ernest Michael of two fundamental papers [10]. It is important to notice that the axiom of choice ensures the existence of a selection for any nonempty family of subsets of X [9]. Yet, the axiom of choice does not guarantee the continuity of the selection. Michael studies are more concerned about continuous selections for correspondences $\varphi : X \to 2^Y$. He guarantees the continuity under specific structures on X (paracompact spaces, perfectly normal spaces, collectionwise normal spaces,$\cdots$) and on Y (Banach spaces, separable Banach spaces, Fréchet spaces,$\cdots$) [11].

[1] This Chapter is based on "*A convex selection theorem with a non separable Banach space* " co-authored with Pascal Gourdel [13].

Without any doubt, the most known selection theorems are: closed-convex valued, compact valued, zero dimensional and finite dimensional theorems [6, 7, 8].

The closed-convex valued theorem is considered as one of the most famous Michael's contribution to the continuous selection theory for correspondences. This theorem gives sufficient conditions for the existence of a continuous selection with the paracompact domain: Paracompactness of the domain is a necessary condition for the existence of continuous selections of lower semicontinuous correspondences into Banach spaces with convex closed values [9].

However, despite their importance, all the theorems mentioned above were obtained for closed-valued correspondences. One of the selection theorems obtained by Michael in order to relax the closeness restriction is the convex valued selection theorem [Theorem $3.1'''$] [6]. The result was obtained by an alternative assumption on X (Perfect normality), a separability assumption on Y and an additional assumption involving **three** alternative conditions on the images. Besides, Michael shows that when $Y = \mathbb{R}$, then perfect normality is a necessary and sufficient condition in order to get a continuous selection of any convex valued lower semicontinuous correspondence. The proof of the convex valued selection theorem is based on the existence of a dense family of selections. The technique is quite involved and exploits the characterization of perfect normality of X and separability of Y.

An interesting question is the following: is it possible to relax the separability of Y? To answer this question, Michael provided in his paper [6] a counter example (Example 6.3) showing that the separability of Y can not be omitted. Even though, the correspondence satisfies one of the three conditions, Michael established an overall conclusion.

One question arises naturally: Is it possible to omit the separability of Y when the images satisfy **one of the two** remaining conditions? This study aims to prove that the answer is affirmative.

The paper is organized as follows. In Section 2, we begin with some definitions and results which will be very useful in the sequel. Section 3 is dedicated to recall the two Michael's selection theorems that will be used later: the closed-convex valued and the convex valued theorems. In section 4, we first state a partial result when the dimension of the images is finite and constant. Then, we introduce and motivate the concept of peeling. Finally, we state the general case and Section 5 and 6 provide the proofs of our results.

3.2 Preliminaries and notations

We start by introducing some notations which will be useful throughout this paper.

3.2.1 Notations

Let Y be a normed space and $C \subset Y$. We shall denote by

1. $\overline{C}$ the closure of C in Y.

2. $\mathrm{co}(C)$ the convex hull of C and $\mathrm{aff}(C)$ the affine space of C.

3. $\dim_a(C) = \dim \mathrm{aff}(C)$ the dimension of C which is by definition the dimension of $\mathrm{aff}(C)$.

4. If C is finite dimensional[2], then $\mathrm{ri}(C)$ the relative interior of C in $\mathrm{aff}(C)$ is given by,

   $$\mathrm{ri}(C) = \{x \in C, \exists \text{ a neighborhood } V_x \text{of } x \text{ such that } V_x \cap \mathrm{aff}(C) \subset C\}.$$

5. $B_C(a, r) := B(a, r) \cap \mathrm{aff}(C)$, where $B(a, r)$ is the open ball of radius $r > 0$ centered at a point $a \in X$, and $\overline{B}_C(a, r) := \overline{B}(a, r) \cap \mathrm{aff}(C)$, where $\overline{B}(a, r)$ is the closed ball of radius $r > 0$ centered at a point $a \in X$.

6. $S_{i-1}(0, 1) := \{x = (x_1, \cdots, x_i) \in \mathbb{R}^i, \|x\| = 1\}$ the unit $(i-1)-$ sphere of $\mathbb{R}^i$ embedded with the euclidean norm.

7. $(Y^p_{ai})_{p\in\mathbb{N}}$ the set of affinely independent families of $(Y^p)_{p\in\mathbb{N}}$.

We recall that if $\{x^0, x^1, \cdots, x^i\}$ is a set of $(i + 1)$ affinely independent points of Y. We call an $i-$ simplex the convex hull of $\{x^0, x^1, \cdots, x^i\}$ given by

$$S_i = \{z \in Y, z = \sum_{k=0}^{i} \alpha_k x^k, \alpha_k \geq 0, \sum_{k=0}^{i} \alpha_k = 1\} = \mathrm{co}(x^0, \cdots, x^i).$$

[2] C is said to be finite dimensional if C is contained in a finite dimensional subspace of Y.

3.2.2 Classical definitions

We go on with formal definitions and related terms of correspondences. Let us consider nonempty topological spaces X and Y.

Definition 5 Let $\varphi : X \to 2^Y$ be a correspondence, $B \subset Y$. We define by

$$\varphi^+(B) = \{x \in X \mid \varphi(x) \subset B\}, \qquad \varphi^-(B) = \{x \in X \mid \varphi(x) \cap B \neq \emptyset\}.$$

Definition 6 Let $\varphi : X \to 2^Y$ be any correspondence. A correspondence $\psi : X \to 2^Y$ satisfying $\psi(x) \subset \varphi(x)$, for each $x \in X$, is called a selection of φ. In particular, if ψ is single-valued (associated to some function $f : X \to Y$), then f is a single-valued selection when $f(x) \in \varphi(x)$, for each $x \in X$.

We recall some alternatives characteristics of lower semicontinuous correspondences.

Definition 7 [4] Let $\varphi : X \to 2^Y$ be a correspondence. We say that φ is lower semicontinuous (abbreviated to lsc) if one of the equivalent conditions is satisfied.

1. For all open set $V \subset Y$, we have $\varphi^-(V)$ is open.
2. For all closed set $V \subset Y$, we have $\varphi^+(V)$ is closed.

In the case of metric spaces, an alternative characterization is given by the following proposition.

Proposition 3.2.1 [4] Let X and Y be metric spaces and $\varphi : X \to 2^Y$ a correspondence. We have φ is lsc on X if and only if for all $x \in X$, $(x_n)_{n\in\mathbb{N}}$ convergent sequence to x, and all $y \in \varphi(x)$, there exists a sequence $(y_n)_{n\geq n_0}$ in Y such that $y_n \to y$ and for all $n \geq n_0$, $y_n \in \varphi(x_n)$.

3.3 Michael's selection theorems (1956)

Let us first recall one of the main selection theorems: the closed-convex valued selection theorem.

Theorem 3.3.1 (Closed-Convex valued selection theorem) Let X be a paracompact space, Y a Banach space and $\varphi : X \to 2^Y$ a lsc correspondence with nonempty closed convex values. Then φ admits a continuous single-valued selection.

Before stating the next theorem, we recall[3] that a topological space is perfectly normal if it is normal and every closed subset is a G-delta subset (G_δ).

The following Michael selection theorem dedicated for non-closed valued correspondences is much more difficult to prove. The assumption on Y is reinforced by adding the separability. We recall that perfectly normality does not imply paracompactness nor the converse.

Theorem 3.3.2 (Convex valued selection theorem) Let X be a perfectly normal space, Y a separable Banach space and $\varphi : X \to Y$ a lsc correspondence with nonempty convex values. If for any $x \in X$, $\varphi(x)$ is either finite dimensional, or closed, or have an interior point, then φ admits a continuous single-valued selection.

Note that as explained before, in his paper [6] (Example 6.3), Michael provided the following counter example showing that the assumption of separability of Y can not be omitted in Theorem 3.3.2.

Example 3.3.1 There exists a lsc correspondence φ from the closed unit interval X to the non empty, open, convex subsets of a Banach space Y for which there exists no selection.

Indeed, let X be the closed unit interval $[0, 1]$ and $Y = \ell_1(X) = \{y : X \to \mathbb{R}, \sum_{x \in X} |y(x)| < +\infty\}$. Michael showed that the correspondence $\varphi : X \to 2^Y$ given by $\varphi(x) = \{y \in Y \mid y(x) > 0\}$ has open values, consequently, images have an interior point but Michael proved that there does not exists a continuous selection.
The case where the correspondence is either finite dimensional or closed values still remain to be dealt with. In order to provide an answer, we now state the main results of this paper.

3.4 The results

We start by recalling that if X is a metric space, then it is both paracompact and perfectly normal. In many applications, both paracompactness and perfect normality aspects are ensured by the metric character. Therefore, throughout this section, we assume that (X, d) is a metric space. In addition, let $(Y, \|.\|)$ be a Banach space. We recall that the relative interior of a convex set C is a convex set of

[3]A subset of a topological space is termed a G_δ subset if it is expressible as a countable intersection of open subsets.

same dimension and that $\mathrm{ri}(\overline{C}) = \mathrm{ri}(C)$ and $\overline{\mathrm{ri}(C)} = \overline{C}$. In the first instance, in order to prove the main result, we focus on the case of constant (finite) dimensional images. In addition, compared with Theorem 3.3.2 of Michael, we suppose first that X is a metric space and omit the separability of Y. We denote by D_i, the following set $D_i := \{x \in X, \dim_a \varphi(x) = i\}$. Then, we state the following theorem.

Theorem 3.4.1 Let $\varphi : X \to 2^Y$ be a lsc correspondence with nonempty convex values. Then, for any $i \in \mathbb{N}$, the restriction of $\mathrm{ri}(\varphi)$ to D_i admits a continuous single-valued selection $h_i : D_i \to Y$. In addition, if $i > 0$, then there exists a continuous function $\beta_i : D_i \to]0, +\infty[$ such that for any $x \in D_i$, we have $\overline{B}_{\varphi(x)}(h_i(x), \beta_i(x)) \subset ri(\varphi(x))$.

Once we have Theorem 3.4.1, we will be able to prove our main result given by

Theorem 3.4.2 Let X be a metric space and Y a Banach space. Let $\varphi : X \to 2^Y$ be a lsc correspondence with nonempty convex values. If for any $x \in X$, $\varphi(x)$ is either finite dimensional or closed, then φ admits a continuous single-valued selection.

Note that the property of φ being either closed or finite dimensional values is not inherited by $\mathrm{co}(\varphi)$. Consequently, we can not directly convert Theorem 3.4.2 in terms of the convex hull. Yet, first, a direct consequence of both Theorem 3.4.2 and Theorem 3.3.1 is the following.

Corollary 3.4.1 Let X be a metric space and Y a Banach space. Let $\varphi : X \to 2^Y$ be a lsc correspondence with nonempty values. Then $\overline{\mathrm{co}}(\varphi(x))$ admits a continuous single-valued selection.

Second, we can also deduce from Theorem 3.4.2 the following result.

Corollary 3.4.2 Let X be a metric space and Y a Banach space. Let $\varphi : X \to 2^Y$ be a lsc correspondence with nonempty values. If for any $x \in X$, $\varphi(x)$ is either finite dimensional or closed convex, then $\mathrm{co}(\varphi)$ admits a continuous single-valued selection.

It is also worth noting that under the conditions of Theorem 3.4.2, we may have a l.s.c correspondence with both closed and finite dimensional values. This is made clear in the following example.

Example 3.4.1 Let Y be an infinite Banach dimensional space and $X = [0,1]$. Consider $(e_0, \cdots, e_n, \cdots)$ some linearly independent normed family of Y. Let $\varphi : \mathbb{R} \to 2^Y$ defined by

$$\varphi(x) = \begin{cases} \{0\} & \text{if } x \in \{0,1\} \\ \\ \mathrm{ri}(\mathrm{co}(0, e_0)) & \text{if } x \in [\frac{1}{2}, 1) \\ \\ \mathrm{ri}(\mathrm{co}(0, e_0, e_1)) & \text{if } x \in [\frac{1}{3}, \frac{1}{2}) \\ \vdots & \\ Y & \text{if } x \in X^c \end{cases}$$

Remark that φ is l.s.c on $]0,1[$ since it is locally increasing. In other terms, for any $x \in X$, there exists V_x such that $\forall x' \in V_x$, $\varphi(x) \subset \varphi(x')$. Besides, φ is l.s.c at the point $x = 0$. It suffices to remark that for $\varepsilon > 0$, we have $\varphi^{-}(B(0,\varepsilon)) = \mathbb{R}$. Indeed, since for any $x \in \mathbb{R}$, $0 \in \overline{\varphi}(x)$, then $B(0,\varepsilon) \cap \varphi(x) \neq \emptyset$. The same argument is used for $x = 1$. Therefore, we conclude that φ is l.s.c.
By Theorem 3.4.2, we can conclude that φ admits a continuous selection. It should also be noted that we can even build an explicit selection.

The proof of Theorem 3.4.1 and Theorem 3.4.2 are postponed respectively in Section 5 and 6. The proof of Theorem 3.4.1 is based on the concept of “peeling" that we will introduce and motivate here.

Definition 8 Let C be a nonempty finite dimensional subset of Y. We say that C' is a peeling of C of parameter $\rho \geq 0$ if

$$C' = \Gamma(C, \rho) := \left\{ y \in C \text{ such that } \overline{B}_C(y, \rho) \subset \mathrm{ri}(C) \right\}.$$

In order to gain some geometric intuition, the concept is illustrated by Figure 3.1.

Definition 9 Let η be a non negative real-valued function defined on X, and φ a correspondence from X to Y. We will say that the correspondence $\varphi_\eta : X \longrightarrow 2^Y$, is a peeling of φ of parameter η if for each $x \in X$, we have $\varphi_\eta(x) = \Gamma(\varphi(x), \eta(x))$.

The motivation of the peeling concept is given by the next proposition (whose proof is postponed in the next section) where we show that when the dimension is constant, continuous peeling of a lsc correspondence is also (possibly empty) lsc correspondence. This proposition is a key argument for the proof of Theorem 3.4.1.

Proposition 3.4.1 Let $\varphi : X \to 2^Y$ be a lsc correspondence with nonempty convex values. If there exists $i \in \mathbb{N}^*$ such that for any $x \in X$, $\dim_a \varphi(x) = i$, then the continuity of the function η implies the lower semicontinuity of φ_η.

Remark 3.4.1 The following simple example shows that the above proposition is no more valid if the dimension of φ is equal to zero. Let $\varphi : \mathbb{R} \to 2^{\mathbb{R}}$, defined by $\varphi(x) = \{0\}$ and $\eta(x) = |x|$. Obviously φ is lsc and η is continuous, but $\varphi_\eta : \mathbb{R} \to 2^{\mathbb{R}}$ is not lsc since $\varphi_\eta(0) = \{0\}$ while for $x \neq 0$, $\varphi_\eta(x) = \emptyset$.

Remark 3.4.2 Modifying slightly the previous example, we also show that the above proposition does not hold true if the dimension of φ is not constant. Let us consider the case where $X = \mathbb{R}$, $Y = \mathbb{R}^2$, and $\varphi : X \to 2^Y$, is the lsc correspondence defined by $\varphi(x) = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \geq 0 \text{ and } y_2 \geq \tan(2\arctan(|x|))y_1\}$ if $x \neq 0$ and $\varphi(0) = [0, 1/2] \times \{0\}$. Using the same $\eta(x) = |x|$, we obtain that when $x \neq 0$, $\varphi_\eta(x)$ is a translation of $\varphi(x)$. More precisely, $\varphi_\eta(x) = \{(1, |x|)\} + \varphi(x)$, (See Figure 3.2). In particular $d(\varphi(0), \varphi_\eta(x)) \geq 1/2$, which allows us to conclude that φ_η is not lsc.

3.5 Proof of Theorem 3

In the first subsection, we first present elementary results about the "peeling" of a set. Subsection 2 is dedicated to prove some affine geometry results used to prove Proposition 3.4.1 in Subsection 3. Finally, we deduce Theorem 3 from this proposition in the last subsection.

3.5.1 Elementary results on a set "Peeling"

In this subsection, C is a finite dimensional set of a Banach set Y.

Definition 10 We define[4] the internal radius of a finite dimensional set C by

$$\alpha(C) := \sup\left\{\rho \in \mathbb{R}_+, \exists y \in C \text{ such that } \overline{B}_C(y,\rho) \subset \mathrm{ri}(C)\right\}.$$

[4] Note that α may value $+\infty$. The proofs need to distinguish whether α is finite or infinite.

Lemma 3.5.1 Let C be a nonempty convex set, one has $\Gamma(C, 0) = \mathrm{ri}(C)$. Yet, if $\alpha(C)$ is finite, then $\Gamma(C, \alpha(C)) = \emptyset$.

Proof. The equality on $\Gamma(C, 0)$ is a simple consequence of the definition. Let us prove by contradiction that $\Gamma(C, \alpha(C)) = \emptyset$. Indeed if $y \in \Gamma(C, \alpha(C))$, we have $\overline{B}_C(y, \alpha(C)) \subset \mathrm{ri}(C)$. By a compactness argument on the circle of center y and radius $\alpha(C)$ and the openness of $ri(C)$ in aff(C), we can prove the existence of some $\varepsilon > 0$ such that $\overline{B}_C(y, \alpha(C) + \varepsilon) \subset \mathrm{ri}(C)$. □

We first establish that a peeling of a convex set remains convex. In addition, we can characterize the nonemptiness.

Lemma 3.5.2 Let C be a nonempty convex set and $\rho \in [0, +\infty[$. Then, the set $\Gamma(C, \rho)$ is convex. In addition, the set $\Gamma(C, \rho)$ is nonempty if and only if $\rho < \alpha(C)$.

Proof. First, we have $\Gamma(C, \rho)$ is convex. Let $x_1, x_2 \in \Gamma(C, \rho)$ and $\lambda \in [0, 1]$. We claim that $\overline{B}_C((\lambda x_1 + (1 - \lambda)x_2), \rho) \subset \mathrm{ri}(C)$. By triangle inequality, it is easy to see that $\overline{B}(\lambda x_1 + (1 - \lambda)x_2, \rho) = \lambda\overline{B}(x_1, \rho) + (1 - \lambda)\overline{B}(x_2, \rho)$. Yet, since $\mathrm{ri}(C)$ is convex, then we have $(\lambda\overline{B}(x_1, \rho) + (1 - \lambda)\overline{B}(x_2, \rho)) \cap \mathrm{aff}(C) \subset \lambda\,\mathrm{ri}(C) + (1 - \lambda)\,\mathrm{ri}(C) = \mathrm{ri}(C)$, which establishes the result.
Now, remark that if $\Gamma(C, \rho) \neq \emptyset$, then by definition of Γ, we have $\rho \leq \alpha(C)$. We have to distinguish two cases. First, if $\alpha(C) = +\infty$, then $\rho < \alpha(C)$. Second, if $\alpha(C)$ is finite, then in view of Lemma 3.5.1, we have also $\rho < \alpha(C)$. It remains to prove the converse. Using the definition of α, there exists $y_\rho \in C$ such that $\overline{B}_C(y_\rho, \rho) \subset \mathrm{ri}(C)$. Therefore, $y_\rho \in \Gamma(C, \rho)$. □

Lemma 3.5.3 Let C be a convex set and ρ_1, ρ_2 non negative real numbers such that $\rho_1 < \rho_2$. Then, we have $\overline{\Gamma}(C, \rho_2) \subset \Gamma(C, \rho_1) \subset \mathrm{ri}(C)$.

Proof. Let $\overline{y} \in \overline{\Gamma}(C, \rho_2)$, since $\varepsilon = \rho_2 - \rho_1 > 0$, then there exists $y \in \Gamma(C, \rho_2) \cap B(\overline{y}, \varepsilon)$. Consequently, by triangle inequality, $\overline{B}_C(\overline{y}, \rho_1) \subset \overline{B}_C(y, \rho_2)$, and therefore $\overline{y} \in \Gamma(C, \rho_1)$. Finally, it comes from the definition that $\Gamma(C, \rho_1) \subset \mathrm{ri}(C)$.

3.5.2 Affine geometry

Next, we will use known results about linear independence in order to raise a series of results about affine independence and barycentric coordinates.

Lemma 3.5.4 Y_{ai}^{i+1} is an open set of Y^{i+1}.

Proof. Let $(z_0, \cdots, z_i) \in Y_{\text{ai}}^{i+1}$. By contradiction, we suppose that $\forall r > 0, \forall k \in \{0, \cdots, i\}$, $\exists z'_k \in B(z_k, r)$ such that $(z'_0, \cdots, z'_i)$ are affinely dependent. In particular, $\forall p \in \mathbb{N}^*, \forall k \in \{0, \ldots, i\}$, there exists $z'_{k,p} \in B(z_k, 1/p)$ such that $(z'_{0,p}, \ldots, z'_{i,p})$ are affinely dependent. Thus, the family of vectors $(v_1^p, \ldots, v_i^p) = (z'_{1,p} - z'_{0,p}, \ldots, z'_{i,p} - z'_{0,p})$ is linearly dependent. Hence, $\exists \lambda^p = (\lambda_1^p, \cdots, \lambda_i^p) \in \mathbb{R}^i \backslash \{0\}$ such that $\sum_{k=1}^i \lambda_k^p v_k^p = 0$. We can normalize by letting $\mu^p = \lambda^p / \|\lambda^p\| \in S_{i-1}(0,1)$. By a compactness argument, the sequence μ^p admits a convergent subsequence $\mu^{\varphi(p)}$ to $\overline{\mu} \in S_{i-1}(0,1)$.
Since $\forall k \in \{0, \cdots, i\}$, $\exists z'_{k,p} \in B(z_k, 1/p)$, then the sequence $(z'_{k,p})_{p \in \mathbb{N}^*}$ converges to z_k. Therefore, we have $\sum_{k=1}^i \mu^{\varphi(p)} v_k^{\varphi(p)} = \sum_{k=1}^i (\lambda_k^{\varphi(p)} / \|\lambda^{\varphi(p)}\|) v_k^{\varphi(p)} = 0 \to \sum_{k=1}^i \overline{\mu}_k (z_k - z_0)$. That is, $\sum_{k=1}^i \overline{\mu}_k (z_k - z_0) = 0$. Since, by the starting assumption, $(z_1 - z_0, \cdots, z_i - z_0)$ is linearly independent, then we obtain $\overline{\mu} = 0$, absurd. $\square$

We recall that if $y^n = (y_0^n, \cdots, y_i^n) \in Y_{ai}^{i+1}$, then every point z_n of $\text{aff}(y^n)$ has a unique representation

$$z^n = \sum_{k=0}^{i} \lambda_k^n y_k^n, \lambda^n = (\lambda_0^n, \cdots, \lambda_i^n) \in \mathbb{R}^{i+1}, \sum_{k=0}^{i} \lambda_k^n = 1,$$

where $\lambda_0^n, \cdots, \lambda_i^n$ are the barycentric coordinates of the point z^n relative to $(y_0^n, \ldots, y_i^n)$. Using the previous notations and adopting obvious ones for the limits, our next result is formulated in the following way, where the assumption $\overline{y} \in Y_{ai}^{i+1}$ can not be omitted.

Lemma 3.5.5 Let $y^n \in Y_{ai}^{i+1}$ tending to $\overline{y} \in Y_{ai}^{i+1}$. Let $z^n \in \text{aff}(y^n)$. Hence, we have

1. If z^n is bounded, then λ^n is bounded and z^n has a cluster point in $\text{aff}(\overline{y})$.

2. If z^n converges to $\overline{z}$, then λ^n converges to $\overline{\lambda}$.

Proof. We start by proving Assertion 1. We denote by $w^n = z^n - y_0^n$. Since $\sum_{k=0}^i \lambda_k^n = 1$, it follows that $w^n = \sum_{k=0}^i \lambda_k^n (y_k^n - y_0^n)$. Hence, $w^n = \sum_{k=1}^i \lambda_k^n (y_k^n - y_0^n)$.
We denote by $\tilde{\lambda}^n = (\lambda_1^n, \cdots, \lambda_i^n) \in \mathbb{R}^i$, where the first component is omitted. First, we will prove by contradiction that $\tilde{\lambda}^n$ is bounded. Assume the contrary. Then, there exists a subsequence $\tilde{\lambda}^{\psi(n)}$ of $\tilde{\lambda}^n$ such that $\|\tilde{\lambda}^{\psi(n)}\|$ diverges to infinity. Since

z^n is bounded, then $\frac{w^{\psi(n)}}{\|\tilde{\lambda}^{\psi(n)}\|} \to 0$. Moreover, by normalizing, the sequence $\mu^{\psi(n)} = \tilde{\lambda}^{\psi(n)}/\|\tilde{\lambda}^{\psi(n)}\|$ belongs to the compact set $S_{i-1}(0,1)$. Therefore, the sequence $\mu^{\psi(n)}$ admits a convergent subsequence $\mu^{\psi(\varphi(n))}$ converging to $\overline{\mu}$ in $S_{i-1}(0,1)$. Thus, we obtain that $\frac{w^{\psi(\varphi(n))}}{\|\tilde{\lambda}^{\psi(\varphi(n))}\|} \to \sum_{k=1}^{i} \overline{\mu}_k(\overline{y_k} - \overline{y_0})$. By uniqueness of the limit, we deduce that $\sum_{k=1}^{i} \overline{\mu}_k(\overline{y_k} - \overline{y_0}) = 0$. Since $(\overline{y_1} - \overline{y_0}, \cdots, \overline{y_i} - \overline{y_0})$ is independent, then we obtain $\overline{\mu} = 0$, which is impossible.
Now it suffices to remark that since $\lambda_0^n = 1 - \sum_{k=1}^{i} \lambda_k^n$, then the boundedness of $\tilde{\lambda}^n$ implies the one of λ_0^n and therefore of the whole vector λ^n.
It remains to check that z^n has a cluster point. We have already established that λ^n is bounded. Therefore, there admits a convergent subsequence $\lambda^{\psi(n)}$ converging to $\overline{\lambda}$. Hence, $z^{\psi(n)} = \sum_{k=0}^{i} \lambda_k^{\psi(n)} y_k^{\psi(n)} \to \overline{z} = \sum_{k=0}^{i} \overline{\lambda}_k \overline{y}_k$. That is $\overline{z}$ is a cluster point of z^n. Moreover, observe that since $\sum_{k=0}^{i} \lambda_k^{\psi(n)} = 1$ converges to $\sum_{k=0}^{i} \overline{\lambda}_k$, then we have $\overline{z} \in \text{aff}(\overline{y})$, which establish the result.

Now, we are able to prove the second assertion. By Assertion 1, we know that λ^n is bounded. By [1], in order to prove that λ^n converges to $\overline{\lambda}$, we claim that the bounded sequence λ^n has a unique cluster point. Using the same notations, let $a \in F$, where F is the set of cluster points of λ^n. Then, there exists $\lambda^{\varphi(n)}$ a subsequence of λ^n such that $\lambda^{\varphi(n)} \to a$.
Consequently, $w^{\varphi(n)} = \sum_{k=1}^{i} \lambda_k^{\varphi(n)}(y_k^{\varphi(n)} - y_0^{\varphi(n)}) \to \sum_{k=1}^{i} a_k(\overline{y}_k - \overline{y}_0)$. Alternatively, we have $w^{\varphi(n)} = z^{\varphi(n)} - y_0^{\varphi(n)}$ converges to $\overline{z} - \overline{y}_0 = \sum_{k=0}^{i} \lambda_k \overline{y}_k - \overline{y}_0 = \sum_{k=1}^{i} \lambda_k(\overline{y}_k - \overline{y}_0)$, since $\sum_{k=0}^{i} \lambda_k = 1$. By uniqueness of the limit and given that $(\overline{y_1} - \overline{y_0}, \cdots, \overline{y_i} - \overline{y_0})$ is independent, we obtain that $a_k = \overline{\lambda}_k$, $\forall k \in \{1, \cdots, i\}$. Therefore, we have a unique cluster point. Consequently, $\lambda_k^n \to \overline{\lambda}_k$, $\forall k \in \{1, \cdots, i\}$. In addition, for the first component, we obtain $\lambda_0^n = 1 - \sum_{k=1}^{n} \lambda_k^n \to 1 - \sum_{k=1}^{n} \overline{\lambda}_k = \overline{\lambda}_0$. Thus, $\forall k \in \{0, \cdots, i\}$, $\lambda_k^n \to \overline{\lambda}_k$. We complete the proof. □

3.5.3 Proof of Proposition 3.4.1

The key argument we will use is based on the following lemma. Note that Tan and Yuan ([12]) have a similar but weaker result, since they only treated the case of a finite dimensional set Y when the images have a non empty interior which avoids to introduce the relative interior.

Lemma 3.5.6 [Fundamental Lemma]
Under the assumptions of Theorem 3, let $\overline{x} \in D_i$, $\gamma > 0$ and $\overline{y} \in \Gamma(\varphi(\overline{x}), \gamma)$. Then, for any $\varepsilon \in]0, \gamma[$, there exists a neighborhood V a of $\overline{x}$, such that $\forall x \in V \cap D_i$, we have $\Gamma(\varphi(x), \gamma - \varepsilon) \cap B(\overline{y}, \varepsilon) \neq \emptyset$.

Proof. First, let us fix some $\varepsilon \in]0, \gamma[$. In order to simplify the notations, we will assume within this proof that $X = D_i$. We will start by proving the following claim.

Claim 1: There exists a neighborhood V_1 of $\overline{x}$, such that for all $x \in V_1$, we have $\overline{B}_{\varphi(x)}(\overline{y}, \gamma - \varepsilon/3) \subset \varphi(x)$.

Proof of Claim 1: Let us denote by r the positive quantity $r = \gamma - \varepsilon/3$. By contradiction, assume that $\forall n > 0$, $\exists x_n \in B(\overline{x}, 1/n)$, $\exists z^n \in \overline{B}_{\varphi(x_n)}(\overline{y}, r)$ such that $z^n \notin \varphi(x_n)$.
Since $\forall \overline{x} \in X$, we have $\dim_a \varphi(\overline{x}) = i$, then there exists $(\widehat{y}_0, \cdots, \widehat{y}_i) \in Y_{ai}^{i+1} \cap \varphi(\overline{x})$. Moreover, we have $x_n \to \overline{x}$ and φ is lsc. Therefore, for n sufficiently large, there exists $y_k^n \to \widehat{y}_k$ such that $y_k^n \in \varphi(x_n)$, for all $k \in \{0, \cdots, i\}$. Using Lemma 3.5.4, for n large enough, we obtain that $\dim_a (y_0^n, \cdots, y_i^n) = i$.
Besides, we have $z^n \in \text{aff}(\varphi(x_n)) = \text{aff}(y_0^n, \cdots, y_i^n)$. Since $z_n \in \overline{B}(\overline{y}, r)$, then applying the first assertion of Lemma 3.5.5, we conclude that z^n has a cluster point $\overline{z}$ in $\text{aff}(\varphi(\overline{x})) = \text{aff}(\widehat{y}_0, \cdots, \widehat{y}_i)$. Moreover, since $z^n \in \overline{B}(\overline{y}, r)$, then $\overline{z} \in \overline{B}(\overline{y}, r) \subset B(\overline{y}, \gamma)$. It follows that $\overline{z}$ belongs to $\text{ri}(\overline{B}_{\varphi(\overline{x})}(\overline{y}, \gamma))$.

Hence, in view of the dimension of $\overline{B}_{\varphi(\overline{x})}(\overline{y}, \gamma)$, there exists an $i-$simplex $S_i = co(\overline{U}_0, \cdots, \overline{U}_i)$ contained in $\overline{B}_{\varphi(\overline{x})}(\overline{y}, \gamma)$ such that $\overline{z} \in \text{ri}(S_i)$. We can write the affine decomposition $\overline{z} = \sum_{k=0}^{i} \overline{\mu}_k \overline{U}_k$, for some $\mu \in \mathbb{R}^{i+1}$ such that $\sum_{k=0}^{i} \overline{\mu}_k = 1$ and $\overline{\mu}_k > 0$. In particular, in view of the assumption of the lemma, $\overline{U}_k \in \overline{B}_{\varphi(\overline{x})}(\overline{y}, \gamma) \subset \text{ri}(\varphi(\overline{x}))$. Using again that $x_n \longrightarrow \overline{x}$ and the lower semicontinuity of φ, we obtain that there exists a sequence $U_k^n \longrightarrow \overline{U}_k$ such that $U_k^n \in \varphi(x_n), \forall k \in \{0, \cdots, i\}$ and n large enough. Then, we consider $\mu^n = (\mu_0^n, \cdots, \mu_i^n)$ the affine coordinates of z^n in $(U_0^n, \cdots, U_i^n)$. Since z^n has a cluster point $\overline{z}$, then there exists $z^{\psi(n)}$ a subsequence of z^n converging to $\overline{z}$. By Assertion 2 of Lemma 3.5.5, we have $\mu_k^{\psi(n)} \to \overline{\mu}_k$, $\forall k \in \{0, \cdots, i\}$.
However, $\overline{\mu}_k > 0$, then $\mu_k^{\psi(n)} \geq 0$, for n large enough. Consequently, $z^{\psi(n)}$ is a convex combination of $U_k^{\psi(n)}$. By convexity of $\varphi(x_{\psi(n)})$, we conclude that $z^{\psi(n)} \in \varphi(x_{\psi(n)})$, contradiction. This proves Claim 1.

Note than an obvious consequence of Claim 1 is that for all $x \in V_1$, we have $\overline{B}_{\varphi(x)}(\overline{y}, \gamma - \frac{2\varepsilon}{3}) \subset \mathrm{ri}(\varphi(x))$.

Now, since $\varepsilon > 0$, we have $\overline{y} \in \varphi(\overline{x}) \cap B(\overline{y}, \varepsilon/3)$. From the lower semicontinuity of φ, there exists a neighborhood V_2 of $\overline{x}$, such that for all $x \in V_2$, we have the existence of some $y \in \varphi(x) \cap B(\overline{y}, \varepsilon/3)$. Therefore, for any $x \in V = V_1 \cap V_2$, we have

$$\overline{B}_{\varphi(x)}(y, \gamma - \varepsilon) \subset \overline{B}_{\varphi(x)}(\overline{y}, \gamma - \frac{2\varepsilon}{3}) \subset \mathrm{ri}(\varphi(x)).$$

Thus, we finish the proof. □

Note that Proposition 3.4.1 was already stated without proof in Section 4. We are now ready to prove it.

Proof of Proposition 3.4.1. Let us consider an open set O and $\overline{x} \in \varphi_\eta^-(O)$. This means that $\varphi_\eta(\overline{x}) \cap O$ is nonempty and contains some $\overline{y}$. Since O is an open set containing $\overline{y}$, we can first remark that there exists $r > 0$ such that $\overline{B}(\overline{y}, r) \subset O$.

On one hand, by the condition on η, letting $\gamma = \frac{\alpha(\overline{x}) - \eta(\overline{x})}{3} > 0$ and $\eta_i = \eta(\overline{x}) + i\gamma$, for any $i = \{0, \cdots, 3\}$, we have

$$\eta(\overline{x}) = \eta_0 < \eta_1 < \eta_2 < \eta_3 = \alpha(\overline{x}).$$

By definition of α, there exists $\overline{z} \in \Gamma(\varphi(\overline{x}), \eta_2)$. Applying the fundamental lemma, we obtain that there exists a neighborhood V_1 of $\overline{x}$ such that for any $x \in V_1$, there exists $z \in \Gamma(\varphi(x), \eta_1) \cap B(\overline{z}, \gamma)$.

On the other hand, we will distinguish two cases depending whether at point x, there is or not a significant peeling. Let us denote by $M = 1 + \frac{2}{\gamma}(\|\overline{y} - \overline{z}\| + \gamma)$, we will consider the positive number $\overline{\varepsilon} = \min(r/M, \gamma)$

- **First case,** $\eta(\overline{x}) > 0$

  Let us denote $\varepsilon = \min(\overline{\varepsilon}, \eta(\overline{x})/2)$. Since $\overline{y} \in \varphi_\eta(\overline{x}) = \Gamma(\varphi(\overline{x}), \eta(\overline{x}))$, then once again, by the fundamental lemma, we have the existence of a neighborhood V_2 of $\overline{x}$ such that for any $x \in V_2$, there exists $y \in \Gamma(\varphi(x), \eta(\overline{x}) - \varepsilon) \cap B(\overline{y}, \varepsilon)$.

  Now, let us consider $\lambda = \frac{2\varepsilon}{\gamma + \varepsilon} \in]0, 1]$ and $y_\lambda = (1 - \lambda)y + \lambda z$.

- **Second case,** $\eta(\overline{x}) = 0$

  In this case, $\varphi_\eta(\overline{x}) = \mathrm{ri}(\varphi(\overline{x}))$. Let $\varepsilon = \overline{\varepsilon}$, since φ is lsc, then there exists a neighborhood V_2 of $\overline{x}$ such that for any $x \in V_2$, there exists $y \in \varphi(x) \cap B(\overline{y}, \varepsilon)$ satisfying

  $$\overline{B}_{\varphi(x)}(y, 0) \subset \mathrm{ri}(\varphi(x)).$$

  Now, let us consider $\lambda = \frac{\varepsilon}{\gamma} \in]0, 1]$ and $y_\lambda = (1-\lambda)y + \lambda z$.

In both cases, since the set $\varphi(x)$ is convex, in view of our choice of λ, by a simple computation, we can prove that if $x \in V_1 \cap V_2$, then $y_\lambda \in \Gamma(\varphi(x), \eta(\overline{x}) + \varepsilon)$.

Note that in both cases $\lambda \leq 2\varepsilon/\gamma$. The following computation will show that our choice of ε implies that $y_\lambda \in O$.

$$\begin{aligned} \|y_\lambda - \overline{y}\| &\leq (1-\lambda)\|y - \overline{y}\| + \lambda\|z - \overline{y}\| \\ &\leq \|y - \overline{y}\| + \lambda(\|z - \overline{z}\| + \|\overline{z} - \overline{y}\|) \\ &\leq \varepsilon + \tfrac{2\varepsilon}{\gamma}(\|z - \overline{z}\| + \|\overline{z} - \overline{y}\|) \\ &\leq \varepsilon + \tfrac{2\varepsilon}{\gamma}(\gamma + \|\overline{y} - \overline{z}\|) = \varepsilon M \leq r. \end{aligned}$$

Finally, by continuity of η, there exists a neighborhood V_3 of $\overline{x}$ such that for any $x \in V_3$, we have $\eta(\overline{x}) - \varepsilon < \eta(x) < \eta(\overline{x}) + \varepsilon$. Summarizing the previous results, for all $x \in V_1 \cap V_2 \cap V_3$, there exists $y_\lambda \in O$ such that $\overline{B}_{\varphi(x)}(y_\lambda, \eta(x)) \subset \overline{B}_{\varphi(x)}(y_\lambda, \eta(\overline{x}) + \varepsilon) \subset \mathrm{ri}(\varphi(x))$, which means that $y_\lambda \in \varphi_\eta(x) \cap O$ and establishes the result. $\square$

3.5.4 Proof of Theorem 3

Using the above lemma 3.5.6, we are able to prove the following result on the regularity of the internal radius of a lsc correspondence.

Lemma 3.5.7 Let us consider $\varphi : X \to 2^Y$ such that there exists some positive integer i such that for all $x \in X$, $\dim_a \varphi(x) = i$, then $\alpha \circ \varphi : X \to \mathbb{R}^+ \cap \{+\infty\}$ is a lower semicontinuous function.

Proof. Let us first recall that

$$\alpha(\varphi(x)) := \sup\left\{\rho \in \mathbb{R}^+, \exists y \in \varphi(x) \text{ such that } \overline{B}_{\varphi(x)}(y, \rho) \subset \mathrm{ri}(\varphi(x))\right\},$$

Let us fix $\overline{x}$ in X, and $\rho < \alpha(\varphi(\overline{x}))$. We can consider γ such that $\rho < \gamma < \alpha(\varphi(\overline{x}))$. By definition of α, there exists $y \in \Gamma(\varphi(\overline{x}), \gamma)$. Now, by the fundamental lemma, there exists a neighborhood V of $\overline{x}$, such that $\forall x \in V$, $\Gamma(\varphi(x), \rho) \neq \emptyset$. Using Lemma 3.5.2, we obtain that for all x in V, $\alpha(\varphi(x)) > \rho$, which establishes the result. □

An immediate consequence is the following result

Corollary 3.5.1 For all positive integer i, There exists $\eta_i : D_i \to \mathbb{R}^+$ continuous such that $0 < \eta_i(x) < \alpha(\varphi(x))$.

Proof. Since on D_i, $\alpha \circ \varphi$ is a lower semicontinuous, then the correspondence given by $\Lambda : D_i \to 2^{\mathbb{R}}$, $x \to]0, \alpha(\varphi(x))[$ is lsc. Applying Michael' selection theorem 3.3.2, we deduce that there exists η_i a single-valued continuous selection of Λ. □

Proof of Theorem 3.4.1.

- If $i = 0$, then φ is reduced to a singleton on D_0. Moreover, on D_0, the correspondences φ and $\mathrm{ri}(\varphi)$ coincide. Therefore, there exists a mapping $h_0 : D_0 \to Y$ such that for all $x \in D_0$, $\varphi(x) = \{h_0(x)\}$. Since φ is lsc on D_0, it is well known that h_0 is a continuous function.

- If $i > 0$, let us first apply Corollary 3.5.1 in order to get a continuous function η_i. Consequently by Proposition 3.4.1, we have for any $i \in \mathbb{N}^*$, φ_{η_i} is lsc on D_i. By a classical result, this implies that $\overline{\varphi}_{\eta_i}$ is also lsc on D_i. Moreover, in view of the double inequality satisfied by η_i, we can apply Lemma 3.5.2, in order to state that $\overline{\varphi}_{\eta_i}$ is a correspondence with nonempty convex closed values. Therefore, applying Theorem 3.3.1 of Michael gives a continuous single-valued selection h_i of $\overline{\varphi}_{\eta_i}$. That is for any $x \in D_i$, $h_i(x) \in \overline{\Gamma}(\varphi(x), \eta_i(x))$. Since $\eta_i(x) > 0$, we can apply Lemma 3.5.3 in order to show that $h_i(x) \in \Gamma(\varphi(x), \eta_i(x)/2)$. Now, let for all $x \in D_i$, $\beta_i(x) := \eta_i(x)/2$, the previous condition can be rewritten as $\overline{B}_{\varphi(x)}(h_i(x), \beta_i(x)) \subset ri(\varphi(x))$, which proves the result. □

It is worth noting that the idea of peeling should be distinguished from the approximation method introduced by Cellina [2]. Indeed, mainly Cellina's method consists of approximating an upper semi continuous correspondence φ by a lower semi continuous one.[5] In addition, unlike the peeling concept which can be seen as an "inside approximation" (the approximated set is a subset of the original

[5]The approximation is given by the following, $\varphi_\varepsilon(x) = \mathrm{co}(\cup_{z \in (B(x,\varepsilon) \cap X)} \varphi(z))$.

one), the approximation of Cellina is an "outside one" (the original set is a subset of the approximated one). As it is well known (see [2]), applying selection theorems to Cellina approximations is used to deduce Kakutani's fixed point theorem.

We are now ready to prove the main result of this paper.

3.6 Proof of Theorem 4

3.6.1 Notations and Preliminaries

We denote by

1. $D_\infty = \{x \in X \mid \varphi(x)$ has an infinite dimension$\}$.
2. $D_{\leq i} = D_0 \cup D_1 \cup \cdots \cup D_i$.
3. $D_{\geq i} = (D_i \cup D_{i+1} \cup \cdots) \cup D_\infty$.

As in the previous section, we begin with listing a series of parallel results which will be used in order to prove Theorem 3.4.2. The first one is a classical result (see for example [3]).

Lemma 3.6.1 Let C be a convex set such that $\mathrm{ri}(C) \neq \emptyset$. Then, for any α such that $0 < \alpha < 1$, we have $(1-\alpha)\overline{A} + \alpha\,\mathrm{ri}(A) \subset \mathrm{ri}(A)$.

Lemma 3.6.2 $D_{\geq i}$ is an open set of X.

Proof. Let $x \in D_{\geq i}$. That is $\dim_a \varphi(x) \geq i$. Therefore, there exists $y \in Y_{ai}^{i+1} \cap \varphi^{i+1}(x)$ where $\varphi^{i+1}(x)$ is the cartesian product $\varphi^{i+1}(x) = \varphi(x) \times \cdots \times \varphi(x)$. Since the lower semicontinuity of φ implies that φ^{i+1} is also lsc, in view of the openness of Y_{ai}^{i+1}, there exists a neighborhood V_x of x such that for any $x' \in V_x$, $Y_{ai}^{i+1} \cap \varphi^{i+1}(x') \neq \emptyset$. This implies that $\dim_a \varphi(x') \geq i$ and finishes the proof. $\square$

Lemma 3.6.3 $D_{\leq(i-1)}$ is closed in $D_{\leq i}$.

Proof. In view of the partition, $D_{\leq i} = D_{\leq i-1} \cup D_i$, in order to prove the result, it suffices to prove that D_i is an open set in $D_{\leq i}$. Using the previous remark, we already know that $D_{\geq i}$ is an open set of X. Yet, since $D_i = D_{\geq i} \cap D_{\leq i}$, the result is established.

Lemma 3.6.4 Let X and Y be two topological spaces and F a closed subset of X. Suppose that $\varphi : X \longrightarrow 2^Y$ is a lsc correspondence and

$f : F \to Y$ is a continuous single-valued selection of $\varphi_{|F}$. Then, the correspondence ψ given by

$$\psi(x) = \begin{cases} \{f(x)\} & \text{if } x \in F \\ \varphi(x) & \text{if } x \notin F \end{cases}$$

is also lsc.

Proof. Let V be a closed subset of Y. We have, $\psi^+(V) = \{x \in F, \psi(x) \subset V\} \cup \{x \in X \backslash F, \psi(x) \subset V\} = \{x \in F, f(x) \in V\} \cup \{x \in X \backslash F, \varphi(x) \subset V\}$. Since f is a selection of $\varphi_{|F}$, then we deduce that $\{x \in X, \varphi(x) \subset V\} = \{x \in X \backslash F, \varphi(x) \subset V\} \cup \{x \in F, \varphi(x) \subset V\} = \{x \in X \backslash F, \varphi(x) \subset V\} \cup \{x \in F, f(x) \in \varphi(x) \subset V\}$. Therefore, $\psi^+(V) = \{x \in F, f(x) \in V\} \cup \{x \in X, \varphi(x) \subset V\} = f^{-1}(V) \cup \varphi^+(V)$. Since φ is lsc, then $\varphi^+(V)$ is closed. Moreover, since V is a closed subset of Y and f is continuous, then $f^{-1}(V)$ is a closed subset of F. Now, since F is closed in X, then $f^{-1}(V)$ is closed in X. Hence, $\psi^+(V)$ is closed, as required. □

3.6.2 Proof of Theorem 4

The proof of Theorem 3.4.2 is ruled out in three steps as follows.

> **Step** 1: For any $k \in \mathbb{N}$, we have $\mathrm{ri}(\varphi)$ admits a continuous selection $h_{\leq k}$ on $D_{\leq k}$.
>
> **Step** 2: For any $k \in \mathbb{N}$, there exists $j^k : X \to Y$ such that
>
> - j^k is a continuous selection of $\overline{\varphi}$ on X.
> - j^k is a continuous selection of $\mathrm{ri}(\varphi)$ on $D_{\leq k}$.
>
> **Step** 3: There exists f a continuous selection of φ on X.

Proof of Step 1. Let us apply for any $k \in \mathbb{N}$ Theorem 3.4.1 in order to get the existence of a continuous single-valued function h_k defined on D_k such that $\forall x \in D_k$, $h_k(x) \in \mathrm{ri}(\varphi_k(x))$.

Let P_n be the following heredity property: the restriction of $\mathrm{ri}(\varphi)$ to $D_{\leq n}$ admits a continuous selection $h_{\leq n}$.

- For $n = 0$, it suffices to notice that $D_{\leq 0} = D_0$. Therefore, we can let $h_{\leq 0} := h_0$ which is a continuous selection of $\mathrm{ri}(\varphi)$. Thus, P_0 is true.
- Let $n \geq 1$, suppose that P_{n-1} holds true and let us prove that P_n is true. By the heredity hypothesis, we have $\mathrm{ri}(\varphi)$ admits a continuous selection $h_{\leq(n-1)}$ on $D_{\leq(n-1)}$. We will introduce

an auxiliary mapping $\widetilde{\varphi}_n$ defined on $D_{\leq n}$, lsc and such that the graph is contained in the graph of $\overline{\varphi}$, by taking:

$$\widetilde{\varphi}_n : D_{\leq n} \to 2^Y \text{ defined by } \widetilde{\varphi}_n(x) = \begin{cases} \{h_{\leq(n-1)}(x)\} & \text{if } x \in D_{\leq n-1} \\ \overline{\varphi}(x) & \text{if } x \in D_n \end{cases}$$

By Lemma 3.6.3 and Lemma 3.6.4, we conclude that $\widetilde{\varphi}_n$ is lsc. Moreover, $\widetilde{\varphi}_n$ has closed convex values. By Michael selection Theorem 3.3.1, there exists g_n continuous such that $g_n(x) \in \widetilde{\varphi}_n(x) \subset \overline{\varphi}(x), \forall x \in D_{\leq n}$.
In the following, we will construct a continuous single-valued selection $h_{\leq n}$ of $\mathrm{ri}(\varphi)$ on $D_{\leq n}$. We consider the continuous application $\lambda : D_n \longrightarrow]0,1[$ given by,

$$\lambda(x) = \frac{\min(d(x, D_{\leq n-1}), 1)}{2 + \|h_n(x)\|}$$

Then, we define $h_{\leq n} : D_{\leq n} \to Y$ given by,

$$h_{\leq n}(x) = \begin{cases} g_n(x) & \text{if } x \in D_{\leq n-1} \\ (1-\lambda(x))g_n(x) + \lambda(x)h_n(x) & \text{if } x \in D_n \end{cases}$$

Using the heredity property, we know that $h_{\leq(n-1)}$ is a selection of $\mathrm{ri}(\varphi)$ on $D_{\leq(n-1)}$.
On the other hand, on D_n, the function $h_{\leq n}$ is a strict convex combination of $g_n \in \overline{\varphi}$ and $h_n \in \mathrm{ri}(\varphi)$, thus by Lemma 3.6.1, $h_{\leq n} \in \mathrm{ri}(\varphi)$. Therefore, we conclude that $h_{\leq n}$ is a selection of $\mathrm{ri}(\varphi)$ on $D_{\leq n}$.
It remains to check the continuity of $h_{\leq n}$. It is clear that the restriction of $h_{\leq n}$ on D_n (respectively on $D_{\leq n-1}$) is continuous. Since D_n is open in $D_{\leq n}$, then it suffices to consider the case of a sequence $x_k \in D_n$ such that $x_k \to \overline{x} \in D_{\leq n-1}$. It is obvious that since $d(x_k, D_{\leq n-1})$ tends to zero, then $\lambda(x_k)$ tends to zero when x_k tends to $\overline{x}$. Besides, we have that $\|h_n(x)\|/(2 + \|h_n(x)\|)$ is bounded. That is $\lambda(x_k)h_n(x_k)$ tends to zero when x_k tends to $\overline{x}$. Therefore, combining the previous remarks and the continuity of g_n allow us to conclude that $h_{\leq n}(x_k)$ tends to $g_n(\overline{x}) = h_{\leq n}(\overline{x})$, which establishes the result.

Proof of Step 2. Using Lemma 3.6.2, we have already proved that $D_{\geq i}$ is an open set of X. Consequently, $D_{\leq i-1}$ is closed in X. Using again Lemma 3.6.3, we can define a lsc $T^k : X \to 2^Y$ given by

$$T^k(x) = \begin{cases} \{h_{\leq k}(x)\} & \text{if } x \in D_{\leq k} \\ \overline{\varphi}(x) & \text{if } x \in D_{\geq k+1} \end{cases}$$

By Michael selection theorem 3.3.1, there exists j^k continuous such that $\forall x \in X$, $j^k(x) \in T^k(x) \subset \overline{\varphi}(x)$. That is j^k is a selection of $\overline{\varphi}$. On the other hand, for any $x \in D_{\leq k}$, we have $j^k(x) \in T^k(x) = \{h_{\leq k}(x)\}$, which finishes the proof of Step 2.

Proof of Step 3. We can write X as a partition between X_1 and D_∞, where $X_1 = \cup_{k\in\mathbb{N}} D_k$. Under the hypothesis of Theorem 3.4.2, we have D_∞ is a subset of X_2, where $X_2 = \{x \in X \text{ such that } \varphi(x) \text{ has closed values}\}$.

Now, in the spirit of Michael's proof, for any $k \in \mathbb{N}^*$, we define $f^k(x) = \lambda_k(x) j^k(x) + (1 - \lambda_k(x)) j^0(x)$, where $\lambda_k(x) = \frac{1}{\max(1, \|j^k(x) - j^0(x)\|)}$. It is easy to check in view of Lemma 3.6.1 that for each $k \in \mathbb{N}^*$, f^k is also a selection of $\overline{\varphi}$ on X satisfying for any $x \in D_{\leq k}$, $f^k(x) \in \mathrm{ri}(\varphi(x))$. In addition, we have $f^k(x)$ is bounded since $f^k(x)$ can be written as $f^k(x) = j^0(x) + \lambda_k(x)(j^k(x) - j^0(x))$ and our choice of $\lambda_k(x)$ ensures that $\|f^k(x)\| \leq \|j_0(x)\| + 1$.

We can now define for any $k \in \mathbb{N}^*$,

$$\widetilde{f^n}(x) = \sum_{k=1}^{n} \frac{1}{2^k} f^k(x) \qquad f(x) = \sum_{k \in \mathbb{N}^*} \frac{1}{2^k} f^k(x).$$

First, we claim that f is continuous at any $\overline{x} \in X$. As a consequence of the continuity of j_0, the set $W_{\overline{x}} = \{x \in X \mid \|j_0(x)\| \leq \|j_0(\overline{x})\| + 1\}$ is an open neighborhood of $\overline{x}$. Indeed, each f^k is continuous and the series $\widetilde{f^n}$ converges uniformly to f on $W_{\overline{x}}$

$$\|f(x) - \widetilde{f^n}(x)\| \leq \sum_{k=n+1}^{\infty} \frac{1}{2^k} \left\|f^k(x)\right\| \leq (\|j^0(\overline{x})\| + 2) \sum_{k=n+1}^{\infty} \frac{1}{2^k}$$

Moreover, we have for any $x \in X$, since f is an "infinite convex combination"[6] of elements of $\overline{\varphi}(x)$, then $f(x) \in \overline{\varphi}(x)$.

[6] We recall that if C is a convex subset of Y then for any bounded sequence $(c_k)_{k\in\mathbb{N}} \in C$ and for any sequence of non negative real numbers $(\lambda_k)_{\in\mathbb{N}}$ with $\sum_{k\in\mathbb{N}} \lambda_k = 1$, the series $\sum_{\in\mathbb{N}} \lambda_k c_k$ converges to an element of $\overline{C}$.

Now, we claim that for all $x \in X$, $f(x)$ is an element of $\varphi(x)$. We have to distinguish three cases.

- If $x \in D_0$, then $f(x) = j_0(x)$ since $\varphi(x)$ is a singleton.
- If $x \in X_1 \setminus D_0$, then there exists $k_0(x) > 0$ such that $x \in D_{k_0(x)} \subset D_{\leq k_0(x)}$. Let us remark that $f(x)$ can be written as

  $$f(x) = \mu f^{k_0(x)}(x) + (1-\mu) \sum_{k \neq k_0(x)} \frac{f^k(x)}{2^k(1-\mu)},$$

  where $\mu = \frac{1}{2^{k_0(x)}}$.
  Using once again Footnote 6, we easily check that $\sum_{k \neq k_0(x)} \frac{f^k(x)}{2^k(1-\mu)}$ belongs to $\overline{\varphi}(x)$. Therefore, by Lemma 3.6.1, $f(x)$ is an interior point of $\varphi(x)$ then a selection of φ, which finishes the proof.
- If $x \in D_\infty$, then $x \in X_2$. That is $\overline{\varphi}(x) = \varphi(x)$. Since, we have already established that $\forall x \in X$, $f(x) \in \overline{\varphi}(x)$, then the result is immediate. $\square$

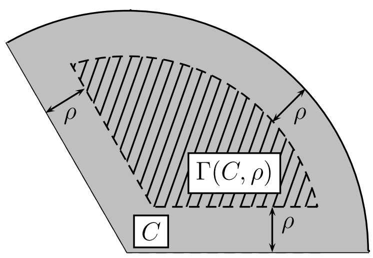


Figure 1: Peeling Concept

FIGURE 3.1: Peeling Concept

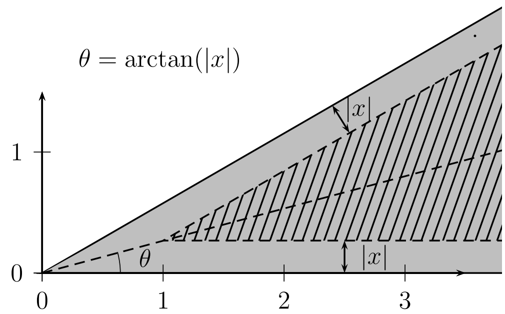


FIGURE 3.2: Graphic Illustration

Bibliography

[1] N. Bourbaki, Eléments de Mathématiques: Topologie générale, Springer Science & Business Media, 2007.

[2] A. Cellina, Approximation of set valued functions and fixed point theorems. Ann. Mat. Pura Appl (1969), 17-24.

[3] M. Florenzano and C. Le Van, *Finite Dimensional convexity and optimization*, Springer, Heidelberg, 2001.

[4] M. Florenzano, *General Equilibrium Analysis*, Springer, 2003.

[5] J. Dugundji and A. Granas, Fixed point Theory, Monogor. Math., vol. 61, PWN, Warsaw, 1982.

[6] E. Michael, Continuous selections I, Ann. of Math. (2) 63 (1956), 361-382.

[7] E. Michael, Continuous selections II, Ann. of Math. (2) 64 (1956), 562-580.

[8] E. Michael, Continuous selections III, Ann. of Math. (2) 65 (1957), 375-390.

[9] D. Repovs and P.V. Semenov , Continuous selections of multivalued mappings, in: Recent Progress in General Topology II (M. Husek and J. van Mill, Editors), Elsevier, Amsterdam, 2002, pp. 423-461.

[10] D. Repovs and P.V. Semenov , Ernest Michael and theory of continuous selections. in: Topology Appl. 155 (2008) 755-763.

[11] N. Makala, Normality-like Properties, Paraconvexity and Selections, 2102, manuscript.

[12] K. Tan and X. Yuan, Lower semicontinuity of multivalued mappings and equilibrium points, in: Proceedings of the first world congress on World congress of nonlinear analysts, volume II, Tampa, Florida (1992), 1849–1860.

[13] P. Gourdel, N. Mâagli,"A convex selection theorem with a non separable Banach space", (2016), Accepted for publication in Advances in Nonlinear Analysis.

Chapter 4

Bargaining over a common categorisation

Abstract. Two agents endowed with different categorisations engage in bargaining to reach an understanding and agree on a common categorisation. We model the process as a simple non-cooperative game and demonstrate three results. When the initial disagreement is focused, the bargaining process has a zero-sum structure. When the disagreement is widespread, the zero-sum structure disappears and the unique equilibrium requires a retraction of consensus: two agents who individually associate a region with the same category end up rebranding it under a different category. Finally, we show that this last equilibrium outcome is Pareto dominated by a cooperative solution that avoids retraction; that is, the unique equilibrium agreement may be inefficient. [1]

Keywords: categorical reasoning, conceptual spaces, semantic bargaining, organisational codes, shared cognitive maps.

4.1 Introduction

It is widely documented that agents organise information by means of categories, with significant implications over their behaviour (Cohen and Lefebvre, 2005). This paper is a theoretical foray in a strictly related but still poorly explored territory: what kind of outcome may emerge when two agents endowed with individual categorisations interact and develop a common categorisation?

There exist different families of models for categorical reasoning; see Section 1 in Kruschke (2008) for a concise overview. The

[1] This Chapter is based on *"Bargaining over a common categorisation "* co-authored with Marco LiCalzi [13].
This paper was featured on LesAffaires.com.

model developed in this paper borrows from the theory of conceptual spaces, proposed in Gärdenfors (2000) as an alternative approach for the modelling of cognitive representations. A tenet of this theory is the claim that natural concepts may be associated with convex regions of a suitable space and, in particular, that a conceptual space consists of a collection of convex regions. This underlying geometric structure resonates with early theories of categorisation based on prototypes (Rotsch, 1975; Mervis and Rotsch, 1981), and has recently been given both evolutionary (Jäger, 2007) and game-theoretic foundations (Jäger et al., 2011).

Conceptual spaces, on the other hand, provide a representational framework that may accommodate different notions. Recently, Gärdenfors (2014) has expanded their scope towards semantics and the study of meaning. In particular, Warglien and Gärdenfors (2013) suggest an interpretation of semantics as a mapping between individual conceptual spaces. People negotiate meaning by finding ways to map their own personal categorisations to a common one; see Warglien and Gärdenfors (2015) for an insightful discussion with references. A well-known example is the integration of different cultures within an organisation, when different communication codes blend into a commonly understood language (Wernerfelt, 2004).

Warglien and Gärdenfors (2013) rely on the theory of fixed points to argue for the plausibility of two individuals achieving a "meeting of minds" and sharing a common conceptual space. Their approach, however, is merely existential and thus offers no insight in the structure of the possible outcomes associated with establishing a common conceptual space. We shed a constructive light by framing the problem of how two agents reach a common understanding as the equilibrium outcome of a bargaining procedure.

We borrow from the theory of conceptual spaces the assumption that agents' categorisations correspond to a collection of convex categories or, for short, to a convex categorisation. However, the neutrality of this latter term is meant to help the reader keeping in mind that our results are consistent with, but logically independent from, the theory of conceptual spaces.

We analyse a simple non-cooperative game where two agents, endowed with their own individual convex categorisations, negotiate over the construction of a common convex categorisation. Agents exhibit stubbornness as they are reluctant to give up on their own categorisation, but they are engaged in a dialectic process that must ultimately lead to a common categorisation. The common convex categorisation emerges as the (unique) equilibrium of the game, aligning it with the argument that meaning is constructed and shared via an equilibrating process (Parikh, 2010).

We demonstrate two main phenomena, depending on whether the disagreement between agents' individual spaces is focused or widespread. Under focused disagreement, the bargaining process has a zero-sum structure: agents' stubbornness leads to a unique equilibrium where each concedes as little as possible, and the agents who has a larger span of control over the process ends up being better off. Under widespread disagreement, the zero-sum structure disappears and each agent confronts a dilemma: holding on to one of his individual categories weakens his position on another one. At the unique equilibrium, these conflicting pressures force a retraction of consensus: two agents who individually agree on a region falling under the same category end up relabeling it in order to minimise conflict. Moreover, we uncover that convex categorisations may be a source of inefficiency: the equilibrium outcome is Pareto dominated by the Nash bargaining solution without retraction.

The rest of the paper is organized as follows. Section 4.2 describes our game-theoretic model. Section 4.3 defines two forms of disagreement (focused and widespread) and states our results as theorems. Section 4.4 provides concluding comments. All proofs are relegated in the appendix.

4.2 Model

There are two agents. Each agent $i = 1, 2$ has his own binary convex categorisation over the closed unit disk C in $\mathbb{R}^2$. Our qualitative results carry through for any convex compact region C in $\mathbb{R}^2$, but this specific choice is elegant and analytically advantageous because C is invariant to rotations. Interestingly, Jäger and van Rooij (2007) also choose to develop their second case study under the assumption that the meaning space is circular. Conventionally, we label the two concepts L for Left and R for Right and use them accordingly in our figures.

The agents agree on the classification of two antipodal points in C: they both label $l = (-1, 0)$ as L and $r = (0, 1)$ as R, respectively. Intuitively, this implies that the agents' categorisations are not incompatible. More formally, define the intersection of the agents' initial categorisations as their shared (partial) categorisation. Under our assumption, the individual categorisations are compatible because the shared categorisation is not empty. On the other hand, since in general the individual categorisations are different, the shared categorisation is only partial. The agents' problem is to move from their (partial) shared categorisation to a common (total) categorisation.

The categorisation of Agent i over C consists of two convex regions L_i and R_i. Dropping subscripts for simplicity, this may look like in Figure 4.1. Clearly, the representation is fully characterized by

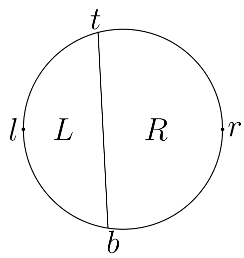


FIGURE 4.1: A binary convex categorisation.

the chord $\overline{tb}$ separating the two convex regions. The endpoints t and b for the chord are located in the top and in the bottom semicircumference, respectively. (To avoid trivialities, assume that the antipodal points l and r are interior.) The two convex regions of the categorisation may differ in extension and thus the dividing chord need not be a diameter for C.

Consider the categorisations of the two agents. Unless $\overline{t_1 b_1} = \overline{t_2 b_2}$, the regions representing the concepts are different and the shared categorisation is partial. If the agents are to reach a common categorisation, they must negotiate an agreement and go through a bargaining process over categorisations, where each agent presumably tries to push for preserving as much as possible of his own original individual categorisation. Figure 4.2 provides a pictorial representation for the process: Agent 1 (Primus) and Agent 2 (Secunda) negotiate a common categorisation as a compromise between their own individual systems of categories.

We provide a simple game–theoretic model for their interaction and study the equilibrium outcomes. We do not claim any generality for our model, but its simplicity should help making the robustness of our results transparent.

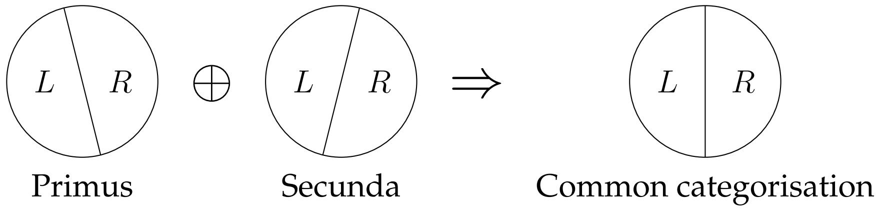


FIGURE 4.2: The search for a common categorisation.

The two agents play a game with complete information, where the endpoints (t_i, b_i) of each agent i are commonly known. Without any loss of generality, let Primus be the agent for whom t_1 precedes t_2 in the clockwise order. Primus picks a point t in the arc interval $[t_1, t_2]$ from the top semicircumference, while Secunda simultaneously chooses a point b between b_1 and b_2 from the bottom semicircumference. The resulting chord $\overline{tb}$ defines the common categorisation. Under our assumption that the antipodal points l and r are interior, the agents cannot pick either of them.

Each agent evaluates the common categorisation against his own. Superimposing these two spaces, there is one region where the common categorisation and the individual one agree and (possibly) a second region where they disagree. For instance, consider the left-hand side of Figure 4.3 where the solid and the dotted chords represent the agent's and the common categorisation, respectively. The two clas-

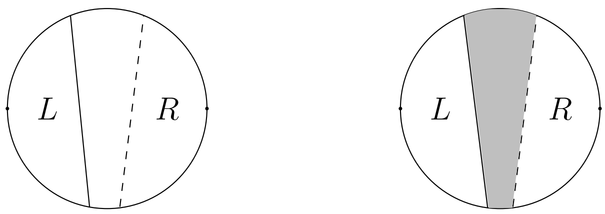


FIGURE 4.3: The disagreement area.

sifications disagree over the central region, coloured in grey on the right-hand side.

Each agent wants to minimise the disagreement between his own individual and the common categorisation. For simplicity, assume that the payoff for an agent is the opposite of the area of the disagreement region D; that is, $u_i = -\lambda(D_i)$ where λ is the Lebesgue measure. (Our qualitative results carry through for any absolutely continuous measure μ.) Note that the region D need not be convex: when the chords underlying the agent's and the common categorisation intersect inside the disc, D consists of two opposing circular sectors.

4.3 Results

The study of the equilibria is greatly facilitated if we distinguish three cases. First, when $t_1 = t_2$ and $b_1 = b_2$, the two individual categorisations (as well as the initial shared categorisation) are identical: the unique Nash equilibrium has $t^* = t_1$ and $b^* = b_2$, and the common categorisation agrees with the individual ones. This is a trivial case, which we consider no further. From now on, we assume that the two individual categorisations disagree and thus the initial shared categorisation is only partial; that is, either $t_1 \neq t_2$ or $b_1 \neq b_2$ (or both).

The other two cases depend on the shape of the disagreement region D. When $\overline{t_1 b_1}$ and $\overline{t_2 b_2}$ do not cross inside the disc, then D is a convex set as in the left-hand side of Figure 4.4. We define this situ-

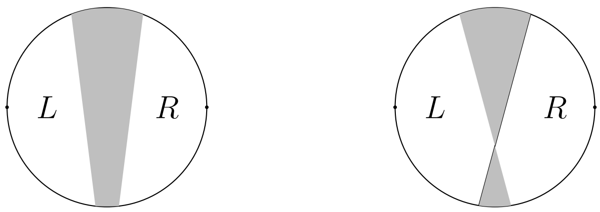


FIGURE 4.4: Focused (left) and widespread disagreement (right).

ation as focused disagreement, because one agent labels D as L and the other as R. The disagreement is focused on whether D should be construed as L or R.

Instead, when $\overline{t_1 b_1}$ and $\overline{t_2 b_2}$ cross strictly inside the disc, then D is the union of two circular sectors as in the right-hand side of Figure 4.4. This is the case of widespread disagreement, because the two agents label the two sectors in opposite ways: the top sector is L for one and R for the other, while the opposite holds for the bottom sector.

4.3.1 Focused disagreement

Under focused disagreement, t_1 precedes t_2 and b_2 precedes b_1 in the clockwise order. The disagreement region is convex and the interaction is a game of conflict: as Primus's choice of t moves clockwise, his disagreement region (with respect to the common categorisation) increases, while Secunda's decreases. In particular, under our simplifying assumption that payoffs are the opposites of the disagreement areas, this is a zero-sum game.

Intuitively, players have opposing interests over giving up on their individual categorisations. Therefore, we expect that in equilibrium each player concedes as little as possible. In our model, this leads to the stark result that they make no concessions at all over whatever is under their control. That is, they exhibit maximal stubbornness. This is made precise in the following theorem, that characterises the unique equilibrium. All proofs are relegated in the appendix.

Theorem 4.3.1 Under focused disagreement, the unique Nash equilibrium is $(t^*, b^*) = (t_1, b_2)$. Moreover, the equilibrium strategies are dominant.

Figure 4.5 illustrates the equilibrium outcome corresponding to the situation depicted on the left-hand side of Figure 4.4. The thick

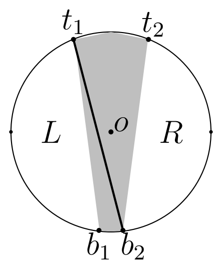


FIGURE 4.5: The unique equilibrium outcome under focused disagreement.

line defines the common categorisation. In this example, Primus and Secunda give up the small grey area on the left and on the right of the thick line, respectively. Note how Primus and Secunda stubbornly stick to their own original t_1 and b_2. Moreover, Primus gives up a smaller area and thus ends up being better off than Secunda. This shows that, in spite of its simplicity, the game is not symmetric. Our next result elucidates which player has the upper hand in general. Formally, let (t^s, b^s) be the Nash bargaining solution, with t^s and b^s being the midpoints of the two players' strategy sets. We say that in equilibrium Primus is stronger than Secunda if $u_1(t^*, b^*) \geq u_1(t^s, b^s) = u_2(t^s, b^s) \geq u_2(t^*, b^*)$.

To gain intuition, consider again Figure 4.5. The thick line defining the common categorisation divides the disagreement region into two sectors $S_1(t_1 t_2 b_2)$ and $S_2(b_2 b_1 t_1)$. Primus wins S_1 and loses S_2; so he is stronger when $\lambda(S_1) \geq \lambda(S_2)$. The area of S_1 depends on the angular distance $\tau = \widehat{t_1 o t_2}$ controlled by Primus and on the angular distance $\theta_R = \widehat{t_2 o b_2}$ underlying the arc that is commonly labeled

R; similarly, the area of S_2 depends on $\beta = \widehat{b_1 o b_2}$ and $\theta_L = \widehat{t_1 o b_1}$. Primus is advantaged when $\tau \geq \beta$ and $\theta_R \geq \theta_L$. The first inequality implies that his span of control is higher. The second inequality makes the common categorisation for R less contestable than for L, so that Primus' stubborn clinging to t_1 is more effective than Secunda's choice of b_2. The next result assumes that a player (say, Primus) has the larger span of control: then Primus is stronger when his span of control is sufficiently large, or when R is more contestable than L but the opponent's span of control is small enough.

Proposition 4.3.1 Suppose $\tau \geq \beta$. If $\tau \geq \beta + (\theta_L - \theta_R)$, then Primus is stronger. If $\tau < \beta + (\theta_L - \theta_R)$, then there exists $\overline{\beta}$ such that Primus is stronger if and only if $\beta \leq \overline{\beta}$.

4.3.2 Widespread disagreement

Under widespread disagreement, t_1 precedes t_2 and b_1 precedes b_2 in the clockwise order. The disagreement region is not convex and the interaction is no longer a zero-sum game. We simplify the analysis by making the assumption that the two chords characterising the players' categorisations are diameters. Then the two angular distances $\tau = \widehat{t_1 o t_2}$ and $\beta = \widehat{b_1 o b_2}$ are equal, the players have the same strength and the game is symmetric.

Players' stubbornness now has a double-edged effect, leading to a retraction of consensus at the unique equilibrium. Before stating it formally, we illustrate this result with the help of Fig. 4.6, drawn for the special case $\tau = \beta = \pi/2$. The thick line depicts the common

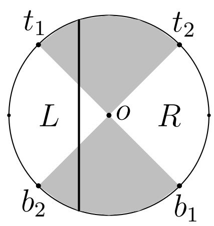


FIGURE 4.6: The unique equilibrium outcome under widespread disagreement.

categorisation at the unique equilibrium for this situation.

Consider Primus. Choosing t very close to t_1 concedes little on the upper circular sector, but exposes him to the risk of a substantial loss in the lower sector. This temperates Primus' stubbornness and, in equilibrium, leads him to choose a value of t^* away from t_1. However, as his opponent's choice makes the loss from the lower sector smaller than the advantage gained in the upper sector, the best reply t^* stays closer to t_1 than to t_2. An analogous argument holds for Secunda.

A surprising side-effect of these tensions is that, in equilibrium, the common categorisation labels the small white triangle between the thick line and the origin as R, in spite of both agents classifying it as L in their own individual systems of categories. That is, in order to reach an agreement, players retract their consensus on a small region and agree to recategorize part of their initial shared categorisation. The following theorem characterise the unique equilibrium by means of the two angular distances $\widehat{t^*ot_1}$ and $\widehat{b^*ob_2}$. It is an immediate corollary that the retraction of consensus always occurs, unless $\tau = 0$ and the two agents start off with identical categorisations.

Theorem 4.3.2 Suppose that the individual categorisations are supported by diameters, so that $\tau = \beta$. Under widespread disagreement, there is a unique Nash equilibrium (t^*, b^*) characterised by

$$\widehat{t^*ot_1} = \widehat{b^*ob_2} = \arctan\left(\frac{\sin\tau}{\sqrt{2} + 1 + \cos\tau}\right).$$

As the equilibrium necessitates a retraction of consensus, it should not be surprising that we have an efficiency loss that we call the cost of consensus. The equilibrium strategies lead to payoffs that are Pareto dominated by those obtained under different strategy profiles. The following result exemplifies the existence of such cost using the natural benchmark provided by the Nash bargaining solution (t^s, b^s), with t^s and b^s being the midpoints of the respective arc intervals.

Proposition 4.3.2 Suppose that the individual categorisations are supported by diameters. Under widespread disagreement, $u_i(t^*, b^*) \leq u_i(t^s, b^s)$ for each player $i = 1, 2$, with the strict inequality holding unless $\tau = 0$.

4.4 Concluding comments

The game-theoretic model presented and solved in this paper is a mathematically reduced form, consistent with different interpretations. As discussed in the introduction, our motivation originates with a few recent contributions about the negotiation of meaning. Accordingly, we suggest to interpret the convex regions of a conceptual space as the (simplified) representation of lexical meanings for words (Gärdenfors, 2014a). Each agent enters the negotiation with his own mapping between words and their meaning, and the purpose of their interaction is to generate a common mapping. This is a first step in the ambitious program of "modelling communication between agents that have different conceptual models of their current context", as proposed by Honkela et al. (2008).

If one also accepts the classical view that concepts have definitional structures, it is possibile to expand the scope of our model to the negotiation of concepts. However, we believe that the underlying philosophical difficulties make this a slippery path and we prefer to confine our discussion to the negotiation of lexical meaning for words. This places our contribution within the recent literature emphasising a game–theoretic approach to the analysis of language (Benz et al., 2005; Clark, 2012; Parikh, 2010).

Finally, we mention some advantages and limitations in our model. The use of noncooperative game theory highlights the "mixed motives" described in Warglien and Gärdenfors (2015): the negotiation agents have a common interest in achieving coordination on a common categorisation, tempered by individual reluctance in giving up their own categories. This conflict is a channel through which egocentrism affects pragmatics (Keysar, 2007), and we show that it may impair efficiency. On the other hand, the simplicity of our model leaves aside important issues of context, vagueness and dynamics in the negotiation of the lexicon (Ludlow, 2014).

.1 Proofs

.1.1 Proof of Theorem 4.3.1

The proof is a bit long, but straightforward. It is convenient to introduce some additional notation. The endpoints (t_i, b_i) for the two agents' chords and their choices for t and b identify six sectors. Proceeding clockwise, these are numbered from 1 to 6 on the left-hand side of Figure 7. For each sector i, we denote its central angle by θ_i; that is, we let $\theta_1 = \widehat{t_1 o t}$, $\theta_2 = \widehat{t o t_2}$, $\theta_3 = \widehat{t_2 o b_2}$, $\theta_4 = \widehat{b_2 o b}$, $\theta_5 = \widehat{b o b_1}$, and $\theta_6 = \widehat{b_1 o t_1}$. The following lemma characterises the disagreement area of each player as a function of the six central angles.

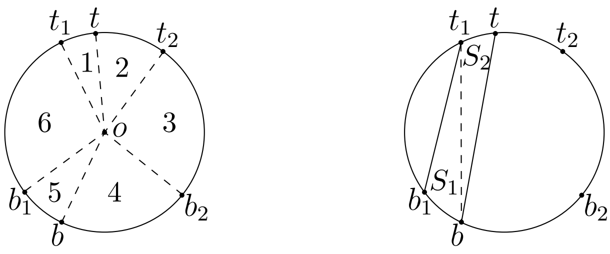


FIGURE 7: Visual aids for the proof of Theorem 4.3.1.

Lemma .1.1 The disagreement areas for Primus and Secunda are, respectively:

$$\lambda(D_1) = \frac{\theta_1 + \theta_5 + \sin\theta_6 - \sin(\theta_1 + \theta_5 + \theta_6)}{2}, \tag{1}$$

and

$$\lambda(D_2) = \frac{\theta_2 + \theta_4 + \sin\theta_3 - \sin(\theta_2 + \theta_3 + \theta_4)}{2}.$$

Proof. The disagreement region D_1 for Primus can be decomposed into the two sector-like regions $S_1(t_1 b b_1)$ and $S_2(t_1 t b)$ as shown on the right-hand side of Figure 7. (The figure illustrates a special case, but the formulas hold in general.) We compute the areas $\lambda(S_1)$ and $\lambda(S_2)$, and then add them up to obtain $\lambda(D_1)$.

Consider $S_1(t_1 b b_1)$. It can be decomposed into two regions: the circular segment from b to b_1 with central angle θ_5, and the triangle $T(t_1 b b_1)$. The area of a circular segment with central angle θ and radius r is $r^2(\theta - \sin\theta)/2$, which in our case reduces to $(\theta_5 - \sin\theta_5)/2$. Concerning the triangle, the inscribed angle theorem implies that the angle $\widehat{b_1 t_1 b} = \theta_5/2$; hence, by the law of sines, its area can be written as

$$\frac{\overline{t_1 b} \cdot \overline{t_1 b_1} \cdot \sin(\theta_5/2)}{2}. \tag{2}$$

Finally, by elementary trigonometry, $\overline{t_1 b} = 2\sin[(\theta_5 + \theta_6)/2]$ and $\overline{t_1 b_1} = 2\sin[(\theta_6)/2]$. Substituting into (2) and adding up the areas of the two regions, we obtain

$$\lambda(S_1) = \frac{\theta_5 - \sin\theta_5}{2} + 2\sin\left(\frac{\theta_5}{2}\right)\sin\left(\frac{\theta_6}{2}\right)\sin\left(\frac{\theta_5 + \theta_6}{2}\right).$$

By a similar argument, we obtain

$$\lambda(S_2) = \frac{\theta_1 - \sin\theta_1}{2} + 2\sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_5 + \theta_6}{2}\right)\sin\left(\frac{\theta_1 + \theta_5 + \theta_6}{2}\right).$$

Summing up $\lambda(S_1)$ and $\lambda(S_2)$, we find

$$\begin{aligned}\lambda(D_1) = & \frac{\theta_1 - \sin\theta_1}{2} + \frac{\theta_5 - \sin\theta_5}{2} \\ & + 2\sin\left(\frac{\theta_5+\theta_6}{2}\right)\left[\sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_1+\theta_5+\theta_6}{2}\right) + \sin\left(\frac{\theta_5}{2}\right)\sin\left(\frac{\theta_6}{2}\right)\right].\end{aligned} \tag{3}$$

After some manipulations shown separately in the following Lemma .1.2, this expression simplifies to

$$\lambda(D_1) = \frac{\theta_1 + \theta_5 + \sin\theta_6 - \sin(\theta_1+\theta_5+\theta_6)}{2}.$$

The derivation of a specular formula for $\lambda(D_2)$ is analogous. ■

Lemma .1.2 The expression in (3) for $\lambda(D_1)$ can be rewritten as

$$\lambda(D_1) = \frac{\theta_1 + \theta_5 + \sin\theta_6 - \sin(\theta_1+\theta_5+\theta_6)}{2}.$$

Proof. Let $p = \theta_5/2$ and $q = \theta_6/2$. Then

$$\begin{aligned}\lambda(S_1) &= \frac{2p - \sin(2p)}{2} + 2\sin(p)\sin(q)\sin(p+q) \\ &= \frac{2p - \sin(2p)}{2} + 2\sin(p+q)\left[\cos(p-q) - \cos(p+q)\right] \\ &= \frac{2p - \sin(2p)}{2} + 2\sin(p+q)\cos(p-q) - \frac{\sin\left[2(p+q)\right]}{2} \\ &= \frac{2p - \sin(2p)}{2} + \frac{\sin(2p)+\sin(2q)}{2} - \frac{\sin\left[2(p+q)\right]}{2} \\ &= \frac{2p + \sin(2q) - \sin\left[2(p+q)\right]}{2} \\ &= \frac{\theta_5 + \sin(\theta_6) - \sin(\theta_5+\theta_6)}{2}.\end{aligned}$$

An analogous derivation with $p = \theta_1/2$ and $q = (\theta_5+\theta_6)/2$ leads to

$$\lambda(S_2) = \frac{\theta_1 + \sin(\theta_5+\theta_6) - \sin\left[(\theta_1+\theta_5+\theta_6)\right]}{2}.$$

Summing up $\lambda(S_1)$ and $\lambda(S_2)$ we obtain the target formula for $\lambda(D_1)$. ■

Proof of Theorem 4.3.1 We compute Primus' best reply function. Given t_1, b_1, t_2, b_2, and b, Primus would like to choose t in order to minimise $\lambda(D_1)$. Because of the 1–1 mapping between t and θ_1, we can reformulate this problem as the choice of the optimal angle θ_1

and compute his best reply with respect to θ_1. Differentiating (1) from Lemma .1.1, we find

$$\frac{\partial \lambda(D_1)}{\partial \theta_1} = \frac{1 - \cos(\theta_1 + \theta_5 + \theta_6)}{2} > 0$$

for any argument, because $0 < |\theta_1 + \theta_5 + \theta_6| < 2\pi$ under the assumption that l and r are interior. Since $\lambda(D_1)$ is (strictly) increasing in θ_1, minimising θ_1 by choosing $t = t_1$ is a dominant strategy for Primus. By a similar argument, $b = b_2$ is a dominant strategy for Secunda. Thus, the unique Nash equilibrium (in dominant strategies) is $(t^*, b^*) = (t_1, b_2)$.$\square$

.1.2 Proof of Proposition 4.3.1

We use the same notation of the previous proof. Hence, $\tau = \widehat{t_1 o t_2} = \theta_1 + \theta_2$ and $\beta = \widehat{b_1 o b_2} = \theta_4 + \theta_5$. Moreover, $\theta_R = \theta_3$ and $\theta_L = \theta_6$.

Proof. The thick line defining the common categorisation divides the disagreement region into two sectors $S_1(t_1 t_2 b_2)$ and $S_2(b_2 b_1 t_1)$. The area $\lambda(S_1)$ is the difference between the areas of the circular segment from t_1 to b_2 with central angle $(\tau + \theta_3)$ and of the circular segment from t_2 to b_2 with central angle θ_3. Hence,

$$\lambda(S_1) = \frac{\tau + \sin\theta_3 - \sin(\tau + \theta_3)}{2}.$$

Similarly,

$$\lambda(S_2) = \frac{\beta + \sin\theta_6 - \sin(\beta + \theta_6)}{2}.$$

Note that $(\tau+\theta_3)+(\beta+\theta_6) = 2\pi$; consequently, $\sin(\tau+\theta_3) = -\sin(\beta+\theta_6)$.

Clearly, Primus is stronger if and only if $\lambda(S_1) - \lambda(S_2) \geq 0$. The sign of the difference

$$\lambda(S_1) - \lambda(S_2) = \frac{\tau - \beta + \sin\theta_3 - \sin\theta_6 - 2\sin(\tau + \theta_3)}{2} \tag{4}$$

is not trivial. We distinguish two cases and study such sign.

1) Assume $\tau+\theta_3 \geq \pi \geq \beta+\theta_6$. We consider two sub-cases, depending on the sign of $\theta_6 - \theta_3$. Let us begin with $\theta_6 \geq \theta_3$. We have

$$\begin{aligned}\lambda(S_1) - \lambda(S_2) &= \frac{\tau - \beta + \sin\theta_3 - \sin\theta_6 - 2\sin(\tau + \theta_3)}{2} \qquad (5)\\ &= \frac{2(\tau + \theta_3 - \pi) + [(\theta_6 - \sin\theta_6) - (\theta_3 - \sin\theta_3)] - 2\sin(\tau + \theta_3)}{2}.\end{aligned}$$

Since $\tau + \theta_3 \geq \pi$ by assumption, the first and the last term in the numerator are positive. Moreover, as the function $x - \sin x$ is increasing on $(0, \pi)$, the term in square brackets is also positive. Hence, $\lambda(S_1) - \lambda(S_2) \geq 0$.

Consider now the sub-case $\theta_6 < \theta_3$. Decomposing S_1 into the circular segment from t_1 to t_2 with central angle τ and the triangle $T(t_1 t_2 b_2)$, we obtain

$$\lambda(S_1) = \frac{\tau - \sin\tau}{2} + 2\sin\left(\frac{\theta_3}{2}\right)\sin\left(\frac{\tau+\theta_3}{2}\right)\sin\left(\frac{\tau}{2}\right),$$

and, similarly,

$$\lambda(S_2) = \frac{\beta - \sin\beta}{2} + 2\sin\left(\frac{\theta_6}{2}\right)\sin\left(\frac{\beta+\theta_6}{2}\right)\sin\left(\frac{\beta}{2}\right).$$

Hence,

$$\begin{aligned}\lambda(S_1) - \lambda(S_2) = {} & \frac{(\tau - \sin\tau) - (\beta - \sin\beta)}{2} \\ & +2\sin\left(\frac{\tau+\theta_3}{2}\right)\left[\sin\left(\frac{\theta_3}{2}\right)\sin\left(\frac{\tau}{2}\right) - \sin\left(\frac{\theta_6}{2}\right)\sin\left(\frac{\beta}{2}\right)\right].\end{aligned} \tag{6}$$

The first term is positive by the increasing monotonicity of the function $(x - \sin x)$ on $(0, \pi)$. We claim that the second term is also positive. If $\theta_3 \leq \pi$, this follows because $\sin x$ is increasing in $(0, \pi/2)$, and thus $\sin(\theta_3/2)\sin(\tau/2) \geq \sin(\theta_3/2)\sin(\beta/2) \geq \sin(\theta_6/2)\sin(\beta/2)$. If $\theta_3 > \pi$, then $\theta_6 \leq \tau + \beta + \theta_6 = 2\pi - \theta_3 < \pi$; thus, $\sin(\theta_6/2) \leq \sin(\pi - \theta_3/2) = \sin(\theta_3/2)$, which suffices to establish the claim. From the positivity of the two terms, we conclude that $\lambda(S_1) \geq \lambda(S_2)$.

2) Assume $\tau + \theta_3 < \beta + \theta_6$. Since by assumption $\tau \geq \beta$, we have $\theta_6 \geq \theta_3$. By (4), using the identity $\tau + \beta + \theta_3 + \theta_6 = 2\pi$, we have

$$2\left[\lambda(S_1) - \lambda(S_1)\right] = \tau - \beta + \sin\theta_3 + \sin(\tau + \beta + \theta_3) - 2\sin(\tau + \theta_3)$$

and it suffices to study the sign of the right-hand term. Fix t_2 and b_2. Given τ in $(0, \pi)$, consider the function $f(\beta) = \tau - \beta + \sin\theta_3 + \sin(\tau + \beta + \theta_3) - 2\sin(\tau + \theta_3)$ for β in $(0, \pi)$. Since $f'(\beta) = -1 + \cos(\tau + \beta + \theta_3) < 0$, the function is strictly decreasing on $[0, \tau]$. Moreover,

$$f(0) = \tau + \sin\theta_3 - \sin(\tau + \theta_3) = [(\tau + \theta_3) - \sin(\tau + \theta_3)] - (\theta_3 - \sin\theta_3) \geq 0$$

by the increasing monotonicity of $(x - \sin x)$ on $[0, \pi]$. Finally, we have

$$\begin{aligned} f(\tau) &= \sin\theta_3 + \sin(\theta_3 + 2\tau) - 2\sin(\tau + \theta_3) && (7)\\ &= \sin(\theta_3) + [\sin(\theta_3)\cos(2\tau) + \cos(\theta_3)\sin(2\tau)] - 2\left[\sin(\tau)\cos(\theta_3) + \cos(\tau)\sin(\theta_3)\right] && (8)\\ &= \sin(\theta_3)\left[1 + \cos(2\tau) - 2\cos\tau\right] + \cos(\theta_3)\left[\sin(2\tau) - 2\sin\tau\right] \end{aligned}$$

Using the identities $\cos(2\tau) = 2\cos^2\tau - 1$ and $\sin(2\tau) = 2\sin\tau\cos\tau$, we obtain

$$f(\tau) = 2\left[\cos\tau - 1\right]\sin\left(\tau + \theta_3\right) \leq 0.$$

By the intermediate value theorem, there exists a unique $\overline{\beta}$ in $[0, \tau]$ such that $f(\overline{\beta}) = 0$. For $\beta \leq \overline{\beta}$, $\lambda(S_1) \geq \lambda(S_2)$ and Primus is stronger. For $\beta > \overline{\beta}$, the opposite inequality holds and Secunda is stronger. ∎

.1.3 Proof of Theorem 4.3.2

Similarly to the above (except for switching b_1 and b_2), the endpoints (t_i, b_i) for the two agents' chords and their choices for t and b identify six sectors. Proceeding clockwise, these are numbered from 1 to 6 on the left-hand side of Figure 8.

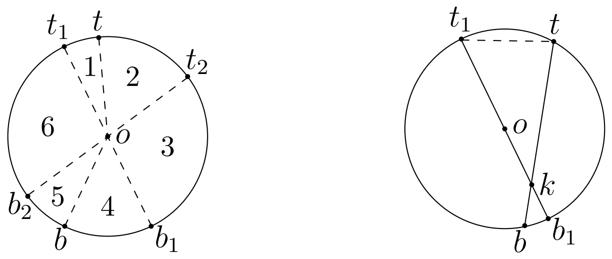


FIGURE 8: Visual aids for the proof of Theorem 4.3.2.

For each sector i, we denote its central angle by θ_i. The notation is similar, except that now $\theta_3 = \widehat{t_2 o b_1}$, $\theta_4 = \widehat{b_1 o b}$, $\theta_5 = \widehat{b o b_2}$, and $\theta_6 = \widehat{b_2 o t_1}$. Recall that $\tau = \theta_1 + \theta_2$ and $\beta = \theta_4 + \theta_5$; moreover, since the categorisations are characterised by diameters, $\tau = \beta$. The following lemma characterises the disagreement area of each player as a function of the six central angles.

Lemma .1.3 The disagreement areas for Primus and Secunda are, respectively:

$$\lambda(D_1) = \frac{\theta_1 - \sin\theta_1}{2} + \frac{\theta_4 - \sin\theta_4}{2} + 2\cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_4}{2}\right)\frac{\sin^2(\theta_1/2) + \sin^2(\theta_4/2)}{\sin(\theta_1/2 + \theta_4/2)}, \quad (9)$$

and

$$\lambda(D_2) = \frac{\theta_2 - \sin\theta_2}{2} + \frac{\theta_5 - \sin\theta_5}{2} + 2\cos\left(\frac{\theta_2}{2}\right)\cos\left(\frac{\theta_5}{2}\right)\frac{\sin^2(\theta_2/2) + \sin^2(\theta_5/2)}{\sin(\theta_2/2 + \theta_5/2)}.$$

Proof. The disagreement region D_1 for Primus can be decomposed into the two sector-like regions $S_1(t_1tk)$ and $S_2(kb_1b)$ as shown on the right-hand side of Figure 8. We compute the areas $\lambda(S_1)$ and $\lambda(S_2)$, and then add them up to obtain $\lambda(D_1)$.

The region $S_1(t_1tk)$ can be decomposed into two parts: the circular segment from t_1 to t with central angle θ_1, and the triangle $T(t_1tk)$. The area of the circular segment is $(\theta_1 - \sin\theta_1)/2$. The computation of the area of the triangle needs to take into account that the position of k depends on t. We use the ASA formula: given the length a of one side and the size of its two adjacent angles α and γ, the area is $(a^2 \sin\alpha\sin\gamma)/(2\sin(\alpha+\gamma))$. We pick $a = \overline{tt_1}$, $\alpha = \widehat{kt_1t}$, and $\gamma = \widehat{t_1tk}$. By the inscribed angle theorem, $\alpha = (\pi - \theta_1)/2$ and $\gamma = (\pi - \theta_4)/2$. Recall that $\overline{tt_1} = 2\sin(\theta_1/2)$; moreover, $\sin\alpha = \sin((\pi - \theta_1)/2) = \cos(\theta_1/2)$ and, similarly, $\sin\gamma = \cos(\theta_4/2)$. Hence,

$$\lambda(T) = \frac{2\left(\sin(\theta_1/2)\right)^2 \cdot \cos(\theta_1/2) \cdot \cos(\theta_4/2)}{\sin(\theta_1/2 + \theta_4/2)}.$$

Adding up the two areas, we obtain

$$\lambda(S_1) = \frac{\theta_1 - \sin\theta_1}{2} + \frac{2\left(\sin(\theta_1/2)\right)^2 \cdot \cos(\theta_1/2) \cdot \cos(\theta_4/2)}{\sin(\theta_1/2 + \theta_4/2)}.$$

By a similar argument,

$$\lambda(S_2) = \frac{\theta_4 - \sin\theta_4}{2} + \frac{2\left(\sin(\theta_4/2)\right)^2 \cdot \cos(\theta_1/2) \cdot \cos(\theta_4/2)}{\sin(\theta_1/2 + \theta_4/2)}.$$

Summing up $\lambda(S_1)$ and $\lambda(S_2)$ provides the formula for $\lambda(D_1)$. The derivation of a specular formula for $\lambda(D_2)$ is analogous. ■

A direct study of the sign of the derivative $\partial\lambda(D_1)/\partial\theta_1$ is quite involved, but the following lemma greatly simplifies it. An analogous result holds for Secunda.

Lemma .1.4 Let $a = \cos(\theta_4/2)$, $b = \sin(\theta_4/2)$, $c = ab = \sin(\theta_4)/2$, and $x = \tan(\theta_1/4)$. Then

$$\operatorname{sgn}\left[\frac{\partial\lambda(D_1)}{\partial\theta_1}\right] = \operatorname{sgn}\left[P(x)\right], \tag{10}$$

where

$$P(x) = -\Big[c\left(1+x^2\right)^2 - 2\left(\sqrt{2}+1\right)x\left(1-x^2\right)\Big].$$

Proof. Differentiating (9) from Lemma .1.3 and using a few trigonometric identities, we obtain

$$\begin{aligned}\frac{\partial\lambda(D_1)}{\partial\theta_1} &= \frac{1-\cos\theta_1}{2} + \frac{2\sin(\theta_1/2)\cos^2(\theta_1/2)\cos(\theta_4/2)}{\sin(\theta_1/2+\theta_4/2)} - \frac{\cos(\theta_4/2)\left[\sin^2(\theta_1/2)+\sin^2(\theta_4/2)\right]}{\sin^2(\theta_1/2+\theta_4/2)} \\ &\quad\cdot\Big[\sin(\theta_1)\sin(\theta_1/2+\theta_4/2)+\cos(\theta_1)\cos(\theta_1/2+\theta_4/2)\Big] \\ &= \sin^2(\theta_1/2) + \frac{\sin(\theta_1)\cos(\theta_1/2)\cos(\theta_4/2)}{\sin(\theta_1/2+\theta_4/2)} - \frac{\cos^2(\theta_4/2)\left[\sin^2(\theta_1/2)+\sin^2(\theta_4/2)\right]}{\sin^2(\theta_1/2+\theta_4/2)}\end{aligned}$$

Let $a = \cos(\theta_4/2)$, $b = \sin(\theta_4/2)$, and $x = \tan(\theta_1/4)$. Recall the double angle formulas $\sin(\theta_1/2) = 2x/(1+x^2)$ and $\cos(\theta_1/2) = (1-x^2)/(1+x^2)$. Then

$$\sin\left(\frac{\theta_1+\theta_4}{2}\right) = a\left(\frac{2x}{1+x^2}\right) + b\left(\frac{1-x^2}{1+x^2}\right) = \frac{2ax+b(1-x^2)}{1+x^2}.$$

Substituting with respect to the new variable x, we find

$$\begin{aligned}\frac{\partial\lambda(D_1)}{\partial\theta_1} &= \left(\frac{2x}{1+x^2}\right)^2 + \frac{4ax(1-x^2)^2}{(1+x^2)^2\left[2ax+b(1-x)^2\right]} - \frac{a^2\left[4x^2+b^2(1+x^2)^2\right]}{\left[2ax+b(1-x)^2\right]^2} \\ &= -\frac{N(x)}{(1+x^2)^2\left[2ax+b(1-x^2)\right]^2}, \end{aligned} \tag{11}$$

where, using the identity $a^2+b^2=1$, the polynomial in the numerator can be written as

$$N(x) = a^2\left(1+x^2\right)^2\left[4x^2+b^2\left(1+x^2\right)^2\right] - 4ax\left(1-x^2\right)^2\left[2ax+b(1-x^2)\right] - 4x^2[2ax+b(1-x^2)]^2.$$

Let $c = ab = \sin(\theta_4)/2$ and rewrite $N(x)$ after collecting terms with respect to c:

$$\begin{aligned}N(x) &= c^2\left(1+x^2\right)^4 - 4cx\left(1-x^2\right)\left(1+x^2\right)^2 - 4x^2\left(1-x^2\right)^2 \\ &= \left[c\left(1+x^2\right)^2 - 2x\left(1-x^2\right)\right]^2 - \left[2\sqrt{2}x\left(1-x^2\right)\right]^2 \\ &= \left[c\left(1+x^2\right)^2 - 2\left(\sqrt{2}+1\right)x\left(1-x^2\right)\right]\cdot\left[c\left(1+x^2\right)^2 + 2\left(\sqrt{2}-1\right)x\left(1-x^2\right)\right].\end{aligned}$$

As both θ_1 and θ_4 are in the open interval $(0,\pi)$ by construction, we have $x = \tan(\theta_1/4) > 0$ and $c = \sin(\theta_4)/2 > 0$; hence, the second term in the multiplication is strictly positive. Returning to (11), this implies

$$\operatorname{sgn}\left[\frac{\partial\lambda(D_1)}{\partial\theta_1}\right] = -\operatorname{sgn}\Big[N(x)\Big] = \operatorname{sgn}\Big[P(x)\Big],$$

with $P(x) = -\left[c(1+x^2)^2 - 2(\sqrt{2}+1)x(1-x^2)\right]$, as it was to be shown. ■

It is convenient to work with the central angles subtended by the points on the circumference. Recall that, given t_1, t_2, b_1, and b_2, Primus and Secunda simultaneously choose t and b, respectively. Then Secunda's choice of b is in a 1-1 mapping with the angle $\theta_5 = \widehat{b_2 o b}$, while Primus' choice of t has a similar relation to $\theta_1 = \widehat{t_1 o t}$.

The following lemma characterizes Primus' best reply using the central angles θ_1 and θ_5, rather than the endpoints t and b. As it turns out, such best reply is always unique; hence, with obvious notation, we denote it as the function $\theta_1 = r_1(\theta_5)$. Correspondingly, let $\theta_5 = r_2(\theta_1)$ be the best reply function for Secunda. Finally, recall our assumption that the individual categorisations are supported by diameters: this implies that the two angular distances $\tau = \theta_1 + \theta_2$ and $\beta = \theta_4 + \theta_5$ are equal with $0 \leq \tau = \beta < \pi$; moreover, players' initial positions have the same strength and the game is symmetric.

Lemma .1.5 The best reply functions for the two players are

$$r_1(\theta_5) = \arcsin\left(\frac{\sin(\beta - \theta_5)}{\sqrt{2}+1}\right) \qquad \text{and} \qquad r_2(\theta_1) = \arcsin\left(\frac{\sin(\tau - \theta_1)}{\sqrt{2}+1}\right),$$

with $0 \leq \theta_5 \leq \beta$ and $0 \leq \theta_1 \leq \tau$.

Proof. Consider Primus. (The argument for Secunda is identical.) For any θ_5 in $[0, \beta]$, we search which value of θ_1 in $[0, \tau]$ minimises $\lambda(D_1)$. We distinguish two cases.

First, suppose $\theta_5 = \beta$. Then $\theta_4 = 0$ and $\lambda(D_1) = (\theta_1 + \sin\theta_1)/2$. As this function is increasing in θ_1, the optimal value is $\theta_1^* = 0$.

Second, suppose $\theta_5 < \beta$. We begin by finding the stationary points of $\lambda(D_1)$. Recall that we let $x = \tan(\theta_1/4)$. By Lemma .1.4, $\partial\lambda(D_1)/\partial\theta_1 = 0$ if and only if $P(x) = 0$; that is, if and only if

$$c = \frac{2\left(\sqrt{2}+1\right)x\left(1-x^2\right)}{\left(1+x^2\right)^2}.$$

Replacing the double angle formulæ $\sin(\theta_1/2) = 2x/(1+x^2)$ and $\cos(\theta_1/2) = (1-x^2)/(1+x^2)$, we obtain

$$c = \left(\sqrt{2}+1\right)\sin\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_1}{2}\right) = \left(\sqrt{2}+1\right)\frac{\sin\theta_1}{2}.$$

On the other hand, since $c = (\sin\theta_4)/2$ by definition and $\theta_4 + \theta_5 = \beta$, this yields

$$\sin\theta_1 = \frac{\sin\theta_4}{\sqrt{2}+1} = \frac{\sin(\beta - \theta_5)}{\sqrt{2}+1}.$$

Since $\theta_5 \in [0, \beta]$, the only solutions to this equation are the supplementary angles θ_1' and $\theta_1'' = \pi - \theta_1'$ with

$$\theta_1' = \arcsin\left(\frac{\sin(\beta - \theta_5)}{\sqrt{2}+1}\right) < \frac{\pi}{2} < \pi - \theta_1' = \theta_1''.$$

These are the stationary points for $\lambda(D_1)$.

Clearly, $\theta_1' \geq 0$. We claim that $\theta_1' < \tau$. If $\pi/2 \leq \tau$, this is obvious. Suppose instead $\tau < \pi/2$. Since $\theta_4 < \beta = \tau < \pi/2$, we have $\sin\theta_1' = \left(\sqrt{2}-1\right)\sin(\theta_4) < \sin\theta_4 < \sin\tau$ and thus $\theta_1' < \tau$. We conclude that the stationary point θ_1' belongs to the interval $[0, \tau]$.

For $\theta_1 = 0$, we have $x = 0$ and $P(x)|_{x=0} = -c = -(\sin\theta_4)/2 < 0$. Therefore, we have by continuity that $P(x)$ changes sign from negative to positive in θ_1' and from positive to negative in θ_1''. By Lemma .1.4, this implies that the only local minimisers for $\lambda(D_1)$ in the compact interval $[0, \tau]$ are $\theta = \theta_1'$ and $\theta = \tau$. Comparing the corresponding values for $\lambda(D_1)$, we find

$$\lambda(D_1)|_{\theta_1 = \theta'} < \lambda(D_1)|_{\theta_1 = 0} < \lambda(D_1)|_{\theta_1 = \tau},$$

where the first inequality follows from the (strict) negativity of $\partial\lambda(D_1)/\partial\theta_1$ in $[0, \theta - 1')$ and the second inequality from a direct comparison. Hence, the global minimiser is θ'. Combining the two cases, it follows that, for any θ_5 in $[0, \beta]$, the unique best reply is $r_1(\theta_5) = \arcsin\left[\sin(\beta - \theta_5) / \left(\sqrt{2}+1\right)\right]$. ■

Proof of Theorem 4.3.2 A Nash equilibrium is any fixed point (θ_1, θ_5) of the map

$$\begin{pmatrix}\theta_1 \\ \theta_5\end{pmatrix} = \begin{pmatrix} r_1(\theta_5) \\ r_2(\theta_1)\end{pmatrix}$$

from $[0, \tau] \times [0, \beta]$ into itself. Substituting from Lemma .1.5 and using $\tau = \beta$, we obtain the system of equations

$$\begin{cases} \sin(\theta_1) = \dfrac{\sin(\tau - \theta_5)}{\sqrt{2}+1} \\[2ex] \sin(\theta_5) = \dfrac{\sin(\tau - \theta_1)}{\sqrt{2}+1} \end{cases} \tag{12}$$

Multiplying across gives

$$\sin(\theta_1)\sin(\tau-\theta_1) = \sin(\theta_5)\sin(\tau-\theta_5);$$

or, using the prosthaphaeresis formula,

$$\cos(2\theta_1-\tau)-\cos\tau = \cos(2\theta_5-\tau)-\cos\tau$$

from which we get that the only two possible solutions in $[0,\tau]$ are

$$\theta_1=\theta_5 \qquad \text{or} \qquad \theta_1=\tau-\theta_5.$$

When $\tau>0$, the second possibility can be discarded because, when replaced in (12), it would yield the contradiction $\theta_1=\theta_5=\tau=0$. (When $\tau=0$, we trivially obtain $\theta_1=-\theta_2$ as in the first case.) Hence, we are left with $\theta_1=\theta_5$.

Substituting in the first equation of (12), we obtain

$$\sin(\theta_1)=\frac{\sin(\tau-\theta_1)}{\sqrt{2}+1}=\frac{\sin\tau\cos\theta_1-\cos\tau\sin\theta_1}{\sqrt{2}+1}.$$

As $0\le\theta_1<\pi/2$, dividing by $\cos\theta_1$ yields

$$\tan(\theta_1)=\frac{\sin\tau}{\sqrt{2}+1+\cos\tau}$$

and the result follows.

.1.4 Proof of Proposition 4.3.2

Proof. Recall that the payoff for an agent is the opposite of the area of the disagreement region. Consider Primus. (The proof for Secunda is analogous.) Let D^* and D^s be the region of disagreement between Primus' and the common categorisation at the equilibrium and, respectively, at the Nash cooperative solution. For $\tau=0$, $D^*=D^s$. Hence, we assume $\tau\neq 0$ and show that $\lambda(D^*)-\lambda(D^s)>0$.

At the Nash bargaining solution, $\theta_1^s=\theta_2^s=\tau/2$; replacing these into (10), we find $\lambda(D^s)=\tau/2$. At the Nash equilibrium, $\theta_1^*=\theta_5^*$ and thus $\theta_4^*=\tau-\theta_1^*$; substituting these into (10) and dropping superscripts and subscripts for simplicity, we obtain

$$\lambda(D^*)=\frac{\tau}{2}-\left[\frac{\sin\theta+\sin(\tau-\theta)}{2}\right]+2\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\tau-\theta}{2}\right)\frac{\sin^2(\theta/2)+\sin^2((\tau-\theta)/2)}{\sin(\tau/2)}.$$

Hence, using standard trigonometric identities,

$$
\begin{aligned}
\lambda(D^*) - \lambda(D^s) &= -\left[\frac{\sin\theta + \sin(\tau-\theta)}{2}\right] \\
&\qquad + 2\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\tau-\theta}{2}\right)\frac{\sin^2(\theta/2) + \sin^2((\tau-\theta)/2)}{\sin(\tau/2)} \\
&= -\sin\left(\frac{\tau}{2}\right)\cos\left(\theta - \frac{\tau}{2}\right) \\
&\qquad + \left[\frac{1}{\sin(\tau/2)}\right]\left[\cos\left(\frac{\tau}{2}\right) + \cos\left(\theta - \frac{\tau}{2}\right)\right]\left[1 - \frac{\cos\theta}{2} - \frac{\cos(\tau-\theta)}{2}\right] \\
&= -\left[\frac{1}{\sin(\tau/2)}\right]\left[1 - \cos^2\left(\frac{\tau}{2}\right)\right]\cos\left(\theta - \frac{\tau}{2}\right) \\
&\qquad + \left[\frac{1}{\sin(\tau/2)}\right]\left[\cos\left(\frac{\tau}{2}\right) + \cos\left(\theta - \frac{\tau}{2}\right)\right]\left[1 - \cos\left(\frac{\tau}{2}\right)\cos\left(\theta - \frac{\tau}{2}\right)\right] \\
&= \left[\frac{\cos(\tau/2)}{\sin(\tau/2)}\right]\sin^2\left(\theta - \frac{\tau}{2}\right),
\end{aligned}
$$

from which we obtain

$$\operatorname{sgn}\left[\lambda(D^*) - \lambda(D^s)\right] = \operatorname{sgn}\left[\tan(\tau/2)\right].$$

Since $0 < \tau < \pi$, $\tan(\tau/2) > 0$, and the claim follows. ■

Bibliography

[1] Benz, A., Jager, G., van Rooy, R. (2005), editors. *Game Theory and Pragmatics*. Basingstoke, UK: Palgrave Macmillan.

[2] Clark, R. (2012). *Meaningful Games: Exploring Language with Game Theory*. Cambridge, MA: The MIT Press.

[3] Cohen, H. & Lefebvre, C. (2005), editors. *Handbook of Categorization in Cognitive Science*. Amsterdam: Elsevier.

[4] Gärdenfors, P. (2000). *Conceptual Spaces: The Geometry of Thought*. Cambridge, MA: The MIT Press.

[5] Gärdenfors, P. (2014). *The Geometry of Meaning: Semantics based on Conceptual Spaces*. Cambridge, MA: The MIT Press.

[6] Gärdenfors, P. (2014a). Levels of communication and lexical semantics. *Synthese*, accepted for publication.

[7] Honkela, T., Könönen, V., Lindh–Knuutila, T., Paukkeri, M.-S. (2008). Simulating processes of concept formation and communication. *Journal of Economic Methodology* **15**, 245–259.

[8] Jäger, G. (2007). The evolution of convex categories. *Linguistics and Philosophy* **30**, 551–564.

[9] Jäger, G., & Van Rooij, R. (2007). Language structure: Psychological and social constraints. *Synthese* **159**, 99–130.

[10] Jäger, G., Metzger, L.P., Riedel, F. (2011). Voronoi languages: Equilibria in cheap-talk games with high-dimensional types and few signals, *Games and Economic Behavior* **73**, 517–537.

[11] Keysar, B. (2007). Communication and miscommunication: The role of egocentric processes, *Intercultural Pragmatics* **4**, 71–84.

[12] Kruschke, J.K. (2008). *Models of categorization*. In: R. Sun (Ed.), The Cambridge Handbook of Computational Psychology, New York: Cambridge University Press, 267–301.

[13] LiCalzi, M. & Mâagli, N. (2016) *Bargaining over a common categorisation*. Synthese 193, 705-723.

[14] Ludlow, P. (2014). *Living Words: Meaning Underdetermination and the Dynamic Lexicon*. Oxford, UK: Oxford University Press.

[15] Mervis, C., & Rotsch, E. (1981). Categorisation of natural objects. *Annual Review of Psychology* **32**, 89–115.

[16] Parikh, P. (2010). *Language and equilibrium*. Cambridge, MA: The MIT Press.

[17] Rotsch, E. (1975). Cognitive representations of semantic categories. *Journal of Experimental Psychology: General* **104**, 192–233.

[18] Warglien, M. & Gärdenfors, P. (2013). Semantics, conceptual spaces, and the meeting of minds. *Synthese* **190**, 2165–2193.

[19] Warglien, M. & Gärdenfors, P. (2015). Meaning negotiation. In: Zenker, F. & Gärdenfors, P. (eds.), *Applications of Conceptual Spaces: The Case for Geometric Knowledge Representation*, Dordrecht: Springer, 79–94.

[20] Wernerfelt, B. (2004). Organizational languages. *Journal of Economics and Management Strategy* **13**, 461–472.