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Essays on decision theory and bargaining theory

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“L’Université Paris 1 - Panthéon Sorbonne n’entend donner aucune approbation, ni improbation aux opinions émises dans cette thèse ; elles doivent être considérées comme propres à leur auteur."

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Chapitre 1

Introduction (Version française)

Les modèles mathématiques sont des outils très puissants pour l'étude du fonctionnement de l'économie et, plus généralement, des interactions sociales. Puisque chaque société est composée d'un ensemble d'individus, la compréhension des mécanismes à la base des choix des agents est fondamentale.

Quand nous nous intéressons aux décisions faites par des agents économiques deux aspects sont particulièrement importants et doivent être pris en considération, le *temps* et le *risque* (ou, de façon plus générale, l'incertitude).

La plupart de nos choix ont des conséquences temporelles. Considérons par exemple un agent, avec une dotation initiale d'un certain bien de consommation, qui vit dans une économie de deux périodes. Il doit décider comment répartir sa consommation. Le plus souvent, on fait l'hypothèse qu'il préfère consommer davantage dans la première période que dans la deuxième. Ce comportement est connu sous le nom d'*impatience*. Si l'on considère un modèle avec N périodes, alors on peut représenter le flux des revenus (ou des biens de consommation) comme un vecteur dans $\mathbb{R}^N$. À partir de l'article fondateur de Samuelson [1937], le *modèle d'utilité escomptée* a été le paradigme pour représenter l'impatience. Dans ce modèle, un flux inter-temporel $(x_0, x_1, \ldots, x_N)$ est évalué par la fonction d'utilité :

$$U(x_0, x_1, \ldots, x_N) = \sum_{n=0}^{N} \delta^n u(x_n).$$

La fonction $u : \mathbb{R} \to \mathbb{R}$ est une fonction d'utilité instantanée qui représente les préférences de l'agent. Pour chaque période, $n = 1, \ldots N$, δ^n, avec $\delta \in (0, 1)$, est un facteur d'actualisation qui indique que le décideur accorde une importance moindre aux périodes éloignées dans le temps. Ce modèle est sans doute celui qui a connu le plus grand succès dans le contexte du choix inter-temporel grâce à sa simplicité et son élégance. Pourtant, comme nous le montrerons plus tard, il n'est pas complètement satisfaisant car, par exemple, il ne permet pas de faire la distinction entre différents concepts d'impatience définis directement à partir des préférences du décideur (notamment il ne permet pas de différentier l'aversion au délai et la myopie [1]). Dans les Chapitre 3 et le Chapitre 4, nous nous intéressons à ce problème, dans un contexte plus général avec une infinité de périodes.

Un deuxième aspect fondamental qu'on rencontre dans tout type de décision est le risque. Comme pour le temps, le risque est une notion très importante en économie car, quand les agents doivent faire des choix, ils ne font pas face à des environnements déterministes. Imaginons une personne qui veut investir de l'argent en bourse : les actifs ressemblent à des loteries dont on ne peut pas déterminer la valeur future avec précision. La première et sans doute la plus importante contribution à cette littérature est celle de von Neumann and Morgenstern [1947], qui ont axiomatisé le *modèle d'utilité espérée* (EU).

Soit $\mathcal{C}$ un ensemble de prix et soit $\mathcal{P}_0(\mathcal{C})$ l'ensemble des probabilités simples sur $\mathcal{C}$ (c'est à dire les probabilités qui sont différentes de 0 seulement pour un nombre fini de $c \in \mathcal{C}$). Alors, sous des axiomes très intuitifs, il existe une fonction $u : \mathcal{C} \to \mathbb{R}$ telle que les préférences sur $\mathcal{P}_0(\mathcal{C})$ peuvent être représentées par la fonctionnelle :

$$U(P) = \sum_{\mathcal{C}} P(c)u(c)$$

pour tout $P \in \mathcal{P}_0(\mathcal{C})$. Dans le Chapitre 5, nous utilisons une version “target-based" de ce modèle, développée par Castagnoli and LiCalzi [1996], dans le cadre de la négociation coopérative.

1.1 Choix inter-temporel

Les deux chapitres après l'introduction traitent de la théorie du choix inter-temporel. Plus précisément, nous analysons d'un point de vue mathématique le comportement des agents ayant des préférences pour une consommation immédiate. Ce type de préférences “impatientes" joue un rôle clé dans le cadre de l'économie théorique et cela dès la fin du $19^{\text{ème}}$ siècle, comme on peut le constater dans la citation de Böhm-Bawerk [1891] :

> “Present goods are, as a rule, worth more than future goods of like kind and number. This proposition is the kernel and center of the interest theory which I have to present.” [2]

Plus dans le détail, nous nous appuyons sur des modèles introduits en économie par les travaux pionniers de Koopmans [1960] et Diamond [1965]. Comme eux, nous considérons un agent doté d'une relation de préférence, notée $\succsim$, sur l'ensemble des suites réelles et bornées (noté l^∞), interprétées comme des flux infinis de biens de consommation. Nous étudions les règles qu'il faut imposer à ses préférences afin qu'il montre de l'impatience.

1. Ces concepts sont définis plus loin.
2. Voir Böhm-Bawerk [1891], p. 237.

Autrement dit, nous voulons savoir sous quelles conditions un décideur, auquel deux quantités de bien de consommation seraient proposées à deux périodes distinctes, choisisse toujours celle plus proche du présent.

Depuis Samuelson [1937], le paradigme pour décrire un agent économique avec un comportement impatient a été le modèle d'utilité escomptée, où l'agent est muni d'une fonction d'utilité du type

$$U(x_0, x_1, \dots) = \sum_t \beta(t)u(x_t). \tag{1.1}$$

Dans ce modèle, l'attitude d'un agent vis à vis du futur est résumée par le facteur d'escompte $\beta : \mathbb{N} \to (0, 1]$, qui est une fonction strictement décroissante. Puisque l'agent accorde des poids décroissant avec le temps, il est impatient au sens de Samuelson.

De la même manière que le modèle Bayésien et le modèle d'espérance d'utilité de Savage [1954] sont les points de référence dans la théorie de la décision dans l'incertain (voir Gilboa and Marinacci [2013]), le modèle de Samuelson est (quasiment) le seul modèle utilisé dans la théorie du choix inter-temporel. Dans le Chapitre 3 et le Chapitre 4, nous dérogeons à l'usage de cette approche classique. Le Chapitre 3 est ainsi centré sur des formes fonctionnelles qui sont utilisées dans le cadre de l'ambiguïté. Le Chapitre 4 suit une approche topologique.

La contribution principale du premier chapitre réside dans la définition de deux concepts qui représentent des préférences impatientes.

La première notion, qu'on appelle *aversion au délai sur le long-terme*, exprime l'idée suivante. Imaginons qu'un agent faisant face à un certain flux de paiements soit invité à choisir entre deux paiements supplémentaires qui seront versés à deux dates différentes. Imaginons aussi que le paiement versé à la date la plus proche soit également le plus faible. Alors cet agent est averse au délai sur le long-terme s'il préfère toujours le paiement plus faible (mais versé plus tôt) dès que le paiement plus important est versé trop loin dans le futur.

La deuxième notion, qu'on appelle *aversion au délai sur le court-terme*, représente une idée très intuitive. Un décideur est averse au délai sur le court-terme si, à chaque fois qu'on lui demande de choisir entre deux paiements égaux versés à deux dates consécutives, il choisit toujours le paiement le plus proche temporellement.

Nous supposons ensuite que la relation de préférence $\succsim$ du décideur peut être représentée numériquement par trois modèles très répandus dans la littérature sur l'ambiguïté : le modèle d'utilité espérée (EU), le modèle d'utilité espérée à la Choquet (CEU) et le modèle d'utilité espérée MaxMin (MMEU). Ces modèles ont été introduits dans le contexte du choix dans l'incertain par les travaux de Schmeidler [1986], Schmeidler [1989] et Gilboa and Schmeidler [1989]. Leur usage pour décrire l'impatience n'est pas standard en économie et représente une des contributions principales du chapitre. Il se trouve en effet que les modèles EU, CEU et MMEU sont des alternatives très puissantes et flexibles au modèle standard d'utilité escomptée.

Nous caractérisons l'aversion au délai sur le long-terme dans ces trois modèles considérés, et nous montrons que c'est un type d'impatience assez faible. Elle ne dépend que des poids que le décideur donne aux périodes de temps, ou aux sous-ensembles des périodes. Il est intéressant de remarquer que l'on peut faire un parallèle entre cette notion et la théorie qui étudie l'impossibilité d'avoir des préférences qui sont en même temps parétienne (au sens strict) et anonymes (les préférences sont anonymes si elles traitent de manière égalitaire toute période de temps).

L'aversion au délai sur le court-terme se trouve être très intéressante pour plusieurs raisons. Premièrement, contrairement à l'aversion au délai sur le long-terme, nous montrons qu'il n'est pas possible de séparer les préférences du décideur et son évaluation du temps. L'aversion au délai sur le court-terme demande des propriétés sur les probabilités (ou la capacité) *et* sur l'utilité (marginale). Deuxièmement, un agent averse au délai sur le court-terme doit donner des poids décroissants dans le temps. Troisièmement, cette notion est la contrepartie comportementale de la définition d'impatience donnée par Fisher [1930]. En effet, nous prouvons que, dans le modèle EU, un individu est averse au délai sur le court-terme si et seulement si son taux marginal de substitution inter-temporelle est toujours plus grand que 1. Enfin, nous définissons le concept de *dominance temporelle* et nous montrons qu'il est équivalent à l'aversion au délai sur le court-terme. La dominance temporelle, liée à l'article de Foster and Mitra [2003] sur la dominance des flux de paiements, est une notion désirable qui rappelle la notion de dominance stochastique dans l'incertain.

Dans le Chapitre 4, nous approfondissons l'étude du concept d'aversion au délai sur le long-terme suivant une approche différente : au lieu de faire l'hypothèse que les préférences sont représentées numériquement (comme au Chapitre 3), nous utilisons un point de vue topologique.

L'espace l^∞ des suites réelles bornées, qui représente l'ensemble des flux infinis de revenus (ou de biens de consommation), est un espace de dimension infinie. Dans ce type d'espace, le choix de la topologie a des conséquences sur le comportement des agents. Ainsi, il ne faudra pas choisir une topologie que pour ses propriétés mathématiques. Mas-Colell and Zame [1991] écrivent :

> "It should be stressed that the choice of the topology can only be dictated by economic, rather than mathematical, considerations."

Le concept économique qui sous-tend les topologies étudiées dans le Chapitre 4 est précisément celui de l'aversion au délai sur le long-terme. Nous définissons deux espaces de Hausdorff localement convexes qui "escomptent" le futur d'une manière cohérente avec l'aversion au délai. Enfin, à la fin du chapitre, nous montrons que l'aversion au délai sur le long-terme est compatible avec la notion d'une plus grande aversion au délai de Benoît and Ok [2007] (qui ont inspiré notre définition).

D'abord, nous comparons ces topologies avec d'autres qui ont la propriété de représenter des préférences impatientes ou patientes. Schématiquement, nous trouvons qu'un décideur averse au délai sur le long-terme se situe "entre" un agent myope et un agent patient. Ensuite, nous nous intéressons à l'espace dual de l^∞ (qui est interprété comme l'espace des prix en économie). Le résultat le plus intéressant dit que le dual est égal à l'espace ba, c'est à dire l'espace des charges (autrement appelées mesures simplement additives) bornées.

Nos résultats ont des implications sur la théorie de l'équilibre général en dimension infinie, et sur les travaux qui considèrent une bulle spéculative comme la partie pathologique (non sigma-additive) d'une charge (voir Gilles and LeRoy [1992]). La comparaison des différentes topologies nous aide à clarifier un résultat de Araujo [1985]. Nous prouvons que, même si les agents sont impatients, il peut y avoir des économies où il n'y a pas d'équilibre. L'étude de l'espace dual implique que, si un équilibre existe, il est possible d'avoir des bulles spéculatives malgré l'impatience (au sens de l'aversion au délai).

Dans le reste de la Section 1.1 nous donnons les principaux résultats mathématiques des Chapitre 3 et 4.

1.1.1 Sur l'aversion au délai

Dans cette section nous illustrons les résultats principaux du Chapitre 3.

Nous étudions une relation de préférence d'un décideur sur l'ensemble $l_+^\infty = \{\mathbf{x} := (x_n)_{n\in\mathbb{N}} | x_n \geq 0 \, \forall n \text{ et } \sup_n x_n < +\infty\}$ des suites réelles, positives et bornées. Les éléments de l_+^∞ sont notés $\mathbf{x}, \mathbf{y}$, etc. et sont interprétés, comme des flux infinis de revenu ou de biens de consommation. L'ensemble des naturels $\mathbb{N}$ représentera le temps.

Dans ce chapitre, la relation de préférence $\succsim$ du décideur est représentée par :

– Le modèle d'utilité espérée (EU) : $U(\mathbf{x}) = \mathbb{E}_P[u(\mathbf{x})]$.

– Le modèle d'utilité espérée à la Choquet (CEU) : $U(\mathbf{x}) = \int u(\mathbf{x}) \, dv$.

– Le modèle d'utilité espérée MaxMin (MMEU) : $U(\mathbf{x}) = \min_{P\in C} \mathbb{E}_P[u(\mathbf{x})]$.

– Le modèle d'utilité escomptée : $U(\mathbf{x}) = \sum_{t=0}^{+\infty} \beta(t) u(x_t)$.

En généralisant la notion des *poids* que le décideur donne au temps, les modèles EU, CEU et MMEU généralisent ainsi le modèle d'utilité escomptée. Pour plus d'informations le lecteur pourra se référer à la Section 3.2.

Aversion au délai sur le long-terme

Dans cette section nous nous intéressons à l'étude de *l'aversion au délai sur le long-terme*. La définition mathématique est la suivante.

Définition 1.1.1. *Soit* $\succsim$ *une relation de préférence sur* l^{∞}_{+}*. On dit que* $\succsim$ *est* averse au délai sur le long-terme *si pour* $0 < a \leq b$*,* $n_0 \in \mathbb{N}$ *et* $\mathbf{x} \in l^{\infty}_{+}$*,* $\exists N := N(\mathbf{x}, n_0, a, b) > n_0$ *tel que* $\forall n \geq N$*,*

$$(x_{n_0} + a, \mathbf{x}_{-n_0}) \succ (x_n + b, \mathbf{x}_{-n}).$$

La Définition 1.1.1 dit qu'un décideur est averse au délai sur le long-terme si, faisant face à deux paiements, $a > 0$ fait à la période n_0 et $b \geq a$ fait à la période n, il choisit toujours le paiement plus immédiat, même si plus faible, chaque fois que l'autre est trop loin dans le futur.

Nous caractérisons ensuite l'aversion au délai sur le long-terme dans les trois modèles définis auparavant.

Proposition 1.1.1. *Soit* $\succsim$ *une relation de préférence représentée par le modèle CEU. Alors* $(i) \Leftrightarrow (ii)$ *:*
(i) $\succsim$ *est averse au délai sur le long-terme ;*
(ii) $\forall A \in 2^{\mathbb{N}}$ $v(A \cup \{n\}) \to_n v(A)$ *et* $\forall A \in 2^{\mathbb{N}}$*,* $\forall t \notin A$ $v(A \cup \{t\}) > v(A)$*.*

Proposition 1.1.2. *Soit* $\succsim$ *une relation de préférence représentée par le modèle MMEU. Alors* $(i) \Leftrightarrow (ii)$ *:*
(i) $\succsim$ *est averse au délai sur le long-terme ;*
(ii) $\forall P \in C$*,* $\forall n \in \mathbb{N}$*,* $P(\{n\}) > 0$*.*

Corollaire 1.1.1. *Soit* $\succsim$ *une relation de préférence représentée par le modèle EU. Alors* $(i) \Leftrightarrow (ii)$ *:*
(i) $\succsim$ *est averse au délai sur le long-terme ;*
(ii) $P(\{n\}) > 0 \, \forall n \in \mathbb{N}$*.*

Les trois caractérisations, qui dépendent seulement des poids que le décideur donne aux périodes de temps, sont liées à la monotonie stricte des préférences [3]. Cette relation est présentée dans le résultat ci-dessous.

Proposition 1.1.3. *Soit* $\succsim$ *une relation de préférence représentée soit par le modèle MMEU soit par le modèle EU. Alors* $(i) \Leftrightarrow (ii)$ *:*
(i) $\succsim$ *est averse au délai sur le long-terme ;*
(ii) $\succsim$ *est strictement monotone.*

Soit $\succsim$ *une relation de préférence représentée par le modèle CEU. Alors* $(i) \Rightarrow (ii)$ *:*
(i) $\succsim$ *est averse au délai sur le long-terme ;*
(ii) $\succsim$ *est strictement monotone.*

3. On dit q'une rélation de préférence sur l^{∞} est *monotone* si $x_k \geq y_k$ pour tout $k \in \mathbb{N}$ implique $\mathbf{x} \succsim \mathbf{y}$ et *strictement monotone* si $x_k \geq y_k$ pour tout $k \in \mathbb{N}$ et $\mathbf{x} \neq \mathbf{y}$ implique $\mathbf{x} \succ \mathbf{y}$.

La Proposition 1.1.3 peut être utilisée pour clarifier un résultat de Basu and Mitra [2003], où les auteurs étudient l'impossibilité d'être strictement parétien et de traiter également toute génération. Considérons la définition suivante.

Définition 1.1.2. (Basu and Mitra [2003]) *Une relation de préférence* $\succsim$ *est* anonyme *si pour tous* $\mathbf{x}, \mathbf{y} \in l_+^\infty$ *tels qu'il existe* $i, j \in \mathbb{N}$ *tels que* $x_i = y_j$ *et* $x_j = y_i$ *et tels que pour* $k \in \mathbb{N} \setminus \{i, j\}$, $x_k = y_k$, *alors* $\mathbf{x} \sim \mathbf{y}$.

Le résultat principal de Basu and Mitra [2003] dit qu'il n'y a pas de représentation numérique pour une relation de préférence qui est en même temps anonyme et strictement monotone. L'intuition derrière ce résultat n'est pas facile à saisir. La Proposition 1.1.3, surtout la partie qui concerne le modèle EU, peut nous guider dans l'interprétation. Dans le modèle EU, la monotonie stricte et l'aversion au délai sur le long-terme sont équivalentes. Il est facile de voir que cette dernière notion est incompatible avec celle de la Définition 1.1.2. Donc, dans le cas spécifique du modèle EU, l'impossibilité est évidente.

Enfin, nous montrons que l'utilisation des modèles EU, CEU et MMEU permet de différencier l'aversion au délai sur le long-terme d'autres types de préférences qui montrent une prédisposition pour une consommation immédiate. Les deux concepts que nous considérons sont celui de la *myopie faible* (qu'on appellera ici myopie) (voir Brown and Lewis [1981]) et celui de *l'impatience* au sens de Chateauneuf and Ventura [2013].

Définition 1.1.3. (Brown and Lewis [1981]) *On dit qu'une relation de préférence* $\succsim$ *est* myope *si* $\forall \mathbf{x}, \mathbf{y} \in l_+^\infty$ *tels que* $\mathbf{x} \succ \mathbf{y}$ *et* $\forall \epsilon > 0$, $\exists n_0(\mathbf{x}, \mathbf{y}, \epsilon) := n_0 \in \mathbb{N}$ *tel que* $n \geq n_0 \Rightarrow \mathbf{x} \succ \mathbf{y} + \epsilon 1_{[n,+\infty)}$.

Définition 1.1.4. (Chateauneuf and Ventura [2013]) *On dit qu'une relation de préférence* $\succsim$ *est* impatiente *si* $\forall \mathbf{x} \in l_+^\infty$, $\forall A > 0$, $\exists N(\mathbf{x}, A) := N \in \mathbb{N}$ *tel que* $n \geq N \Rightarrow (\mathbf{x} + A)1_{[0,n]} \succ \mathbf{x}$.

Il n'est pas difficile de montrer que le modèle d'utilité escomptée est équivalent au modèle EU avec une probabilité sigma-additive (voir Section 3.2). Dans ce cas, les Propositions 3.2 et 3.4 de Chateauneuf and Ventura [2013] montrent que la myopie et l'impatience ne peuvent pas être distinguées. Par contre, nous prouvons qu'il est possible de construire des préférences qui sont averses au délai sur le long-terme, sans être ni myopes ni impatientes. Nous le faisons dans l'exemple qui suit.

Exemple 1.1.1. *Considérons une relation de préférence* $\succsim$ *représentée par le modèle EU avec une probabilité simplement additive définie sur l'algèbre* $\mathcal{A}$ *des ensembles finis et cofinis par :* $P(\{n\}) = \left(\frac{1}{3}\right)^{n+1}$ $\forall n \in \mathbb{N}$, $P(\mathbb{N}) = 1$ *et*

$$P(A) = \begin{cases} \sum_{n \in A} P(\{n\}) & \text{si } A \text{ est fini} \\ 1 - \sum_{n \in A^c} P(\{n\}) & \text{si } A \text{ est cofini.} \end{cases}$$

Il est alors possible de prolonger la probabilité P à une probabilité Q définie sur l'ensemble des parties de $\mathbb{N}$, noté $2^{\mathbb{N}}$, tel que $Q|_{\mathcal{A}} = P$. On peut remarquer que Q n'est pas sigma-additive car

$$1 = Q(\mathbb{N}) = Q(\cup_n \{n\}) \neq \sum_{n=0}^{\infty} Q(\{n\}) = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{n+1} = \frac{1}{2}.$$

Un décideur EU avec une telle probabilité est averse au délai sur le long-terme par le Corollaire 1.1.1. Pourtant, il est possible de montrer qu'il n'est ni myope ni impatient. On peut ainsi avoir des préférences qui sont averses au délai sur le long-terme mais qui n'exhibent ni myopie ni impatience, deux notions beaucoup plus fortes.

Aversion au délai sur le court-terme

Dans cette section nous nous intéressons à l'étude des préférences qui sont *averses au délai sur le court-terme*. Nous commençons avec une définition mathématique.

Définition 1.1.5. *Soit $\succsim$ une relation de préférence sur l^{∞}_{+}. On dit que $\succsim$ est* averse au délai sur le court-terme *si pour tout $a > 0$, $k \in \mathbb{N}$ et $\mathbf{x} \in l^{\infty}_{+}$, on a*

$$(x_k + a, \mathbf{x}_{-k}) \succsim (x_{k+1} + a, \mathbf{x}_{-(k+1)}).$$

La Définition 1.1.5 dit qu'un décideur est averse au délai sur le court-terme si, chaque fois qu'il fait face à deux paiements faits à deux dates successives, il choisit toujours celui le plus proche du présent.

Nous avons réussi à donner des caractérisations complètes pour les modèles EU et CEU, mais pas pour le modèle MMEU. Pour ce dernier, nous avons prouvé deux caractérisations partielles.

Les résultats dans cette section sont valables sous l'hypothèse que la fonction d'utilité $u(\cdot)$ est C^1 et telle que $u'(x) > 0, \forall x \in \mathbb{R}_+$.

Proposition 1.1.4. *Soit $\succsim$ une relation de préférence représentée par le modèle CEU. Alors $(i) \Leftrightarrow (ii)$:*
(i) $\succsim$ est averse au délai sur le court-terme ;
(ii) Les trois conditions ci-dessous sont vérifiées :

1. *$\forall A \in 2^{\mathbb{N}}$, $\forall n \in \mathbb{N}$ t.q. $n, n+1 \notin A$, $v(A \cup \{n\}) \geq v(A \cup \{n+1\})$;*
2. *$\forall x, y \in \mathbb{R}_+$ t.q. $x > y$ $\forall n \in \mathbb{N}$, $\forall A, B \in 2^{\mathbb{N}}$ t.q. $A \subset B$, $n \in B$, $n \notin A$, $n+1 \notin B$,*

$$u'(x)(v(A \cup \{n\}) - v(A)) \geq u'(y)(v(B \cup \{n+1\}) - v(B))$$

3. *$\forall x, y \in \mathbb{R}_+$ t.q. $y > x$ $\forall n \in \mathbb{N}$, $\forall A, B \in 2^{\mathbb{N}}$ t.q. $B \subset A$, $n+1 \in A$, $n \notin A$, $n+1 \notin B$,*

$$u'(x)(v(A \cup \{n\}) - v(A)) \geq u'(y)(v(B \cup \{n+1\}) - v(B))$$

Corollaire 1.1.2. *Soit* $\succsim$ *une relation de préférence représentée par le modèle EU. Alors* $(i) \Leftrightarrow (ii)$ *:*
(i) $\succsim$ *est averse au délai sur le court-terme ;*
(ii) $\forall x, y \in \mathbb{R}_+$, $\forall n \in \mathbb{N}$, $u'(x)P(\{n\}) \geq u'(y)P(\{n+1\})$.

Proposition 1.1.5. *Soit* $\succsim$ *une relation de préférence représentée par le modèle MMEU. Alors* $(i) \Rightarrow (ii)$ *:*
(i) $\forall P \in C$, $\forall n \in \mathbb{N}$ *and* $\forall x, y \in \mathbb{R}_+$, $u'(x)P(\{n\}) \geq u'(y)P(\{n+1\})$ *;*
(ii) $\succsim$ *est averse au délai sur le court-terme.*

Proposition 1.1.6. *Soit* $\succsim$ *une relation de préférence représentée par le modèle MMEU. Alors* $(i) \Rightarrow (ii)$ *:*
(i) $\succsim$ *est averse au délai sur le court-terme ;*
(ii) $\forall P \in C$, $\forall n \in \mathbb{N}$, $P(\{n\}) \geq P(\{n+1\})$.

On peut remarquer que, tandis que l'aversion au délai sur le long-terme ne s'appuyait que sur des propriétés de poids attachés au temps, l'aversion au délai sur le court-terme demande des propriétés sus les poids et sur l'utilité. Dans ce qui suit nous nous concentrons sur le modèle EU, pour lequel l'interprétation est plus pointue.

En premier lieu, il est intéressant de noter qu'un décideur averse au délai sur le court-terme doit escompter le futur, dans le sens qu'il doit donner des poids décroissants. Cela résulte immédiatement du Corollaire 1.1.2. En deuxième lieu, on remarque que la caractérisation implique des limitations considérables sur la fonction d'utilité instantanée. En particulier, l'utilité ne peut pas satisfaire les conditions d'Inada et celles du Corollaire 1.1.2 en même temps. En troisième lieu, l'aversion au délai sur le court-terme se trouve être la contrepartie comportementale de la définition d'impatience de Fisher [1930]. En arrangeant les termes du Corollaire 1.1.2, on trouve que $\forall x, y \in \mathbb{R}_+$ et $\forall n \in \mathbb{N}$:

$$\frac{P(\{n\})}{P(\{n+1\})} \frac{u'(x)}{u'(y)} \geq 1. \tag{1.2}$$

Si l'on considère x comme le revenu de la période n et y comme le revenu de la période $n+1$ alors l'expression (1.2) nous dit qu'un agent est averse au délai sur le court-terme si et seulement si il a un taux marginal de substitution inter-temporel toujours plus grand que 1. Cela est exactement la condition de Fisher.

Nous terminons cette section en faisant le lien entre l'aversion au délai sur le court-terme et la notion de *dominance temporelle.*

Définition 1.1.6. *Soit* $\mathbf{x}, \mathbf{y} \in l^\infty_+$. *On dit que* $\mathbf{x}$ domine temporellement $\mathbf{y}$, *noté* $\mathbf{x} \succsim_T \mathbf{y}$, *si* $\sum_{i=0}^k x_i \geq \sum_{i=0}^k y_i$ $\forall k \in \mathbb{N}$.

Donc, un flux de revenu $\mathbf{x}$ domine un flux $\mathbf{y}$ si pour tout $k \in \mathbb{N}$ la somme partielle des k premiers éléments de $\mathbf{x}$ est plus grande que la somme partielle des k premiers éléments de $\mathbf{y}$. La Définition 1.1.6 généralise la condition (3b), p. 474 de Foster and Mitra [2003]. Le lecteur pourra trouver dans leur papier une discussion sur la relation entre dominance temporelle et dominance stochastique.

Proposition 1.1.7. *Soit $\succsim$ une relation de préférence monotone et transitive sur l^∞_+, continue par rapport à la convergence croissante. Alors $(i) \Leftrightarrow (ii)$:*
(i) $\succsim$ est averse au délai sur le court-terme ;
(ii) $\mathbf{x} \succsim_T \mathbf{y} \Rightarrow \mathbf{x} \succsim \mathbf{y}$.

Sous certaines hypothèses de monotonie et continuité [4], la notion d'aversion au délai sur le court-terme est équivalente au fait que, chaque fois que $\mathbf{x}$ domine temporellement $\mathbf{y}$, l'agent choisit $\mathbf{x}$. En dimension finie, si $\mathbf{x}$ domine temporellement $\mathbf{y}$, alors $\mathbf{x}$ a une valeur actuelle plus élevée que $\mathbf{y}$ pour tout taux d'intérêt. Il semble donc évident qu'un décideur préfère $\mathbf{x}$ par rapport à $\mathbf{y}$. Donc, l'équivalence entre dominance temporelle et aversion au délai sur le court-terme donne encore plus de force à ce dernier concept.

1.1.2 Une approche topologique de l'aversion au délai

Dans cette section nous illustrons les résultats principaux du Chapitre 4. Dans ce chapitre, nous étudions l'aversion au délai sur le long-terme en utilisant une approche topologique. Au lieu de faire l'hypothèse que les préférences peuvent être représentées par quelque fonctionnelle, nous définissons des topologies qui "escomptent" le futur d'une façon compatible avec l'aversion au délai sur le long-terme et nous étudions leurs propriétés.

Le cadre dans lequel nous travaillons dans ce chapitre est le même que celui du Chapitre 3 avec une seule différence. Au lieu de considérer l'espace l^∞_+, nous nous intéressons à tout l'ensemble l^∞. Les quantités négatives sont interprétées comme des dettes d'argent ou de biens de consommation.

Comme nous l'avons dit plus en haut, nous voulons étudier l'aversion au délai sur le long-terme définie dans la Définition 1.1.1. Pour simplifier la notation, nous dirons, plus simplement, aversion au délai.

Étant donnée une topologie $\mathcal{T}$ sur l'espace l^∞, il est possible de définir la notion de continuité des préférences. Nous dirons ainsi qu'une relation de préférence $\succsim$ sur l^∞ est $\mathcal{T}$-continue si les sous-ensembles $\{\mathbf{x} \in l^\infty | \mathbf{x} \succ \mathbf{y}\}$ et $\{\mathbf{x} \in l^\infty | \mathbf{y} \succ \mathbf{x}\}$ sont $\mathcal{T}$-ouverts pour tout $\mathbf{y}$. Dans cette section introductive, nous présentons seulement les résultats pour les espaces séparés localement convexes avec une base monotone [5]

Le but principal du Chapitre 4 est de trouver une topologie qui "escompte" le futur d'une façon cohérente avec l'aversion au délai. Une fois cette topologie définie, nous répondons aux deux questions suivantes. Premièrement, peut-on comparer cette topologie avec

4. Pour plus d'informations sur ces hypothèse le lecteur peut se référer à le Section 3.4.2.
5. Pour plus de détails sur ces espaces, voir le Section 4.2

celles qui sont habituellement utilisées dans l'espace l^∞ (surtout, la topologie de Mackey et la topologie uniforme) ? Deuxièmement, peut-on caractériser le dual topologique ?

Nous nous limitons aux préférences strictement monotones [6]. La monotonie stricte nous place dans le bon cadre pour deux raisons. Premièrement, parce que c'est le même cadre de Benoît and Ok [2007], qui ont inspiré la définition d'aversion au délai. Deuxièmement, parce que si une préférence est averse au délai, alors monotonie et monotonie stricte sont équivalentes.

On définit ce que signifie pour une topologie d'être averse au délai.

Définition 1.1.7. *Une topologie $\mathcal{T}$ sur l^∞ est* averse au délai *si toute préférence strictement monotone et continue par rapport à $\mathcal{T}$ est averse au délai.*

Les deux résultats que nous donnons maintenant sont le point de départ pour définir la topologie que nous étudions par la suite. On note 1_n la suite $1_n := (\underbrace{0, \dots, 0}_{n-1}, 1, 0, \dots)$ et 0 la suite $(0, 0, \dots)$.

Proposition 1.1.8. *Toute topologie localement convexe $\mathcal{T}$ pour laquelle on a $1_n \xrightarrow{\mathcal{T}}_n 0$ est une topologie averse au délai.*

Proposition 1.1.9. *Soit $\mathcal{T}$ une topologie localement convexe. Si toute préférence continue par rapport à $\mathcal{T}$ est averse au délai, alors $1_n \xrightarrow{\mathcal{T}}_n 0$.*

Malheureusement les Propositions 1.1.8 et 1.1.9 ne donnent pas une caractérisation complète des topologies averses au délai. Néanmoins, elles soulignent que la caractéristique fondamentale qu'il faut prendre en considération est la convergence $1_n \xrightarrow{\mathcal{T}}_n 0$.

Ces préliminaires mis en place, nous pouvons maintenant définir la topologie à laquelle nous nous intéressons.

Définition 1.1.8. *On note $\mathcal{T}_{DA}^{mon}$ la topologie la plus fine de Hausdorff avec une base monotone pour laquelle on a $1_n \xrightarrow{\mathcal{T}_{DA}^{mon}}_n 0$.*

L'idée économique qui est à la base de la Définition 1.1.8 est très simple. Considérons un agent qui fait face à un flux qui lui donne une unité de bien à la période n et zéro à toute autre période. Si ses préférences sont continues par rapport à $\mathcal{T}_{DA}^{mon}$, alors en reculant ce paiement dans le futur, ce flux sera arbitrairement proche de la suite $(0, 0, \dots)$. On peut remarquer que la topologie uniforme, notée $\mathcal{T}_\infty$, n'a pas cette propriété. En effet, $\|1_n\| = \sup_k |1_n(k)| = 1$ pour toute période de temps n. Dans ce sens, nous dirons que la topologie uniforme est une topologie apte à décrire un comportement patient (et pas impatient) de l'individu.

Il est facile de montrer que $\mathcal{T}_{DA}^{mon}$ est une topologie averse au délai dans le sens de la Définition 1.1.7. De plus, il est possible de prouver que cette topologie existe.

6. Voir le renvoi 3, page 14

Comparaisons avec d'autres topologies sur l^∞ et espace dual

La topologie dont on muni l'espace l^∞ dans la plupart des cas est la topologie uniforme, $\mathcal{T}_\infty$, définie par la norme $\|\mathbf{x}\| = \sup_k |x_k|$. On a vu plus haut que cette topologie convient si l'on veut étudier des préférences patientes. Peut on comparer les espaces $\mathcal{T}_{DA}^{mon}$ et $\mathcal{T}_\infty$? Si oui, quelle est la relation entre ces deux topologies ?

Par ailleurs, en économie mathématique il est connu que la continuité des préférences par rapport à la topologie de Mackey mène à l'impatience des décideurs. Cette topologie est particulièrement importante pour son usage extensif dans la théorie de l'équilibre général en dimension infinie. Brown and Lewis [1981] ont montré que des préférences continues par rapport à la topologie de Mackey sont impatientes dans le sens précis décrit ci-dessous.

Définition 1.1.9. (Brown and Lewis [1981]) *Soit $\succsim$ une relation de préférence sur l^∞. On dit que $\succsim$ est* fortement myope *si $\forall \mathbf{x}, \mathbf{y} \in l^\infty$ tels que $\mathbf{x} \succ \mathbf{y}$ et $\forall \mathbf{z} \in l^\infty$, $\exists n_1(\mathbf{x}, \mathbf{y}, \mathbf{z}) := n_1 \in \mathbb{N}$ tel que $n \geq n_1 \Rightarrow \mathbf{x} \succ \mathbf{y} + \mathbf{z}1_{[n,+\infty)}$.*

Une fois donnée la définition de myopie forte, il est possible de définir la topologie fortement myope $\mathcal{T}_{SM}$.

Définition 1.1.10. (Brown and Lewis [1981]) *La topologie de Hausdorff localement convexe $\mathcal{T}_{SM}$ sur l^∞ est la topologie la plus fine pour que toute préférence (pas forcement monotone) continue par rapport à $\mathcal{T}_{SM}$ soit fortement myope.*

Brown and Lewis [1981] ont montré que la topologie $\mathcal{T}_{SM}$ est équivalente à la topologie de Mackey. Donc, au lieu de comparer $\mathcal{T}_{DA}^{mon}$ avec la topologie de Mackey, nous travaillerons avec $\mathcal{T}_{SM}$.

Dans le résultat suivant, nous comparons la topologie $\mathcal{T}_{DA}^{mon}$ avec la topologie $\mathcal{T}_{SM}$, qui escompte le futur, et la topologie $\mathcal{T}_\infty$, qui ne l'escompte pas.

Proposition 1.1.10. $\mathcal{T}_{SM} \subset \mathcal{T}_{DA}^{mon} \subset \mathcal{T}_\infty$

La Proposition 1.1.10 montre formellement que l'aversion au délai est une notion plus faible que la myopie forte. En effet, puisque la continuité des préférences est définie en terme d'ouverts, et puisque $\mathcal{T}_{SM} \subset \mathcal{T}_{DA}^{mon}$ signifie que tout ouvert de $\mathcal{T}_{SM}$ est aussi un ouvert pour $\mathcal{T}_{DA}^{mon}$, il est plus facile pour un décideur d'être averse au délai plutôt que fortement myope. D'un point de vue intuitif, la Proposition 1.1.10 peut être interprétée en disant qu'un décideur averse au délai, est "entre" un décideur qui est fortement myope et un autre qui est patient. Néanmoins, il faut faire attention à ne pas trop s'appuyer sur cette interprétation car dans le Chapitre 4 nous n'avons pas défini la notion de patience à partir d'une relation de préférence.

L'inclusion $\mathcal{T}_{DA}^{mon} \subset \mathcal{T}_\infty$, a des implication importantes pour ce qui concerne le dual topologique de l^∞ associé avec $\mathcal{T}_{DA}^{mon}$. En économie théorique, l'espace dual a un rôle clé car il est l'espace des prix, voir Mas-Colell and Zame [1991]. Le corollaire qui suit découle facilement de la Proposition 1.1.10. Nous rappelons que l'espace ba est l'espace des charges bornées sur $2^{\mathbb{N}}$.

Corollaire 1.1.3. $l^1 \subset (l^\infty, \mathcal{T}_{DA}^{mon})^* \subseteq ba.$

Peut-on donner une caractérisation complète de l'espace dual $(l^\infty, \mathcal{T}_{DA}^{mon})^*$? La réponse est oui et est donnée dans la Proposition 1.1.11.

Proposition 1.1.11. $(l^\infty, \mathcal{T}_{DA}^{mon})^* = ba$

D'un point de vue mathématique, ce résultat est très intéressant car il donne une nouvelle caractérisation de l'espace ba. Dans la section prochaine, nous discutons les implications économiques de ces résultats.

Équilibre général et bulles

Les Propositions 1.1.10 et 1.1.11 sont particulièrement intéressantes pour ce qui concerne la théorie de l'équilibre général et celle qui étudie les bulles.

– *Sur l'équilibre général.* La Proposition 1.1.10 peut être considérée comme un perfectionnement d'un résultat de Araujo [1985]. Dans ce papier, l'auteur montre que, si les préférences des agents sont continues par rapport à une topologie $\mathcal{T}$ telle que *l'on n'a pas* $\mathcal{T} \subseteq \mathcal{T}_{SM}$, alors il existe des économies sans équilibre. Araujo conclut donc que l'impatience est une caractéristique nécessaire si l'on veut avoir un équilibre. La Proposition 1.1.10 clarifie cette interprétation. En effet, si un décideur a une relation de préférence continue par rapport à $\mathcal{T}_{DA}$, alors il est clair qu'il montre de l'impatience. Pourtant, l'équilibre pourrait ne pas exister. Donc on peut conclure que le besoin d'impatience évoqué par Araujo [1985] est en réalité un besoin d'un type d'impatience assez fort. Des décideurs seulement averses au délai ne garantiraient pas l'existence d'équilibres.
– *Sur les bulles.* Comme nous l'avons déjà dit, les prix peuvent être identifiés avec des éléments de l'espace dual. Si cet espace est ba, alors en appliquant le théorème de Yoshida–Hewitt il est possible de décomposer le prix comme la somme d'une partie sigma-additive et d'une partie pure. Gilles and LeRoy [1992] définissent une bulle comme étant la partie pure d'une charge. Dans ce papier ils argumentent qu'on ne peut pas avoir de bulles si les décideurs sont impatients (voir [Gilles and LeRoy, 1992, p. 332]). La Proposition 1.1.11 fournit un contre-exemple à cette affirmation. En effet, puisque nous montrons que l'espace dual $(l^\infty, \mathcal{T}_{DA}^{mon})$ est ba, cela implique qu'on peut avoir une partie de bulle dans les prix. Comme pour le cas de l'équilibre général, l'intuition est que les décideurs doivent être assez impatients pour éviter les bulles.

1.2 Sur la théorie de la négociation

Le dernier chapitre de cette thèse porte sur la théorie de la négociation coopérative introduite par l'article fondateur de Nash [1950]. Cette section présente brièvement cette théorie et explique comment nous avons contribué à la littérature.

Dans le cadre original introduit par Nash, un jeu de négociation entre deux personnes se compose d'une paire (S, d) où $S \subseteq \mathbb{R}^2$ est un ensemble compact, convexe et d un point dans S. Les éléments de S sont interprétés comme des paires d'utilités de *von Neumann-Morgenstern* (NM). Le point d est appelé le *point de désaccord*. Il se définit comme un vecteur d'utilités que chaque joueur peut imposer unilatéralement, si un accord n'est pas trouvé. Cette définition est une abstraction d'une situation réelle dans laquelle les négociateurs feraient face à des alternatives possibles. Prenons l'exemple suivant.

Exemple 1.2.1. *Considérons une situation dans laquelle il y a deux agents (dotés de deux fonctions d'utilité de NM) qui doivent se partager 1\$. L'ensemble des partages possibles est l'ensemble des vecteurs* $X := \{(x, 1-x) | 0 \leq x \leq 1\}$ *(en supposant qu'il n'y a pas de gaspillage du dollar). Supposons de plus que, en cas de désaccord, ils obtiennent 0\$. Le modèle de Nash ne considère pas l'ensemble des partages réalisables et prend comme point de départ un ensemble* $S \supseteq \{(u_1(x), u_2(1-x)) | 0 \leq x \leq 1\}$ *et le point de désaccord* $d = (u_1(0), u_2(0))$. *Chaque paire d'utilités représente le niveau de satisfaction des négociateurs associé à une certaine division.*

On se demande maintenant : comment les joueurs doivent partager l'utilité ? Ou mieux, quel point en S devraient-ils choisir ?

L'approche de Nash est axiomatique. Il définit une *solution* comme étant une fonction $f : (S, d) \to \mathbb{R}^2$ qui associe à chaque paire (S, d) un résultat dans S. Nash [1950] considère ensuite les quatre propriétés ci-dessous.

- **Pareto Optimalité** : $\forall y \in S$, $y \ngtr f(S, d)$;
- **Invariance** : Soit $A : \mathbb{R}^2 \to \mathbb{R}^2$ une transformation affine d'utilité, c'est à dire $A(x_1, x_2) = (A_1(x_1), A_2(x_2))$ où $A_i(x)$ a la forme $\alpha_i x + \beta_i$ avec $\alpha_i > 0$, $\beta_i \in \mathbb{R}$, alors $f(A(S), A(d)) = A(f(S, d))$;
- **Symétrie** : Si $d_1 = d_2$, et $(x, y) \in S$ implique $(y, x) \in S$, alors $f_1(S, d) = f_2(S, d)$;
- **Indépendance des Alternatives Non Pertinentes** : Si $T \subseteq S$ et $f(S, d) \in T$ alors $f(S, d) = f(T, d)$.

L'Optimalité de Pareto dit que la solution doit être un vecteur d'utilités qui est dans la frontière de Pareto de S. Puisque une fonction d'utilité NM est unique par rapport aux transformations affines positives, l'Invariance dit que la solution doit être indépendante de toute renormalisation affine positive du problème. L'axiome de Symétrie impose un partage égal chaque fois que l'ensemble de négociation et le point de désaccord sont symétriques. Enfin, l'Indépendance des Alternatives Non Pertinente dit qu'une solution ne devrait pas changer lorsque les alternatives "non pertinentes" sont éliminées d'un jeu. Pour des explications plus détaillées sur ces axiomes voir Osborne and Rubinstein [1990].

Sous ces hypothèses, Nash [1950] montre que l'unique solution est :

$$f(S,d) = \arg\max_{s\in S, s\geq d}(s_1 - d_1)(s_2 - d_2).$$

Cette solution sélectionne la paire d'utilités qui maximise le produit des gains des joueurs à partir du point de désaccord.

La simplicité et la robustesse de cette solution ont favorisé à la fois ses nombreuses applications et son importance théorique. Pourtant, la solution Nash n'a pas une interprétation intuitive. Par exemple, Rubinstein et al. [1992] écrivent :

> "the solution lacks a straightforward interpretation since the meaning of the product of two von Neumann–Morgenstern utility numbers is unclear."

Nash [1950] n'a pas été le seul à axiomatiser une solution. De nombreux auteurs ont adopté le cadre de négociation décrit ci-dessus et ont proposé différentes axiomatisations et solutions. Les plus notables sont la solution Kalai-Smorodinsky, voir Kalai and Smorodinsky [1975], la solution égalitaire, voir Kalai [1977b] et la solution utilitariste relative, d'abord examiné par Arrow [1963]. Beaucoup d'autres solutions sont présentes dans la littérature. Pour plus d'informations sur ce sujet, voir par exemple Thomson et al. [1994]. Une question naturelle est donc la suivante : quelle solution est la meilleure ?

Les deux contributions principales du Chapitre 5 sont :

- Nous fournissons une interprétation unifiée pour les trois plus importantes solutions en négociation coopérative.
- Nous offrons une justification pour la solution de Nash.

Notre modèle fait l'hypothèse qu'il y a deux joueurs engagés dans un processus de négociation et un médiateur. Le rôle du médiateur est de conseiller les négociateurs sur la meilleure alternative possible dans un ensemble X donné. L'objectif du médiateur est de les mettre d'accord sur une solution.

Chaque joueur a des préférences ordinales $\succsim_i$ sur X, $i = 1, 2$. Le processus de négociation est *target-based* : le joueur i accepte $x \in X$ si et seulement si $x \succsim_i t_i$, où $t_i \in X$ est son seuil minimal d'acceptation. Nous disons que t_i est le target (cible) du joueur i.

Le médiateur ne connait pas avec précision les caractéristiques des joueurs. Plus précisément, nous faison l'hypothèse qu'il connait les préférences $\succsim_i$ $i = 1, 2$, mais qu'il est incertain au sujet de la cible t_i de chaque joueur. Cette incertitude sur t_i est représentée par une variable aléatoire T_i et sa fonction de répartition F_i ; c'est à dire que le médiateur est en mesure d'évaluer :

$$P(i \text{ accepte } x) = P(x \succsim_i T_i) = F_i(x).$$

Sous ces hypothèses, nous proposons une caractérisation comportementale pour une classe assez générale des solutions. Si le médiateur utilise une solution dans cette classe, il proposera aux joueurs une alternative qui maximise la probabilité de trouver un accord. Un tel cadre nous permet de caractériser quelques solutions majeures comme des cas particuliers. La seule caractéristique qui les sépare est la nature de la dépendance stochastique entre les cibles des négociateurs.

En particulier, notre approche probabiliste suggère une interprétation simple pour le produit de deux utilités von Neumann Morgenstern préconisé par la solution de Nash. Ceci est révélé comme le produit de deux probabilités, et correspond à une hypothèse implicite d'indépendance stochastique entre les cibles des négociateurs. Enfin, nous montrons qu'un affaiblissement de cette dernière hypothèse génère d'autres alternatives bien connues, mais moins fréquemment utilisées, à savoir la solution égalitaire et une forme de solution utilitariste.

1.2.1 Solutions target-based pour la négociation à la Nash

Nous présentons ici les résultats du Chapitre 5 plus en détail.

Nous définissons un problème de négociation comme un sous-ensemble compact B de $[0,1]^2$. Chaque point $\mathbf{p} = (p_1, p_2)$ en B correspond à une paire de probabilités. Le nombre p_i représente la probabilité que le joueur i accepte une alternative possible x proposée par le médiateur. Plus formellement, étant donné un ensemble X d'alternatives possibles, nous associons tout $x \in X$ à un point (p_1, p_2) dans le carré unitaire grâce à la fonction $x \to (F_1(x), F_2(x))$ (où F_1 et F_2 sont les fonctions de répartition définies précédemment). Nous supposons donc que $B = (F_1(X), F_2(X))$ et nous prenons B comme point de départ. Une solution est une fonction qui pour tout problème B fournit (au moins) un point dans B.

Nous considérons les préférences du médiateur sur l'ensemble des loteries sur les paires des probabilités d'acceptation. Nous dérivons ensuite une caractérisation comportementale de ses préférences telle qu'il évalue chaque alternative par la probabilité jointe d'acceptation des deux joueurs.

Les axiomes et le théorème de représentation

Nous considérons $[0,1]^2$ comme un espace de mixture pour l'opération $\oplus$, que nous interprétons de manière standard : $\alpha\mathbf{p} \oplus (1-\alpha)\mathbf{q}$ est une loterie qui donne $\mathbf{p}$ en $[0,1]^2$ avec probabilité α et $\mathbf{q}$ en $[0,1]^2$ avec probabilité $1-\alpha$, voir Herstein and Milnor [1953]. Au même temps, $[0,1]^2$ est considéré comme un treillis sous l'ordre partiel standard $\geqq$ de $\mathbb{R}^2$. Nous utilisons les notations $\mathbf{p} \vee \mathbf{q} = (\max(p_1, q_1), \max(p_2, q_2))$ et $\mathbf{p} \wedge \mathbf{q} = (\min(p_1, q_1), \min(p_2, q_2))$.

Nous faisons les hypothèses suivantes sur les préférences $\succsim$ du médiateur sur l'espace de mixture/treillis $[0,1]^2$.

A.1 (Régularité) $\succsim$ est un préordre total, continu et indépendant par rapport aux mixtures.

Sur l'interprétation et les implications de A.1, on renvoie le lecteur au Théorème 8.4 de Fishburn [1970]. Il est intéressant de remarquer que dans [Nash, 1950, p. 157] une hypothèse équivalente à A.1 découle du fait que les joueurs suivent le modèle d'utilité espérée.

A.2 (Non-trivialité) $(1,1) \succ (0,0)$.

Cet axiome exclut le cas trivial où le médiateur est indifférent entre une proposition qui est sûrement acceptée et une autre qui est sûrement refusé par les deux négociateurs.

A.3 (Indifférence sur le désaccord) pour tous p, q dans $[0,1]$, $(p,0) \sim (0,q)$.

A.5 s'inspire de l'axiome DI dans Border and Segal [1997], où les auteurs étudient une relation de préférence sur des solutions. Dans le cadre de Nash, l'axiome DI énonce qu'une alternative qui donne à l'un des joueurs la même utilité qu'il obtiendrait au point de désaccord est aussi bonne que le point de désaccord lui-même. Dans notre cadre probabiliste, cet axiome exprime l'idée que si un joueur refuse une proposition, c'est comme si les deux refusaient. Une proposition est acceptée si et seulement si les deux négociateurs y consentent.

A.4 (Consistance pour la probabilité individuelle) pour tout p dans $[0,1]$,

$$p(1,1) \oplus (1-p)(0,1) \sim (p,1) \quad \text{et} \quad p(1,1) \oplus (1-p)(1,0) \sim (1,p).$$

Supposons que le médiateur sait de manière certaine qu'un joueur va accepter. Alors, il est indifférent entre une loterie qui voit le deuxième joueur sûrement accepter avec probabilité p et sûrement refuser avec probabilité $(1-p)$, ou un vecteur où le deuxième joueur accepte avec une probabilité p.

A.5 (Complémentarité faible) pour tous $\mathbf{p}, \mathbf{q}$ dans $[0,1]^2$,

$$(1/2)\,(\mathbf{p} \vee \mathbf{q}) \oplus (1/2)\,(\mathbf{p} \wedge \mathbf{q}) \succsim (1/2)\mathbf{p} \oplus (1/2)\mathbf{q}$$

A.5 découle de l'axiome S dans Francetich [2013]. Il dit qu'un loterie qui donne avec probabilité 0.5 $\mathbf{p}$, et probabilité 0.5 $\mathbf{q}$, est faiblement inférieure à une loterie avec les mêmes probabilités sur les extrêmes (par rapport à l'ordre $\geqq$ introduit plus haut). En gros, l'axiome dit que les probabilités d'acceptation sont faiblement complémentaires.

Il est possible de montrer que, sous A.1, les axiomes A.2–A.5 sont logiquement indépendants.

Le résultat principal du chapitre est donné ci-dessous.

Théorème 1.2.1. *Une relation de préférence $\succsim$ satisfait A.1–A.5 si et seulement si il existe une copule unique $C : [0,1]^2 \to [0,1]$ qui représente $\succsim$.*

Une copule est une application qui décrit la structure de dépendance d'une variable aléatoire bidimensionnelle en fonction de ses marginales. Ce résultat clé dérive d'un théorème de Sklar [1959]. L'interprétation du Théorème 1.2.1 est que, sous A.1–5, le médiateur classe les alternatives par leur probabilité d'acceptation, selon son opinion subjective sur la structure de dépendance des cibles des joueurs. Plus simplement : le médiateur maximise la probabilité que les deux parties acceptent sa proposition.

Les solutions principales

La solution de Nash surgit dès que l'on suppose que les probabilités d'acceptation des individus sont indépendantes. Puisque c'est une hypothèse assez naturelle, la solution de Nash apparaît comme la plus proéminente. Considérons l'axiome suivant.

A.7 (Indifférence par rapport aux diminutions proportionnelles) pour tous α, p, q in $[0, 1]$, $(\alpha p, q) \sim (p, \alpha q)$ [7].

Cela indique que le médiateur est indifférent au fait que la même réduction proportionnelle sur la probabilité d'acceptation soit appliquée à un joueur ou à l'autre.

Théorème 1.2.2. *Une relation de préférence $\succsim$ satisfait A.1–A2, A4–A.5*, et A.7 (A.5* est la version stricte de A.5) si et seulement si elle est représentée par la copule* $\Pi(p, q) = p \cdot q$.

Sous A.7, le médiateur juge que les probabilités d'acceptation des joueurs sont stochastiquement indépendantes et donc il choisit de leur proposer une alternative x telle que

$$\max_{x \in X} P(x \succsim_1 T_1, x \succsim_2 T_2) = \max_{(p_1, p_2) \in B} p_1 \cdot p_2$$

et on retrouve la solution de Nash.

Dans le modèle original (basé sur les utilités), la solution égalitaire de Kalai [1977b], recommande le point maximal auquel les gains d'utilité du point de désaccord sont égaux pour les deux joueurs. Plus simplement, si (S, d) est un jeu de négociation, la solution égalitaire recommande le vecteur dans S qui maximise le min $\{(u_1 - d_1), (u_2 - d_2)\}$ pour (u_1, u_2) dans S et $u_i \geq d_i$ pour $i = 1, 2$. Dans notre cadre probabiliste, considérons l'hypothèse suivante.

A.8 (Indifférence pour le min) pour tous p, q en $[0, 1]$, $(p, p \wedge q) \sim (p \wedge q, q)$.

Cet axiome dit que le médiateur est indifférent entre deux paires des probabilités d'acceptation s'ils ont le même minimum.

7. L'axiome A.6, qui n'est pas présenté dans cette section introductive est un axiome de symétrie. Le lecteur intéressé peur voir la Section 5.3.

Théorème 1.2.3. *Une relation de préférence $\succsim$ satisfait A.1–A2, A4–A.5*, et A.8 (A.5* est la version stricte de A.5) si et seulement si elle est représentée par la copule $M(p,q) = \min(p,q)$.*

Sous A.8, les préférences du médiateur sont représentées par la borne supérieure de Fréchet $M(p,q) = \min(p,q)$, qui indique une dépendance positive maximale entre les deux distributions marginales. Par conséquent, nous pouvons réinterpréter la solution égalitaire de Kalai [1977b] comme la solution que le médiateur doit utiliser s'il pense que les cibles des joueurs sont parfaitement corrélées positivement.

Enfin, nous considérons la solution utilitariste relative de Arrow [1963]. Cette solution consiste à maximiser la somme des utilités après les avoir normalisées entre zéro et un. Considérons l'axiome suivant.

A.9 (Indifférence pour la moyenne) pour tous p,q dans $[0,1]$, $(p,q) \sim (\frac{p+q}{2}, \frac{p+q}{2})$.

A.9 dit que les préférences du médiateur ne changent pas si on baisse la probabilité d'acceptation d'un joueur et simultanément on augmente la probabilité d'acceptation de l'autre du même montant. Nous montrons le résultat suivant.

Théorème 1.2.4. *Une relation de préférence $\succsim$ satisfait A.1–A.5 et A.9 si et seulement si elle est représentée par la copule $W(p,q) = \max(p+q-1, 0)$.*

Le Théorème 1.2.4 caractérise la borne inférieure de Fréchet $W(p,q) = \max(p+q-1, 0)$ qui indique la dépendance négative maximale entre les deux distributions marginales. Par conséquent, nous pouvons réinterpréter cette forme de solution utilitariste (tronquée) comme la solution qui maximise la probabilité d'acceptation jointe lorsque le médiateur pense que les cibles des joueurs sont parfaitement corrélées négativement.

Sur l'interprétation target-based

Le modèle Nash est défini en terme d'utilité NM alors que notre modèle se base sur des probabilités. Une question légitime se pose : est-ce vraiment de la négociation à la Nash ?

Comme nous l'avons déjà dit, la négociation est target-based et, de plus, nous faison l'hypothèse que le médiateur connait $P(i \text{ accepte } x) = P(x \succsim_i T_i) = F_i(x)$. L'observation clé est que le modèle NM peut être reformulé dans une langue probabiliste, comme le montrent les papiers de Castagnoli and LiCalzi [1996] et de Bordley and LiCalzi [2000]. Soit $\mathcal{C} = [c_*, c^*]$ un intervalle compact et convexe de $\mathbb{R}$. Considérons l'ensemble des loteries $\mathcal{P}_0(\mathcal{C})$. Si u_i est une utilité de NM croissante, bornée et continue a droite, après une simple normalisation nous pouvons définir $P(T_i \leq c) := u_i(c)$ et considérer "l'ancienne" fonction d'utilité u_i comme la fonction de répartition de la cible T_i du joueur. Si une loterie X

a pour fonction de répartition G et est stochastiquement indépendante de T_i, alors la chaîne d'égalités

$$\mathbb{E}[u_i(X)] = \int u_i(c)\,\mathrm{d}G(c) = \int P(T_i \leq c)\,\mathrm{d}G(c) = P(X \geq T)$$

montre que l'utilité espérée de la loterie X est équivalente à la probabilité que X soit supérieure à une cible stochastique T_i.

L'hypothèse implicite de Nash [1950] est que les utilités u_i des joueurs sont connaissance commune. Cela revient à supposer que les fonctions de répartition des cibles T_1, T_2 sont connues ex-ante par les deux joueurs, tandis que les cibles ne le sont pas. Par conséquent, si les agents ont une connaissance commune de la distribution conjointe de leurs cibles ex-ante, ils maximisent la probabilité de succès en s'appuyant sur la copule connue ex-ante. En particulier, s'ils savent que leurs deux cibles sont stochastiquement indépendantes, ils doivent utiliser la solution de Nash.

Chapitre 2

Introduction (English version)

Mathematical models offer powerful tools to study the functioning of economics and, more broadly, of social interactions. As societies are composed by a set of individuals, it is fundamental to understand how a single agent makes decisions.

Two aspects are particularly important when making economic decisions and should be taken into consideration : *time* and *risk* (or, more generally, uncertainty).

Most of our choices have consequences through time. Consider for instance an agent with a fixed amount of endowment living in a two periods economy. She should decide how much she is going to consume in the first period and how much in the second one. Usually, it is supposed that the decision maker prefers to consume more in the first period and less in the second. Such behaviour is known in the literature with the term of impatience. If there are N periods in the economy then we may model any N-periods income (or consumption) stream as a vector in $\mathbb{R}^N$. Starting from the seminal paper of Samuelson [1937], the *(exponential) discounting model* has been the paradigm for describing impatient tastes. In this model, an intertemporal stream of income $(x_0, x_1, \dots, x_N)$ is evaluated by the utility function :

$$U(x_0, x_1, \dots, x_N) = \sum_{n=0}^{N} \delta^n u(x_n).$$

The function $u : \mathbb{R} \to \mathbb{R}$ is an instantaneous utility function that describes the tastes of the agent. For every period of time $n = 1, \dots N$, δ^n, with $\delta \in (0, 1)$, is a discount factor that represents the fact that the decision maker is putting decreasing weights on periods of time. This model is widely spread because of its intuitiveness and simplicity. However, as we will see later, it is not completely satisfactory since, for instance, it does not allow to distinguish between different behavioural definitions of impatience (i.e. when the properties are given directly in terms of preferences). In Chapter 3 and Chapter 4 we address this problem, using some suitable generalizations of this framework for infinite dimensional spaces.

A second key aspect of any type of decision, is represented by risk. As for time, risk is of capital importance in economics since, when making a choice, agents usually do not face a deterministic environment. As a simple example, consider someone who is willing to invest in the stock market : assets look like lotteries whose value cannot be precisely determined. The key contribution to this literature is undoubtedly due to von Neumann and Morgenstern [1947] who axiomatized the *expected utility model* (EU).

Let $\mathcal{C}$ be a set of prizes, or outcomes, and let $\mathcal{P}_0(\mathcal{C})$ be the set of simple probability measures over C (i.e. probabilities that are different from 0 only on for finitely many $c \in \mathcal{C}$). Then, under some intuitive axioms, there exists a function $u : \mathcal{C} \to \mathbb{R}$ such that the the preferences over $\mathcal{P}_0(\mathcal{C})$ can be represented by the functional :

$$U(P) = \sum_{\mathcal{C}} P(c)u(c)$$

for all $P \in \mathcal{P}_0(\mathcal{C})$. In Chapter 5, we apply a target-based version of this model due to Castagnoli and LiCalzi [1996] to the theory of cooperative bargaining.

2.1 Intertemporal choice

The two chapters after the introduction deal with the theory of intertemporal choice. More precisely, we are interested in describing and analysing preferences for advancing the time of future satisfaction. Such "impatient" preferences play a key role in economic theory. At the end of the 19^{th} century Böhm-Bawerk [1891] was writing :

> "Present goods are, as a rule, worth more than future goods of like kind and number. This proposition is the kernel and center of the interest theory which I have to present."[1]

We consider a decision maker who has preferences over infinite (discrete) streams of income or consumption. Our setting follows the seminal papers of Koopmans [1960] and Diamond [1965]. We are interested in imposing some rules so that her preferences would show some kind of impatience. Loosely speaking, we want to find some conditions under which a decision maker who is asked to consume some amounts of good at two different points in time, would prefer to consume it at the earlier date. In these two chapters the set of choices will be the set of real-valued, bounded sequences (denoted l^{∞}), interpreted as the set of infinite streams of income or consumption.

Since the seminal paper of Samuelson [1937], the standard way of tackling impatient preferences has been the (exponential) discounting model :

$$U(x_0, x_1, \dots) = \sum_{t=0}^{\infty} \delta(t)u(x_t).$$

1. See Böhm-Bawerk [1891], p. 237.

As Bayesianism and the subjective expected utility model of Savage [1954] are the reference points in decision theory in an uncertain environment (see Gilboa and Marinacci [2013]), the (exponential) discounted utility model is the paradigm in the theory of intertemporal choice. In both Chapter 3 and Chapter 4 we depart from this classical approach. In Chapter 3 we study preferences represented by some functionals used in the context of ambiguity. These functionals can be thought as generalizations of Samuelson's model. In Chapter 4 we follow a topological approach.

The main contribution of Chapter 3 lies in the introduction of two behavioural definitions that represent preferences for anticipating utility.

The first concept, called *long-term delay aversion*, expresses the following idea. Suppose that an agent who is facing some income distribution is asked to choose between two extra payments which will be done at two different dates. Suppose further that the earlier payment is lower than the other one. Then the agent is long-term delay averse if she always chooses the lower (but earlier) payment as soon as the bigger one is done sufficiently far in the future.

The second concept, called *short-term delay aversion*, tries to depict the intuitive idea that a decision maker should show some kind of impatience if, whenever she is asked to choose between two extra payments done at two consecutive dates, she weakly prefers the one done at the earlier date.

We consider then a decision maker endowed with a preference relation represented by three popular models used in the decision theoretic literature : the Expected Utility (EU), the Choquet Expected Utility (CEU) and the MaxMin Expected Utility (MMEU) models. These models were introduced in the context of decision under uncertainty in the pioneer works of Schmeidler [1986], Schmeidler [1989] and Gilboa and Schmeidler [1989]. The very use of these models to describe impatient preferences is not standard in economic theory and represents one of the main contributions of this chapter. We show that the EU, CEU and MMEU prove to be flexible tools and powerful generalizations of the standard discounting model.

Long-term delay aversion can be characterised in the three models considered. The characterisations show that it is a weak notion which depends only on some simple properties of the weights that the decision maker assigns on periods (or subsets of periods) of time. Interestingly, it is possible to draw a parallel between the theory that study the impossibility of being strongly paretian and treating equally all generation. Such a link is made using the fact that long-term delay aversion is related with strong monotonicity of preferences.

Short-term delay aversion proves to be a very interesting notion for several reasons. First, unlike for long-term delay aversion, it is not possible to separate the tastes of the agent and her evaluation of time. Short-term delay aversion require both properties of the probabilities (or the capacity) *and* of the (marginal) utility. Second, an agent exhibiting short-term delay aversion should assign decreasing weights to periods of time. Third, short-term delay aversion is the behavioral counterpart of Fisher's definition of impatience (see Fisher [1930]). In fact, it turns out that in the EU model a decision maker is short-term delay averse if and only if she has a marginal rate of intertemporal substitution always greater than one. Finally, we define the concept of *temporal domination* and we prove that it is equivalent to short-term delay aversion. Temporal domination, linked with the work of Foster and Mitra [2003] on the dominance of a stream of income over another one at all interest rates, is an appealing notion that recalls the one of stochastic dominance in the context of risk.

In Chapter 4, we develop the study on the concept of long-term delay aversion from a different viewpoint. Instead of considering a preference relation represented by some functional (like in Chapter 3), we use a topological approach.

The space of real-valued, bounded streams of income is an infinite dimensional space. In such spaces, the choice of the topology has behavioural implications. Therefore a topology should not be chosen only for technical convenience. To put it with the words of Mas-Colell and Zame [1991] :

> "It should be stressed that the choice of the topology can only be dictated by economic, rather than mathematical, considerations."

The economic concept behind the topologies studied in Chapter 4 is precisely long-term delay aversion. We propose two Hausdorff locally convex topologies that "discount" the future in a way that is consistent with long-term delay aversion. At the end of the chapter, we also show that the definition of long-term delay aversion is compatible with the notion of more delay aversion in Benoît and Ok [2007] (whose work inspired our definition).

First, we compare these topologies with other topologies that have the property of representing impatient, or patient, preferences. We found, loosely speaking, that a long-term delay averse decision maker is behaviourally "in between" a myopic and a patient agent. Second, we study the topological dual spaces of l^∞ (i.e. the space of bounded streams). The most interesting result says that the dual space is equal to *ba*, the set of bounded charges.

Our results bear relevance on the theory of infinite-dimensional general equilibrium and with the works that consider bubbles as the pathological (not countably additive) part of a charge (see Gilles and LeRoy [1992]). The comparison of topologies implies that it is possible to have preference for advancing the time future satisfaction and still an equilibrium may fail to exist. Such a result clarifies a paper of Araujo [1985] : decision makers should be *enough* impatient to insure the existence of an equilibrium. The study of the dual implies that it is possible to have equilibrium price with bubbles, even in spite of the impatient attitude (represented by long-term delay aversion) of the agents.

In the rest of Section 2.1 we provide the main mathematical results of Chapter 3 and Chapter 4.

2.1.1 About delay aversion

In this section we are going to illustrate the main results of Chapter 3.

We consider a decision maker (DM) with a preference relation over the set $l_+^\infty = \{\mathbf{x} := (x_n)_{n\in\mathbb{N}} | x_n \geq 0 \,\forall n \text{ and } \sup_n x_n < +\infty\}$ of real-valued, positive and bounded sequences. The generic elements of l_+^∞ are denoted as $\mathbf{x}, \mathbf{y}$, etc. and are considered as infinite streams of income or consumption. Obviously, the set $\mathbb{N}$ of natural numbers represents time.

In this chapter, the preference relation $\succsim$ over l_+^∞ of the DM is represented by :

– The Expected Utility (EU) model : $U(\mathbf{x}) = \mathbb{E}_P[u(\mathbf{x})]$.

– The Choquet Expected Utility (CEU) model : $U(\mathbf{x}) = \int u(\mathbf{x})\, dv$.

– The MaxMin Expected Utility (MMEU) model : $U(\mathbf{x}) = \min_{P\in C} \mathbb{E}_P[u(\mathbf{x})]$.

– The discounting model : $U(\mathbf{x}) = \sum_{t=0}^{+\infty} \beta(t)u(x_t)$.

The EU, CEU and MMEU models are used in order to generalize the notion of *weights* that the DM assigns to periods of time. All of them are generalizations of the discounting model, which is, as we said above, the reference in the economic literature. For more informations about these models, the reader is referred to Section 3.2.

Long-Term Delay Aversion

In this section we are interested in studying the notion of *long-term delay aversion.* The formal definition follows.

Definition 2.1.1. *A preference relation $\succsim$ over l_+^∞ is* long-term delay averse *if for $0 < a \leq b$, $n_0 \in \mathbb{N}$ and $\mathbf{x} \in l_+^\infty$ $\exists N := N(\mathbf{x}, n_0, a, b) > n_0$ s.t. $\forall n \geq N$,*

$$(x_{n_0} + a, \mathbf{x}_{-n_0}) \succ (x_n + b, \mathbf{x}_{-n}).$$

Definition 2.1.1 states the following : when a DM with a certain distribution of income (or consumption) faces two extra payments, $a > 0$ made in period n_0 and $b \geq a$ made in period n, if the second one happens to be sufficiently far into the future, then she will strictly prefer the payment made before, even if it is lower.

It turns out that long-term delay aversion can be characterized in all the three models considered. The characterizations are the following.

Proposition 2.1.1. *Let $\succsim$ be represented by the CEU model. Then $(i) \Leftrightarrow (ii)$:*
(i) $\succsim$ is long-term delay averse ;
(ii) $\forall A \in 2^{\mathbb{N}}$ $v(A \cup \{n\}) \to_n v(A)$ and $\forall A \in 2^{\mathbb{N}}$, $\forall t \notin A$ $v(A \cup \{t\}) > v(A)$.

Proposition 2.1.2. *Let $\succsim$ be represented by the MMEU model. Then (i) $\Leftrightarrow$ (ii) :*
(i) $\succsim$ is long-term delay averse;
(ii) $\forall P \in C$, $\forall n \in \mathbb{N}$, $P(\{n\}) > 0$.

Corollary 2.1.1. *Let $\succsim$ be represented by the EU model. Then (i) $\Leftrightarrow$ (ii) :*
(i) $\succsim$ is long-term delay averse;
(ii) $P(\{n\}) > 0 \, \forall n \in \mathbb{N}$.

These characterizations, which depend on relatively simple properties of the weights that the DM puts on time periods, are related with strong monotonicity of preference [2]. This link is presented in the following proposition.

Proposition 2.1.3. *Let $\succsim$ be represented by either the MMEU model or the EU model. Then (i) $\Leftrightarrow$ (ii) :*
(i) $\succsim$ is long-term delay averse;
(ii) $\succsim$ is strongly monotonic.

Let $\succsim$ be represented by the CEU model. Then (i) $\Rightarrow$ (ii) :
(i) $\succsim$ is long-term delay averse;
(ii) $\succsim$ is strongly monotonic.

Interestingly, Proposition 2.1.3 can be used to clarify some result of the theory that studies the impossibility of being, at the same time, strongly Paretian and treating equally all generations. Consider the following definition due to Basu and Mitra [2003].

Definition 2.1.2. (Basu and Mitra [2003]) *A preference relation $\succsim$ satisfies* anonymity *if for all $\mathbf{x}, \mathbf{y} \in l^{\infty}_{+}$ s.t. there exists $i, j \in \mathbb{N}$ s.t. $x_i = y_j$ and $x_j = y_i$ and s.t. for $k \in \mathbb{N} \setminus \{i, j\}$, $x_k = y_k$, then $\mathbf{x} \sim \mathbf{y}$.*

The main theorem of Basu and Mitra [2003] states that there is no numerical representation for a preference relation that satisfies both anonymity and strong monotonicity of preferences. We find that the intuition behind their result is not completely easy to grasp (from an economic perspective). Proposition 2.1.3, and especially the part involving the EU model, may help to understand this impossibility result. In the EU model, strong monotonicity is equivalent to long-term delay aversion. This latter notion is clearly incompatible with anonymity. This is the reason why, in the limited framework of EU preferences, strong monotonicity and anonymity are in conflict.

Finally, we show with one example that the use of the EU, CEU and MMEU models allows to differentiate long-term delay aversion from other demanding notions of preferences for anticipating future utility. The two concepts under consideration are the

2. We say that a preference relation over l^{∞} is *monotone* if $x_k \geq y_k$ for every $k \in \mathbb{N}$ implies $\mathbf{x} \succsim \mathbf{y}$ and *strongly monotone* if $x_k \geq y_k$ for every $k \in \mathbb{N}$ and moreover $\mathbf{x} \neq \mathbf{y}$ implies $\mathbf{x} \succ \mathbf{y}$.

one of *weak myopia* (here just called myopia) of Brown and Lewis [1981] and the one of *impatience* of Chateauneuf and Ventura [2013].

Definition 2.1.3. (Brown and Lewis [1981]) *A preference relation* $\succsim$ *is* myopic *if* $\forall \mathbf{x}, \mathbf{y} \in l_+^\infty$ *such that* $\mathbf{x} \succ \mathbf{y}$ *and* $\forall \epsilon > 0$, $\exists n_0(\mathbf{x}, \mathbf{y}, \epsilon) := n_0 \in \mathbb{N}$ *such that* $n \geq n_0 \Rightarrow \mathbf{x} \succ \mathbf{y} + \epsilon 1_{[n,+\infty)}$.

Definition 2.1.4. (Chateauneuf and Ventura [2013]) *A preference relation* $\succsim$ *is* impatient *if* $\forall \mathbf{x} \in l_+^\infty$, $\forall A > 0$, $\exists N(\mathbf{x}, A) := N \in \mathbb{N}$ *such that* $n \geq N \Rightarrow (\mathbf{x} + A)1_{[0,n]} \succ \mathbf{x}$.

It is not difficult to prove that the the discounting model is equivalent to the EU model with a sigma-additive probability (see Section 3.2). In this case, Proposition 3.2 and Proposition 3.4 of Chateauneuf and Ventura [2013] show that myopia and impatience cannot be disentangled. It is possible however to construct preferences which are long-term delay averse but neither myopic nor impatient. We do it in the example that follows.

Example 2.1.1. *Take* $\succsim$ *represented by EU w.r.t. a simply additive probability defined in the algebra* $\mathcal{A}$ *of finite and cofinite sets as :* $P(\{n\}) = \left(\frac{1}{3}\right)^{n+1}$ $\forall n \in \mathbb{N}$, $P(\mathbb{N}) = 1$ *and*

$$P(A) = \begin{cases} \sum_{n \in A} P(\{n\}) & \text{if } A \text{ is finite} \\ 1 - \sum_{n \in A^c} P(\{n\}) & \text{if } A \text{ is cofinite.} \end{cases}$$

It is possible to extend the probability P *to a simply additive probability* Q *over the power set* $2^{\mathbb{N}}$ *s.t.* $Q|_{\mathcal{A}} = P$. *It should be noted that* Q *is not* σ*-additive in fact*

$$1 = Q(\mathbb{N}) = Q(\cup_n \{n\}) \neq \sum_{n=0}^{\infty} Q(\{n\}) = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{n+1} = \frac{1}{2}.$$

By Corollary 2.1.1 the DM is long-term delay averse. However it is possible to prove that she is neither myopic nor impatient. This example therefore shows that it is possible to construct preferences that are long-term delay averse, but that do not display the much stronger notions of impatience or myopia.

Short-Term Delay Aversion

In this section we are interested in studying the notion of *short-term delay aversion*. The formal definition follows.

Definition 2.1.5. *A preference relation* $\succsim$ *over* l_+^∞ *is* short-term delay averse *if for every* $a > 0$, $k \in \mathbb{N}$ *and every* $\mathbf{x} \in l_+^\infty$, *one has*

$$(x_k + a, \mathbf{x}_{-k}) \succsim (x_{k+1} + a, \mathbf{x}_{-(k+1)}).$$

Definition 2.1.5 says that a DM is short-term delay averse if, when facing two payments at two consecutive dates, she will always prefer the payment closer to the present.

We were able to give complete characterizations of short-term delay aversion for the EU and CEU models, but not for the MMEU model. For this latter model we present two partial characterizations.

Our results in this section assume the utility function $u(\cdot)$ to be C^1 and such that $u'(x) > 0, \forall x \in \mathbb{R}_+$.

Proposition 2.1.4. *Let $\succsim$ be represented by the CEU model. Then $(i) \Leftrightarrow (ii)$:*
(i) $\succsim$ is short-term delay averse ;
(ii) The following holds :

1. $\forall A \in 2^{\mathbb{N}}$, $\forall n \in \mathbb{N}$ *s.t.* $n, n+1 \notin A$, $v(A \cup \{n\}) \geq v(A \cup \{n+1\})$;
2. $\forall x, y \in \mathbb{R}_+$ *s.t.* $x > y$ $\forall n \in \mathbb{N}$, $\forall A, B \in 2^{\mathbb{N}}$ *s.t.* $A \subset B$, $n \in B$, $n \notin A$, $n+1 \notin B$,
$$u'(x)(v(A \cup \{n\}) - v(A)) \geq u'(y)(v(B \cup \{n+1\}) - v(B))$$
3. $\forall x, y \in \mathbb{R}_+$ *s.t.* $y > x$ $\forall n \in \mathbb{N}$, $\forall A, B \in 2^{\mathbb{N}}$ *s.t.* $B \subset A$, $n+1 \in A$, $n \notin A$, $n+1 \notin B$,
$$u'(x)(v(A \cup \{n\}) - v(A)) \geq u'(y)(v(B \cup \{n+1\}) - v(B))$$

Corollary 2.1.2. *Let $\succsim$ be represented by the EU model. Then $(i) \Leftrightarrow (ii)$:*
(i) $\succsim$ is short-term delay averse ;
(ii) $\forall x, y \in \mathbb{R}_+$, $\forall n \in \mathbb{N}$, $u'(x)P(\{n\}) \geq u'(y)P(\{n+1\})$.

Proposition 2.1.5. *Let $\succsim$ be represented by the MMEU model. Then $(i) \Rightarrow (ii)$:*
(i) $\forall P \in C$, $\forall n \in \mathbb{N}$ and $\forall x, y \in \mathbb{R}_+$, $u'(x)P(\{n\}) \geq u'(y)P(\{n+1\})$;
(ii) $\succsim$ is short-term delay averse.

Proposition 2.1.6. *Let $\succsim$ be represented by the MMEU model. Then $(i) \Rightarrow (ii)$:*
(i) $\succsim$ is short-term delay averse ;
(ii) $\forall P \in C$, $\forall n \in \mathbb{N}$, $P(\{n\}) \geq P(\{n+1\})$.

While long-term delay aversion involves only properties of the weights that the DM attaches to periods of time, short-term delay aversion requires features of both the capacity, or the probabilities, and the utility function. We focus on the EU model, for which the interpretation is sharper.

First, we can notice that an EU, short-term delay averse DM should discount the future, in the sense that she should attach decreasing weights to periods of time. This readily follows form Corollary 2.1.2 since we need $P(\{n\}) \geq P(\{n+1\})$. Second, the characterization shows that short-term delay aversion entails substantial limitation on the shape of the instantaneous utility function. In particular, the utility function cannot satisfy at the same time the properties of Corollary 2.1.2 and Inada's conditions. Third, short term delay aversion turns out to be a behavioural counterpart of Fisher's definition of impatience (see Fisher [1930]). Rearranging quantities in the characterization we get that $\forall x, y \in \mathbb{R}_+$ and $\forall n \in \mathbb{N}$:

$$\frac{P(\{n\})}{P(\{n+1\})} \frac{u'(x)}{u'(y)} \geq 1. \tag{2.1}$$

Considering x as the DM's income in period n and y as her income in period $n+1$, then the expression above states that : a DM is short-term delay averse if and only if she has a marginal rate of intertemporal substitution that is always greater than 1. This is precisely Fisher's definition.

We end this section linking short-term delay aversion with the concept of *temporal domination.*

Definition 2.1.6. *Given* $\mathbf{x}, \mathbf{y} \in l^\infty_+$, *we say that* $\mathbf{x}$ temporally dominates $\mathbf{y}$, *denoted* $\mathbf{x} \succsim_T \mathbf{y}$, *if* $\sum_{i=0}^k x_i \geq \sum_{i=0}^k y_i \ \forall k \in \mathbb{N}$.

A flow of income $\mathbf{x}$ temporally dominates another flow of income $\mathbf{y}$ if the partial sum of the first k elements of $\mathbf{x}$ is higher or equal to the partial sum of the first k elements of $\mathbf{y}$, for every $k \in \mathbb{N}$. This definition extends condition (3b) of [Foster and Mitra, 2003, p. 474]. The interested reader can refer to their paper for a discussion of the relation of Definition 2.1.6 to the criterion of stochastic dominance.

The link between these two notions is given below.

Proposition 2.1.7. *Let* $\succsim$ *be a monotone and transitive preference relation over* l^∞_+, *continuous w.r.t. monotone increasing convergence. Then* $(i) \Leftrightarrow (ii)$ *:*
(i) $\succsim$ *is short-term delay averse ;*
(ii) $\mathbf{x} \succsim_T \mathbf{y} \Rightarrow \mathbf{x} \succsim \mathbf{y}$.

Under some monotonicity and continuity assumptions [3], the notion of short-term delay aversion is equivalent to the fact that, whenever $\mathbf{x}$ temporally dominates another flow of income $\mathbf{y}$, then the DM prefers $\mathbf{x}$ to $\mathbf{y}$. In finite dimension, if $\mathbf{x}$ temporally dominates $\mathbf{y}$, then $\mathbf{x}$ has an higher present value for every possible interest rate. It seems natural therefore for a DM to prefer $\mathbf{x}$ over $\mathbf{y}$. Hence, such a result provides additional support to the definition of short-term delay aversion.

2.1.2 A topological approach to delay aversion

In this section we are going to illustrate the main results of Chapter 4. In this chapter, we study the concept of long-term delay aversion using a topological approach. Loosely speaking, instead of assuming that the preference relation $\succsim$ of the decision maker can be

3. For more details concerning these conditions, the reader is referred to Section 3.4.2.

represented by some functional, we propose a topology that discounts the future in a way that is consistent with the notion of long-term delay aversion and we study its properties.

The framework in which we will work in Chapter 4 is the same as the one of Chapter 3 except for one thing. Instead of considering the space l^∞_+, we consider the whole space l^∞. Negative quantities represent debts of money or consumption good.

As we anticipated, the concept that we are interested in, is the one of long-term delay aversion, defined in Definition 2.1.1. For terminological convenience, we will drop the adjective long-term from now on.

If a topology $\mathcal{T}$ over the set l^∞ is given, it is possible to make precise the notion of continuity of preferences. We say that a preference relation over l^∞ is $\mathcal{T}$-continuous if the sets $\{\mathbf{x} \in l^\infty | \mathbf{x} \succ \mathbf{y}\}$ and $\{\mathbf{x} \in l^\infty | \mathbf{y} \succ \mathbf{x}\}$ are $\mathcal{T}$-open for all $\mathbf{y}$. In this introductory section, we are going to present only the results concerning Haussdorff locally convex topologies with a monotone base [4].

The main purpose of Chapter 4 is to find a topology that "discounts" the future consistently with the notion of delay aversion. Once such a topology is defined, we are interested in studying two things. First, how does this topology relates with the other topologies usually paired with l^∞ (especially the Mackey and the sup-norm topologies)? Second, is it possible to characterize the topological dual space?

We are going to work with strongly monotonic preference relations [5]. Strong monotonicity places us in the good framework for two reasons. First, because this is the same framework of Benoît and Ok [2007], whose work inspired the definition of delay aversion. Second, because once delay aversion is assumed, then monotonicity and strong monotonicity are equivalent.

We now define what we mean for delay averse topology.

Definition 2.1.7. *A topology $\mathcal{T}$ on l^∞ is said to be* delay averse *if every strongly monotone, $\mathcal{T}$-continuous preference relation is delay averse.*

Given this definition, the two propositions below provide a starting point to define the topology that we will study. In what follows, 1_n denotes the sequence $1_n := (\underbrace{0, \dots, 0}_{n-1}, 1, 0, \dots)$ and 0 denotes the sequence $(0, 0, \dots)$.

Proposition 2.1.8. *Every locally convex topology $\mathcal{T}$ for which $1_n \xrightarrow{\mathcal{T}}_n 0$ is a delay averse topology.*

Proposition 2.1.9. *Given a locally convex topology $\mathcal{T}$, if every $\mathcal{T}$-continuous, preference relation is delay averse then $1_n \xrightarrow{\mathcal{T}}_n 0$.*

4. For more details about these topologies see Section 4.2.
5. See footnote 2, page 34

Unfortunately, Proposition 2.1.8 and 2.1.9 fall short of a complete characterization of delay averse topologies. Nevertheless, they underline that an important feature that should be kept into consideration when modelling delay averse behaviours is the convergence $1_n \xrightarrow{\mathcal{T}}_n 0$.

Given these preliminaries, we can now define the topology that we are going to study.

Definition 2.1.8. *We denote $\mathcal{T}_{DA}^{mon}$ the finest Hausdorff locally convex topology on l^∞ with a monotone base for which we have $1_n \xrightarrow{\mathcal{T}_{DA}^{mon}}_n 0$.*

The economic idea behind Definition 2.1.8 is simple. Consider a DM facing a stream of income or consumption that gives her one unit at period n and zero in every other period. Then continuity w.r.t. $\mathcal{T}_{DA}^{mon}$ says that, if we postpone this amount far away in the future, such a stream can be made arbitrarily close to the sequence $(0, 0, \dots)$. Notice that the sup-norm topology, $\mathcal{T}_\infty$, does not have this property. In fact, $\|1_n\| = \sup_k |1_n(k)| = 1$ for every period of time n. In this sense, we may say that the sup-norm topology is suitable to describe patient rather than impatient preferences.

It is easy to prove that $\mathcal{T}_{DA}^{mon}$ is a delay averse topology in the sense of Definition 2.1.7. Moreover, it is possible to show that such a topology exists.

Comparison with others topologies on l^∞ and dual space

The usual topology considered when studying l^∞ is the sup-norm topology $\mathcal{T}_\infty$ defined by the norm $\|\mathbf{x}\| = \sup_k |x_k|$. We saw just above that such a topology is more suitable to study patient rather than impatient preferences. Hence it is interesting to investigate how the topology $\mathcal{T}_{DA}^{mon}$ is related with the topology $\mathcal{T}_\infty$.

Also, it is known that continuity of preferences with respect to the Mackey topology induces an impatient behaviour of the DM. This topology is particularly relevant to our analysis because of its extensive use in the theory of general equilibrium in infinite dimensional spaces. Part of its glory is due to the work of Brown and Lewis [1981]. These authors show that every preference relation which is continuous with respect to the Mackey topology is impatient in the precise sense described below.

Definition 2.1.9. (Brown and Lewis [1981]) $\succsim$ *is* strongly myopic *if* $\forall \mathbf{x}, \mathbf{y} \in l^\infty$ *such that* $\mathbf{x} \succ \mathbf{y}$ *and* $\forall \mathbf{z} \in l^\infty$, $\exists n_1(\mathbf{x}, \mathbf{y}, \mathbf{z}) := n_1 \in \mathbb{N}$ *such that* $n \geq n_1 \Rightarrow \mathbf{x} \succ \mathbf{y} + \mathbf{z}1_{[n,+\infty)}$.

Related to the concept of strong myopia, it is possible to define the strongly myopic topology $\mathcal{T}_{SM}$.

Definition 2.1.10. (Brown and Lewis [1981]) *The locally convex Hausdorff topology $\mathcal{T}_{SM}$ on l^∞ is the finest topology such that every $\mathcal{T}_{SM}$-continuous (not necessarily monotone) preference relation is strongly myopic.*

Brown and Lewis [1981] proved that the topology $\mathcal{T}_{SM}$ is equivalent to the Mackey topology. Hence, if we want to compare $\mathcal{T}_{DA}^{mon}$ with the Mackey topology, is enough to focus on $\mathcal{T}_{SM}$.

We proceed comparing $\mathcal{T}_{DA}^{mon}$ with $\mathcal{T}_{SM}$, which discounts the future, and $\mathcal{T}_{\infty}$, which does not.

Proposition 2.1.10. $\mathcal{T}_{SM} \subset \mathcal{T}_{DA}^{mon} \subset \mathcal{T}_{\infty}$

Proposition 2.1.10 proves formally that delay aversion is a weaker notion than strong myopia. In fact, since continuity with respect to a topology is defined in terms of openness of upper and lower contour sets, and since $\mathcal{T}_{SM} \subset \mathcal{T}_{DA}^{mon}$ means that all open sets of $\mathcal{T}_{SM}$ are also open for $\mathcal{T}_{DA}^{mon}$, then it is easier for a DM to be delay averse rather than strongly myopic. Loosely speaking, Proposition 2.1.10 may be interpreted as saying that a delay averse DM is "in between" a strongly myopic and a patient agent. However, we should be careful to push this interpretation too far, since in Chapter 4 no concept of patience is given starting from preferences.

The inclusion $\mathcal{T}_{DA}^{mon} \subset \mathcal{T}_{\infty}$, has important consequences for what concerns the topological dual space of l^{∞} when paired with $\mathcal{T}_{DA}^{mon}$. In economic theory, and especially for general equilibrium in infinite dimension, the dual plays a key role since it is the set of possible prices of an economy, see Mas-Colell and Zame [1991]. Form Proposition 2.1.10 the following corollary is immediate. We recall that the set *ba* is the set of bounded charges over $2^{\mathbb{N}}$.

Corollary 2.1.3. $l^1 \subset (l^{\infty}, \mathcal{T}_{DA}^{mon})^* \subseteq ba$.

Now one natural question is whether it is possible to completely characterise the dual space $(l^{\infty}, \mathcal{T}_{DA}^{mon})^*$? The answer is yes and it is provided by Proposition 2.1.11.

Proposition 2.1.11. $(l^{\infty}, \mathcal{T}_{DA}^{mon})^* = ba$

From a mathematical point of view, this is an interesting result, since it yields a new characterization of the space *ba*. We briefly discuss below the economic consequences.

Link with general equilibrium and bubbles

Proposition 2.1.10 and Proposition 2.1.11 have interesting implications when linked with general equilibrium theory in infinite dimension and the study of bubbles.

- *About general equilibrium.* Proposition 2.1.10 can be considered as a refinement of a result of Araujo [1985]. In this paper, the author proved that, if agents have preferences continuous w.r.t. some topology $\mathcal{T}$ and if we *do not* assume $\mathcal{T} \subseteq \mathcal{T}_{SM}$ then an equilibrium may fail to exist. From this result the author concluded that impatience is necessary in order to get an equilibrium. Proposition 2.1.10 clarifies this interpretation. When preferences are continuous with respect to $\mathcal{T}_{DA}$ then it is clear that the DM exhibits some kind of impatience. Nevertheless, an equilibrium in an economy with such agents may fail to exist. Therefore the need for impatience is in reality the need for *enough* impatience : discounting the future just as a delay averse DM may lead to non existence of equilibria.

– *About bubbles.* As we said before, prices can be thought as elements of the dual space. If the dual space is ba, then using the Yoshida–Hewitt theorem is possible to decompose the price function into the sum of a countably additive part and a purely finitely additive part. Gilles and LeRoy [1992] define a bubble as the pure part of a charge. In their paper they argue that if a DM discounts the future, then a bubble cannot occur (see [Gilles and LeRoy, 1992, p. 332]). Proposition 2.1.11 provides a counterexample to the claim of the authors in the following sense : it proves that we can have bubbles even when the DMs discount the future. Again, the point is that in order to avoid bubbles the DMs should discount the future *enough*.

2.2 Bargaining

The last chapter of this thesis focuses on the theory of cooperative bargaining introduced in the seminal paper of Nash [1950]. This section will introduce briefly this theory and will explain how we contributed to the bargaining literature.

In the original Nash's framework, a two-persons bargaining game consists of a pair (S, d) where $S \subseteq \mathbb{R}^2$ is a compact and convex set and $d \in S$. The elements of S are interpreted as pairs of *von Neumann-Morgenstern* (NM) utilities. The point d is called the *disagreement point* and is thought as a pair of utilities that either player can unilaterally enforce. This definition is an abstraction of a real world situation in which bargainers are dealing with feasible alternatives. Consider the following example.

Example 2.2.1. *Consider for instance two agents (endowed with two NM utilities) that are bargaining over 1\$. The feasible alternatives are all the possible division of the dollar, i.e. the set of vectors* $\{(x, 1-x) | 0 \leq x \leq 1\}$ *(supposing that there is no waste of the dollar). Let's suppose further that if the bargainers do not find an agreement they both get 0\$. Instead of focusing on the possible feasible divisions of the dollar, Nash's model takes as a starting point a set* $S \supseteq \{(u_1(x), u_2(1-x)) | 0 \leq x \leq 1\}$ *and the disagreement point* $d = (u_1(0), u_2(0))$*. Every pair of utility clearly represents the level of satisfaction of the bargainers associated to some division.*

The question now is the following : how should the bargainers "share" the utility ? Or, in other words, which point in S should be picked ?

A *solution* is a function $f : (S, d) \to \mathbb{R}^2$ that assigns to each pair (S, d) an outcome in S. Nash [1950] imposed four intuitive properties on the solution function.

– **Pareto Optimality** : $\forall y \in S$, $y \ngtr f(S,d)$;
– **Scale Invariance** : Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be an affine transformation of utilities, that is $A(x_1, x_2) = (A_1(x_1), A_2(x_2))$ where $A_i(x)$ is of the form $\alpha_i x + \beta_i$ for $\alpha_i > 0$, $\beta_i \in \mathbb{R}$, then $f(A(S), A(d)) = A(f(S,d))$;
– **Symmetry** : If $d_1 = d_2$ and $(x,y) \in S$ implies $(y,x) \in S$, then $f_1(S,d) = f_2(S,d)$;
– **Independence of Irrelevant Alternatives** : If $T \subseteq S$ and $f(S,d) \in T$ then $f(S,d) = f(T,d)$.

Pareto Optimality says that the solution should choose a vector of utilities that is in the Pareto frontier of S. Since a NM utility function is unique up to positive affine transformations, Scale Invariance says that the solution should be independent of any positive affine rescaling of the problem. Symmetry imposes an equal outcome whenever the bargaining set and the disagreement point are symmetric. Finally Independence of Irrelevant Alternatives says, roughly speaking, that a bargaing solution should not change when "irrelevant" alternatives are dropped from a bargaining set. For more detailed explanations on these axioms see Osborne and Rubinstein [1990].

Under these appealing axioms, Nash [1950] proves that the unique solution is

$$f(S,d) = \arg\max_{s \in S, s \geq d} (s_1 - d_1)(s_2 - d_2).$$

The solution selects the utility pair that maximizes the product of the players' gains (in utility) over the disagreement outcome.

The simplicity and robustness of this solution have fostered both its widespread application and its theoretical prominence. However, not all is well with the model : the Nash solution cannot stake a claim for being intuitively appealing. For instance Rubinstein et al. [1992] state :

> "the solution lacks a straightforward interpretation since the meaning of the product of two von Neumann–Morgenstern utility numbers is unclear"

Nash [1950] was not the only one who axiomatized a bargaining solution. A large body of literature adopted the bargaining framework described above and proposed different axiomatizations which lead to several alternative solutions. The most notable ones are the Kalai-Smorodinsky solution due to Kalai and Smorodinsky [1975], the egalitarian solution due to Kalai [1977b] and the relative utilitarian solution, first considered by Arrow [1963]. However many more solution concepts are present in the literature. For a survey on this topic see for instance Thomson et al. [1994]. A natural question can be raised : which solution is the best one ?

The two main contributions of Chapter 5 are :
– We provide a unified interpretation for some well-known solutions in cooperative bargaining.
– We offer a rationale for the Nash solution.

Our model assumes that there are two players engaged in bargaining and one mediator. The mediator's role is to advise the bargainers about the best outcome they should pick over a set of *feasible proposals* X. The mediator's goal is to get them to agree on a solution.

Each bargainer has ordinal preferences $\succsim_i$ over X, $i = 1, 2$. Bargaining is *target-based* : Player i accepts $x \in X$ iff $x \succsim_i t_i$, where $t_i \in X$ is his minimum acceptable outcome. We say that t_i is the player's target or his level of toughness.

The mediator is uncertain about which players' conditions would lead them to an agreement. The mediator knows $\succsim_i$ $i = 1, 2$, but she is uncertain about the level t_i of toughness of each bargainer. Her uncertainty about t_i is represented by a random variable T_i, with cumulative distribution function F_i ; that is, we assume that she is able to assess :

$$P(i \text{ accepts } x) = P(x \succsim_i T_i) = F_i(x).$$

Given these assumptions, we offer a behavioural characterisation for a general class of solutions so that the mediator will maximise the probability that the bargainers strike an agreement. Such a framework allows us to characterise a few major solutions as special cases of this approach, where the single feature separating them is the nature of the stochastic dependence between the bargainers' stances.

In particular, our probability-based approach suggests a straightforward interpretation for the product of two von Neumann–Morgenstern utility numbers advocated by the Nash solution. This is revealed as the product of two probabilities, and corresponds to an implicit assumption of stochastic independence between the bargainers' positions. Finally we show how relaxing this assumption generates other well-known but less frequently used alternatives, namely the egalitarian and the (truncated) utilitarian solutions.

2.2.1 Target-based solutions for Nash bargaining

We present here the results of Chapter 5 in more detail.

We define a bargaining problem as a compact set B in $[0, 1]^2$. Each point $\mathbf{p} = (p_1, p_2)$ in B corresponds to a pair of probabilities. The number p_i represents the probability that bargainer i accepts a feasible offer x made by the mediator. More formally, given a set X of feasible alternatives, we map every $x \in X$ to a point (p_1, p_2) in the unit square through the function $x \to (F_1(x), F_2(x))$ (where F_1 and F_2 are the cdf defined above). We assume therefore that $B = (F_1(X), F_2(X))$ and we take B as the starting point. A *solution* is a map that for any problem B delivers (at least) one point in B.

We consider the preferences of the mediator over the set of lotteries on pairs of acceptance probabilities, and derive a behavioural characterisation under which she evaluates a proposal by the probability that both bargainers agree to it.

The axioms and the representation theorem

We view $[0,1]^2$ as a mixture space for the $\oplus$ operation, under the standard interpretation where $\alpha\mathbf{p} \oplus (1-\alpha)\mathbf{q}$ is a lottery that delivers $\mathbf{p}$ in $[0,1]^2$ with probability α in $[0,1]$ and $\mathbf{q}$ in $[0,1]^2$ with probability $1-\alpha$, see Herstein and Milnor [1953]. At the same time, $[0,1]^2$ is a lattice under the standard component-wise monotonic partial ordering $\geqq$ in $\mathbb{R}^2$. We note $\mathbf{p} \vee \mathbf{q} = (\max(p_1, q_1), \max(p_2, q_2))$ and $\mathbf{p} \wedge \mathbf{q} = (\min(p_1, q_1), \min(p_2, q_2))$.

We make the following assumptions about the mediator's preferences $\succsim$ over the mixture/lattice space $[0,1]^2$.

A.1 (Regularity) $\succsim$ is a complete preorder, continuous and mixture independent.

For the interpretation and implications of A.1 we refer the reader to Theorem 8.4 in Fishburn [1970]. Notice that [Nash, 1950, p. 157] explicitly points out how an analogue of A.1 is implied in his model by the assumption that both bargainers are expected utility maximizers.

A.2 (Non-triviality) $(1,1) \succ (0,0)$.

This rules out the trivial case where the mediator is indifferent between a proposal that is accepted for sure by both bargainers and another proposal that is refused for sure by both bargainers.

A.3 (Disagreement indifference) for any p, q in $[0,1]$, $(p,0) \sim (0,q)$.

This is named after Assumption DI in Border and Segal [1997], who study a preference relation over solutions. Framed within the Nash model, Assumption DI states the following : a solution that assigns to either player the same utility he gets at the disagreement point is as good as the disagreement point itself. In our probability-based framework, it states that having one of the bargainers refusing for sure is equivalent to having both refusing for sure. A proposal is accepted if and only if both bargainers agree to it.

A.4 (Consistency over individual probabilities) for any p in $[0,1]$,

$$p(1,1) \oplus (1-p)(0,1) \sim (p,1) \quad \text{and} \quad p(1,1) \oplus (1-p)(1,0) \sim (1,p).$$

This states the following. Assume that one bargainer is known to accept for sure. Then the mediator is indifferent between a lottery that has the second bargainer accepting for sure with probability p and refusing for sure with probability $(1-p)$, or a proposal where the second bargainer accepts with probability p.

A.5 (Weak complementarity) for any $\mathbf{p}, \mathbf{q}$ in $[0,1]^2$,

$$(1/2)\,(\mathbf{p} \vee \mathbf{q}) \oplus (1/2)\,(\mathbf{p} \wedge \mathbf{q}) \succsim (1/2)\mathbf{p} \oplus (1/2)\mathbf{q}$$

This is named after Axiom S in Francetich [2013]. It states that a fifty-fifty lottery between two pairs of acceptance probabilities $\mathbf{p}$ and $\mathbf{q}$ is weakly inferior to a fifty-fifty lottery between their extremes (under the component–wise ordering). Roughly speaking A.5 requires that the individual acceptance probabilities are (weakly) complementary towards getting to an agreement.

It is possible to show that, under A.1, the four assumptions A.2–A.5 are logically independent.

The main representation theorem is given below.

Theorem 2.2.1. *The preference relation* $\succsim$ *satisfies A.1–A.5 if and only if there exists a unique copula* $C : [0,1]^2 \to [0,1]$ *that represents* $\succsim$.

A copula is a mapping that describes the dependence structure for a bivariate random variable as a function of its marginals. This key result is due to Sklar [1959]. The interpreation of Theorem 2.2.1 is that, under A.1–5, the mediator ranks proposals by their probability of joint acceptance given her subjective opinion over the dependence structure of bargainers' thresholds. In plain words : the mediator maximizes the probability that both sides accept the proposal.

The major solutions

The Nash solution arises whenever we assume that the individuals' acceptance probabilities are independent. As this is a fair and natural requirement, the Nash solution appears as the prominent one. Consider the following axiom.

A.7 (Rescaling indifference) for any α, p, q in $[0,1]$, $(\alpha p, q) \sim (p, \alpha q)$ [6] .

This states that the mediator is indifferent whether the same proportional reduction in the acceptance probability is applied to one bargainer or to the other one.

Theorem 2.2.2. *The preference relation* $\succsim$ *satisfies A.1–A2, A4–A.5*, and A.7 (A.5* is the strict version of A.5) if and only if it is represented by the copula* $\Pi(p,q) = p \cdot q$.

Under axiom A.7, the mediator believes that players' acceptances are stochastically independent and therefore she picks a proposal x s.t.

$$\max_{x \in X} P(x \succsim_1 T_1, x \succsim_2 T_2) = \max_{(p_1,p_2) \in B} p_1 \cdot p_2$$

and the Nash solution emerges.

In the original (utility-based) model, the egalitarian solution, see Kalai [1977b], recommends the maximal point at which utility gains from the disagreement point are equal. More simply, for a Nash problem (S,d), the egalitarian solution selects the maximiser of the function $\min\{(u_1 - d_1), (u_2 - d_2)\}$ for (u_1, u_2) in S and $u_i \geq d_i$ for $i = 1, 2$. In our probability-based framework, consider the following assumption.

6. Axiom A.6, not presented in the this introductory section, is an anonymity axiom. The interested reader is referred to Section 5.3

A.8 (Meet indifference) for any p, q in $[0,1]$, $(p, p \wedge q) \sim (p \wedge q, q)$.

This states that the mediator is indifferent between two pairs of acceptance probabilities as far as they have the same meet. Axiom A.8 gives us the following result.

Theorem 2.2.3. *The preference relation $\succsim$ satisfies A.1–A2, A4–A.5*, and A.8 (A.5* is the strict version of A.5) if and only if it is represented by the copula $M(p,q) = \min(p,q)$.*

Under A.8 the mediator's preferences are represented by the Fréchet upper bound $M(p,q) = \min(p,q)$, that provides the strongest possible positive dependence between two marginal distributions. Hence we can reinterpret the egalitarian solution of Kalai [1977b] as the solution that the mediator should use when she believes that players' target are maximally positively correlated.

Finally, we consider the relative utilitarian solution of Arrow [1963]. This solution consists in maximizing the sum of utilities after having normalized them between zero and one. Consider the following axiom.

A.9 (Average indifference) for any p, q in $[0,1]$, $(p,q) \sim (\frac{p+q}{2}, \frac{p+q}{2})$.

Axiom A.9 may be interpreted by saying that the preferences of the mediator are not affected by a decrease of the acceptance probability of one bargainer whenever this reduction is compensated by an increase of the acceptance probability of the other one. We obtain the following result.

Theorem 2.2.4. *The preference relation $\succsim$ satisfies A.1–A.5 and A.9 if and only if it is represented by the copula $W(p,q) = \max(p+q-1, 0)$.*

Theorem 2.2.4 characterises the Fréchet lower bound $W(p,q) = \max(p+q-1,0)$ that provides the strongest possibile negative dependence between two marginal distributions. Therefore, we can reinterpret this form of (truncated) utilitarian solution as the recommendation that maximises the probability of joint acceptance when the mediator assumes that the individual acceptance probabilities are maximally negatively correlated.

The target-based interpretation

The Nash model is framed in terms of NM utilities whereas our model takes probabilities as basis. A natural question arises : is this really Nash bargaining ?

As we said above, bargaining is target-based and moreover we supposed that the mediator knows $P(i \text{ accepts } x) = P(x \succsim_i T_i) = F_i(x)$. The key observation is that the NM model can be recast in a probability-based language as shown by Castagnoli and LiCalzi [1996] and Bordley and LiCalzi [2000]. Let $\mathcal{C} = [c_*, c^*]$ be a compact and convex interval of $\mathbb{R}$ and consider the set of lotteries $\mathcal{P}_0(\mathcal{C})$. If u_i is an increasing NM utility, under boundedness, right-continuity and an obvious normalization we pose $u_i(c) = P(T_i \leq c)$

and view the "old" utility function u_i as the cumulative distribution function for the target T_i of the bargainer. If a lottery X has cumulative distribution function G and is stochastically independent of T_i then the chain of equalities

$$\mathbb{E}[u_i(X)] = \int u_i(c)\, \mathrm{d}G(c) = \int P(T_i \leq c)\, \mathrm{d}G(c) = P(X \geq T)$$

shows that the expected utility of a lottery X is equivalent to the probability that X beats an uncertain target T_i.

The implicit assumption in Nash [1950] is that the bargainers utilities u_1 and u_2 are commonly known. This is tantamount to assume that the distribution of the targets T_1, T_2 is ex-ante commonly known, while the targets are private information. Hence, if the agents have common knowledge of the joint distribution of their targets ex-ante, they maximise the probability of success by settling on the commonly known copula. In particular, if it is common knowledge that their two targets are stochastically independent, they should settle for the Nash solution.

Chapitre 3

About delay aversion

Ce chapitre est issu de l'article "About delay aversion", en collaboration avec Alain Chateauneuf [1].

Abstract. In this paper, we study the behaviour of decision makers who show preferences for advancing the timing of future satisfaction. We give two definitions that are representative of this kind of attitude and investigate their implications in (an intertemporal version of) three popular models used in decision theory : the Expected Utility, the Choquet Expected Utility and the MaxMin Expected Utility models. The first definition reveals interesting links with the theory studying the impossibility of aggregating infinite streams of income, while keeping both strong monotonicity and equality among all generations. Our second definition turns out to be a behavioural characterization of what Irving Fisher called "impatience". Finally, we make a connection with the notion of domination of one stream of income over another, for all interest rates.

3.1 Introduction

"One today is worth two tomorrows." [2]

As an individual's attitude towards time is crucial in almost all economic problems, it is not surprising that the economic literature studying time preferences is extremely rich and dates back to the 19^{th} century with Böhm-Bawerk's *"The Positive Theory of Capital"*, see Böhm-Bawerk [1891]. Our research follows the stream of papers initiated by the seminal works of Koopmans [1960] and Diamond [1965], as we focus on a Decision Maker (DM),

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2. Benjamin Franklin (1706-1790)

or Social Planner, who has preferences over positive, bounded sequences. In Section 3.3, sequences are treated interchangeably as infinite streams of income or consumption. By contrast, in Section 3.4, we think about them just as streams of income. This difference in interpretation will be clarified below when the main concepts are introduced.

Our main concern is the study of preferences for advancing the time of future satisfaction. This general idea of having an inclination for immediate utility over delayed utility has been given different names in the literature, mostly used as synonyms. We would like to stress straight away, to avoid confusion, that when we are dealing with the concepts of (long-term or short-term) *delay aversion*, *impatience* and *myopia*, we are in fact referring to precise behavioural definitions. All these notions will be formally defined in the main body of the paper, starting from the preferences of the DM over infinite, bounded sequences.

Since the seminal paper of Samuelson [1937], the use of the (exponential) discounting model has been the paradigm for describing such intertemporal tastes. In this model, a stream of income $(x_0, x_1, \dots)$ is evaluated by the utility function :

$$U(x_0, x_1, \dots) = \sum_{t=0}^{\infty} \delta^t u(x_t).$$

The attitude towards the future is represented by the discount factor $\delta \in (0, 1)$. We think that this approach is not totally satisfactory. For instance, it does not allow the concepts of impatience, myopia and delay aversion to be distinguished.

We therefore depart from the traditional analysis and we work with three popular, theoretical models of decision making, adapted to our framework : the Expected Utility model, the Choquet Expected Utility model and the MaxMin Expected Utility model. The idea that these models are suitable to study time preferences can already be found in Marinacci [1998] and Chateauneuf and Ventura [2013]. In Marinacci [1998] the author links the MaxMin Expected Utility model with the concept of patience, whereas in Chateauneuf and Ventura [2013] the authors make a study of the Choquet Expected Utility model and its connection with impatience and myopia. Even if these models are mostly used to analyse decisions under uncertainty in the literature, we want to stress that in our framework no uncertainty is involved. These models are used as flexible tools in order to generalize the notion of the weights that a DM uses in order to evaluate different points in time.

The main contribution of the paper is represented by the introduction of two novel definitions that represent preferences for advancing the time of future satisfaction and their mathematical representation in the three models cited above. In the rest of the introduction we describe these two notions and we briefly present our main findings.

Recent work by Benoît and Ok [2007] describes and characterizes situations when one DM is more delay averse (in some precise sense) than another. Starting from their paper, we define the concept of *long-term delay aversion*, which is compatible with their work. Suppose that an agent has to choose between two extra payments of, say, \$1,000 and \$10,000. The \$1,000 are paid within a month whereas the \$10,000 will be paid much later. We believe that if the second and bigger payment is made sufficiently far in the future, then the agent will choose the first one. More formally, let us consider a DM who is supposed to receive two additional amounts of income or consumption good, a and b, with $a \leq b$, delivered respectively in periods n_0 and n with $n_0 < n$. Then she will be long-term delay averse if she prefers a over b provided that n is sufficiently big. The use of the adjective "long-term" underlines the fact that n, the period of time in which the bigger extra amount is given, can be a very large number. We want to emphasize that there is no uncertainty regarding the date at which the payment b is made. The preference for the sooner payment a derives purely from the fact that the DM has to wait too much, according to her intertemporal tastes, in order to obtain b.

The characterizations of long-term delay aversion for the models considered yield interesting features. First, as long-term delay aversion proves to be a very weak notion, it allows a separation between tastes and evaluation of time to be made. In fact we proved that long-term delay aversion depends only upon the "weights" that the DM attaches to periods (or subsets of periods) of time, as long as she has a strictly increasing and continuous utility function.

Let us now turn to a social planner who has preferences over flows of income in which each period represents the wealth of one generation. The usual discounting model implies very demanding notions of intergenerational inequality, analysed in Chateauneuf and Ventura [2013], called impatience and myopia. Impatience states that an increase in wealth for a finite number of generations, with all the future generations receiving zero income afterwards, is preferred to the original income stream as soon as there are enough generations which are better off. Myopia represents the following notion : suppose that one stream of income is strictly preferred to another. Let us further assume that a fixed, arbitrarily-large amount of extra income is added to the second stream for all but a finite number of generations. Then myopia says that the preference order is not reversed whenever this increase in wealth happens to start for a generation sufficiently far into the future.

One of the main reasons that inspired us to define the concept of long-term delay aversion is exactly the aim of proposing a notion that is weaker than the two described above. A good feature of the Choquet, MaxMin and Expected Utility models is that they allow to represent preferences exhibiting long-term delay aversion without neither myopia nor impatience. Such a property is not shared by the discounting model. We show this in Proposition 3.3.6 and Example 3.3.1 in Section 3.3.3.

Finally, and more interestingly, our definition is linked to the literature that studies the contrast between intergenerational inequalities and Pareto optimality. Basu and Mitra [2003] have shown that there is no aggregating function which satisfies strong monotonicity and equality among generations. In fact, we prove that for the Expected Utility and the MaxMin Expected Utility models, long-term delay aversion is equivalent to the strong monotonicity of preferences, while for the Choquet Expected Utility model strong monotonicity is a necessary condition. Since long-term delay aversion is clearly incompatible with treating all generations equally, this result provides an insight about why strong monotonicity of preferences and equality among generations are contradictory.

The other definition that we propose comes from a straightforward observation. An agent should show preferences for advancing time of future satisfaction if, when dealing with an extra payment that can be made in two consecutive periods, she always chooses the one made at the earlier date. We call this notion *short-term delay aversion*. The word "short-term" is used to underline precisely the fact that we are considering two consecutive dates.

Short-term delay aversion is a demanding notion in terms of preferences of the DM. In discussing this concept, we have in mind that preferences are defined only with respect to streams of income, and not of consumption. While it is not plausible that a DM with an uneven distribution of consumption over time would always prefer to consume at earlier dates, we believe that, when facing two extra monetary payments, an agent should invariably choose the one made at the earlier date. As expected, short-term delay aversion implies both properties of the weights that the DM attaches to periods of time and of the (marginal) utility function. If we focus on the constant marginal utility of wealth, then short-term delay aversion becomes equivalent to discounting (attaching decreasing weights to periods of time).

Short-term delay aversion turns out to be a behavioural counterpart to the definition of preferences for advancing the time of future satisfaction given by Fisher [1930]. We call Fisher's definition F-impatience. According to Fisher, an individual is F-impatient if she has a marginal rate of intertemporal substitution that is always greater than 1 (see footnote 18 p. 82 of Benoît and Ok [2007]). This is exactly our characterization of short-term delay aversion for the Expected Utility model.

Linked to the short-term delay aversion, we propose the concept of *temporal domination*, which was studied recently in a paper by Foster and Mitra [2003] who are interested in characterizing when one cash flow dominates another at all interest rates. We say that one stream of income temporally dominates another if the sum of the first k cash flows of the former is always higher than the sum of the first k cash flows of the latter. We find that a DM is short-term delay averse if and only if whenever one sequence temporally dominates another then she prefers the first to the second. Since accordance of preferences with temporal domination is a natural requirement, its equivalence with short-term delay aversion provides an additional justification to this latter notion. Finally, we show that one infinite cash flow temporally dominates another if and only if all the discounters with constant marginal utility prefer the former to the latter.

The rest of the paper is organized as follows. Section 2 gathers some preliminary notions. In Section 3, we define long-term delay aversion and provide the characterizations. The next section presents the results on short-term delay aversion and temporal domination. Section 5 contains some concluding remarks. The proofs which are not in the main body of the paper are given in the Appendix.

3.2 Preliminaries

We study the preferences of a DM over the set $V := B_\infty^+(\mathbb{N}) = \{\mathbf{x} := (x_n)_{n\in\mathbb{N}} | x_n \geq 0\, \forall n \text{ and } \sup_n x_n < +\infty\}$ of real-valued, positive and bounded sequences. The generic elements of V are denoted as $\mathbf{x}, \mathbf{y}$, etc. and are considered as infinite streams of income (Sections 3.3 and 3.4) or consumption (Section 3.3). The p-th element of sequence $\mathbf{x}$ is denoted equivalently x_p or $\mathbf{x}(p)$. Clearly, the set $\mathbb{N}$ of natural numbers represents time.
Given a sequence $\mathbf{x} = (x_0, x_1, \dots)$, $(x_k + a, \mathbf{x}_{-k})$ denotes sequence $\mathbf{y}$ s.t. $y_k = x_k + a$ and $y_n = x_n$ for all $n \neq k$. The sum of two sequences and the multiplication times a scalar are the pointwise sum and multiplication, meaning that if $\mathbf{x}, \mathbf{y} \in V$ and $\lambda \in \mathbb{R}$ then $\mathbf{x} + \mathbf{y} = (x_0 + y_0, x_1 + y_1, \dots)$ and $\lambda\mathbf{x} = (\lambda x_0, \lambda x_1, \dots)$. We also denote $(x_k + a, \mathbf{x}_{-k})$ as $\mathbf{x} + a1_k$, where 1_A is the indicator function of the set $A \subseteq \mathbb{N}$, i.e. $1_A(n) := \begin{cases} 1 & \text{if } n \in A \\ 0 & \text{if } n \in A^c. \end{cases}$
Therefore 1_A denotes the sequence with $1_A(p) = 1$ if $p \in A$ and $1_A(p) = 0$ if $p \notin A$ and 1_k the sequence with all the elements equal to 0, except the element k which is equal to 1. In the same way, $\mathbf{x}1_A$ denotes the sequence $\mathbf{y}$ such that $y_k = x_k$ if $k \in A$ and $y_k = 0$ otherwise.
As for the convergence of sequences, we use the following notation. If $\{a_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers, we use the notation $a_n \to_n l$ to indicate that the sequence converges to $l \in \mathbb{R}$ when n approaches infinity. Whenever we write $a_n \uparrow_n l$ (resp. $a_n \downarrow_n l$), we mean that the sequence converges to $l \in \mathbb{R}$ and it is monotonically increasing, i.e. for every $n \in \mathbb{N}$, $a_n \leq a_{n+1}$ (resp. monotonically decreasing, i.e. for every $n \in \mathbb{N}$, $a_n \geq a_{n+1}$).
Given a collection of sets $\{A_n\}_{n\in\mathbb{N}}$, $A_n \uparrow_n A$ (resp. $A_n \downarrow_n A$) means that for every $n \in \mathbb{N}$, $A_n \subseteq A_{n+1}$ and $\cup_n A_n = A$ (resp. $n \in \mathbb{N}$, $A_n \supseteq A_{n+1}$ and $\cap_n A_n = A$).

The couple $(\mathbb{N}, 2^{\mathbb{N}})$ is treated as a measurable space. In this setting, we say that :

- A set function $v : 2^{\mathbb{N}} \to [0,1]$ is a *(normalized) capacity* if $v(\emptyset) = 0$, $v(\mathbb{N}) = 1$ and $\forall A, B \in 2^{\mathbb{N}}$, s.t. $A \subseteq B$, then $v(A) \leq v(B)$.
- P is a *(finitely additive) probability* if P is a normalized capacity and moreover $\forall A, B \in 2^{\mathbb{N}}$ s.t. $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$.

– A probability P is *countably additive* if whenever $\{A_n\}$ is a countable disjoint collection of subsets of $\mathbb{N}$ then $P(\cup_n A_n) = \sum_n P(A_n)$.
– The *core* of a capacity v is defined by

$$C(v) = \{P | P \text{ finitely additive s.t. } P(A) \geq v(A) \, \forall A \in 2^{\mathbb{N}}\}.$$

– A capacity is *convex* if $\forall C, D \in 2^{\mathbb{N}}$, $v(C \cup D) + v(C \cap D) \geq v(C) + v(D)$.

We recall that given a capacity v on $2^{\mathbb{N}}$, the Choquet integral of $\mathbf{x} \in V$ w.r.t. v is defined by :

$$\int_{\mathbb{N}} \mathbf{x} \, dv := \int_0^{+\infty} v(\mathbf{x} \geq t) \, dt$$

where $(\mathbf{x} \geq t) := \{n \in \mathbb{N} | x_n \geq t\}$. When the capacity v is simply additive we call it P and denote the integral with the usual symbol used for expectation : $\mathbb{E}_P[\cdot] := \int \cdot \, dP$. See Denneberg [1994].

In the following, we endow the DM with a utility function $u : \mathbb{R}_+ \to \mathbb{R}$ which is strictly increasing, continuous and cardinal (i.e. defined up to a positive affine transformation). With a slight abuse of notation, $u(\mathbf{x})$ denotes the sequence $(u(x_0), u(x_1), \dots)$. Finally, throughout the article, we normalize $u(0) = 0$.

In this paper we say that a preference relation $\succsim$ over V is represented by :

– The Expected Utility (EU) model, if there is a simply additive probability P and a utility function $u(\cdot)$ s.t.

$$\mathbf{x} \succsim \mathbf{y} \Leftrightarrow \mathbb{E}_P[u(\mathbf{x})] \geq \mathbb{E}_P[u(\mathbf{y})].$$

– The Choquet Expected Utility (CEU) model, see Schmeidler [1986] and Schmeidler [1989], if there is a capacity v and a utility function $u(\cdot)$ s.t.

$$\mathbf{x} \succsim \mathbf{y} \Leftrightarrow \int u(\mathbf{x}) \, dv \geq \int u(\mathbf{y}) \, dv.$$

– The MaxMin Expected Utility (MMEU) model, see Gilboa and Schmeidler [1989], if there is a convex and compact (in the weak*-topology) set of simply additive probabilities C and a utility function $u(\cdot)$ s.t.

$$\mathbf{x} \succsim \mathbf{y} \Leftrightarrow \min_{P \in C} \mathbb{E}_P[u(\mathbf{x})] \geq \min_{P \in C} \mathbb{E}_P[u(\mathbf{y})].$$

– The discounting model (we denote it with the couple (u, β) to retain the notation used in Benoît and Ok [2007]) if there is a discount function $\beta : \mathbb{N} \to (0, 1]$ s.t. β is strictly decreasing, $\beta(0) = 1$ and $\sum_t \beta(t) < +\infty$ and a utility function $u(\cdot)$ s.t.

$$\mathbf{x} \succsim \mathbf{y} \Leftrightarrow \sum_{t=0}^{+\infty} \beta(t) u(x_t) \geq \sum_{t=0}^{+\infty} \beta(t) u(y_t).$$

We finish this section with a proposition that explains why we want to represent time preferences with models that are usually employed to catch the attitude of DMs towards risk and/or ambiguity.

Proposition. *A preference relation $\succsim$ can be represented by the discounting model if and only if there exists a σ-additive, strictly positive, probability P, with $P(\{n\}) > P(\{n+1\})\, \forall n \in \mathbb{N}$, such that $\mathbb{E}_P[u(\cdot)]$ represents $\succsim$.*

Démonstration. $\Rightarrow$ Let (u, β) represents $\succsim$ then, calling $\sum_t \beta(t) =: b$, we have $(u, \frac{\beta}{b})$ which also represents $\succsim$. In fact $0 < b < +\infty$ and therefore : $\mathbf{x} \succsim \mathbf{y} \Leftrightarrow \sum_{t=0}^{+\infty} \beta(t)u(x_t) \geq \sum_{t=0}^{+\infty} \beta(t)u(y_t) \Leftrightarrow \sum_{t=0}^{+\infty} \frac{\beta(t)}{b}u(x_t) \geq \sum_{t=0}^{+\infty} \frac{\beta(t)}{b}u(y_t)$.
We can therefore define $P : 2^{\mathbb{N}} \to (0, 1]$ such that $A \subseteq \mathbb{N}$, $P(A) := \sum_{n \in A} \frac{\beta}{b}(n)$. Such a probability is clearly what we need.

$\Leftarrow$ We have $\mathbf{x} \succsim \mathbf{y} \Leftrightarrow \mathbb{E}_P[u(\mathbf{x})] \geq \mathbb{E}_P[u(\mathbf{x})] \Leftrightarrow \sum_{t=0}^{+\infty} P(\{t\})u(x_t) \geq \sum_{t=0}^{+\infty} P(\{t\})u(x_t)$. Therefore $\frac{P}{P(\{0\})}$ can be considered as a discount function. □

The Proposition above uncovers the equivalence between probabilities and the weights that the DM attaches to periods of time. Once this equivalence is established, we can go a step further and consider the generalizations of the EU model (namely the CEU model and the MMEU model) within our intertemporal framework. We therefore treat probabilities and capacities as *weights* that the DM attaches to periods (or subsets of periods) of time. Chateauneuf and Ventura [2013] already studied the case of the CEU model in order to characterize the concepts of impatience and myopia. The MMEU model deserves a separate comment. In fact in the MMEU case, there exists a family of weights rather than a single one. We interpret this family as in Marinacci [1998] :

> "Of course, a [...] justification for this model in our temporal context is that the agents are not sure which weight to use, and instead of a single one, they use a set of weights."[3]

3.3 Long-Term Delay Aversion

In this Section we define the concept of *long-term delay aversion.* We proceed with the main definition.

Definition 3.3.1. *A preference relation $\succsim$ over V is* long-term delay averse *if for $0 < a \leq b$, $n_0 \in \mathbb{N}$ and $\mathbf{x} \in V$ $\exists N := N(\mathbf{x}, n_0, a, b) > n_0$ s.t. $\forall n \geq N$,*

$$(x_{n_0} + a, \mathbf{x}_{-n_0}) \succ (x_n + b, \mathbf{x}_{-n}).$$

3. In the interest of full disclosure, we report that the omitted part contains the world "alternative" used by Marinacci [1998] since he also considers an interpretation in terms of natural density, a concept not presented in this paper.

Definition 3.3.1 states the following : when a DM with a certain distribution of income (or consumption) faces two extra payments, $a > 0$ made in period n_0 and $b \geq a$ made in period n, if the second one happens to be sufficiently far into the future, then she will strictly prefer the payment made before, even if it is lower.

3.3.1 Long-Term Delay Aversion in CEU, MMEU and EU models

We characterize the notion of long-term delay aversion in the CEU, EU and MMEU models.

Proposition 3.3.1. *Let $\succsim$ be represented by the CEU model. Then $(i) \Leftrightarrow (ii)$:*
(i) $\succsim$ is long-term delay averse ;
(ii) $\forall A \in 2^{\mathbb{N}}$ $v(A \cup \{n\}) \to_n v(A)$ and $\forall A \in 2^{\mathbb{N}}$, $\forall t \notin A$ $v(A \cup \{t\}) > v(A)$.

Proposition 3.3.2. *Let $\succsim$ be represented by the MMEU model. Then $(i) \Leftrightarrow (ii)$:*
(i) $\succsim$ is long-term delay averse ;
(ii) $\forall P \in C$, $\forall n \in \mathbb{N}$, $P(\{n\}) > 0$.

Corollary 3.3.1. *Let $\succsim$ be represented by the EU model. Then $(i) \Leftrightarrow (ii)$:*
(i) $\succsim$ is long-term delay averse ;
(ii) $P(\{n\}) > 0 \, \forall n \in \mathbb{N}$.

In the three cases, we see that long-term delay aversion depends just on simple properties of the weights, given by the capacity or the probabilities, that the DM attaches to periods of time, or subsets of periods. These weights need to possess two main features. First, the DM should attach a strictly positive weight to every time-period, which means, in simple words, that every time-period is worth something to her. Second, the weight that the DM attributes to a period of time n could be set arbitrarily close to 0 (but still be strictly positive) by taking n sufficiently far in the future. This is made explicit in Proposition 3.3.1 through the requirement that $\forall A \in 2^{\mathbb{N}}$ $v(A \cup \{n\}) \to_n v(A)$, and it is implicit in Proposition 3.3.2 and Corollary 3.3.1 since we are dealing with probabilities and therefore it is immediate that $P(\{n\}) \to_n 0$. For, suppose by contradiction that $P(\{n\}) \not\to_n 0$. Then for some $\epsilon > 0$ there exists a subsequence $(P(\{\Psi(n)\}))_{n \in \mathbb{N}}$ such that $P(\{\Psi(n)\}) \geq \epsilon$ for all $n \in \mathbb{N}$. Hence since P is finitely additive there exists a finite subset A of $\mathbb{N}$ such that $P(A) > 1$ which is impossible since $1 = P(\mathbb{N}) \geq P(B)$ for all $B \in 2^{\mathbb{N}}$.

It is interesting to remark that, besides continuity and increasingness, which are assumed throughout this paper, no property of the instantaneous utility function $u(\cdot)$ is involved. Such a characterization underlines the fact that long-term delay aversion is a rather weak notion of preferences for advancing the time of future satisfaction. This fact could be better understood with one example. Consider a stream of income $\mathbf{x}$ such that $\mathbf{x}_{n_0} = c > 0$ and $\mathbf{x}_n = 0$, i.e. the income of the period in which a is added is equal to some constant $c > 0$ whereas the extra amount b is added in a period when the income of the DM is equal to 0. Suppose that the period n is sufficiently far in the future as required by Definition 3.3.1 so that the DM prefers the extra amount a rather than the extra amount b. In economics, a usual requirement for the instantaneous utility function of the DM is that the marginal utility satisfies the condition $\lim_{x \to 0} u'(x) = +\infty$ (we assume that $u(\cdot)$ is differentiable for sake of simplicity). This means that adding b to the period n in which the DM earns a zero amount of income should increase the overall utility more than adding a in period n_0, in which the DM is already endowed with some income. In principle, such a property could clash with preferences for anticipating utility. Specifically, it could reverse the preferences of the DM so that she would prefer the extra amount b paid in the distant future, rather the extra amount a paid closer to the present. However, the characterizations above show that this is not an issue, when long-term delay aversion is considered in the CEU, MMEU and EU models. In other words, long-term delay aversion applies even when the marginal utility of the DM is infinite at a zero level of income. These considerations show that long-term delay aversion is a rather mild requirement for the preferences of the DM. It just involves the way she weights periods of time and it does not put any constraint on the shape of her instantaneous utility (as soon as it is strictly increasing and continuous).

As a final remark, it should be noted that a discounting utility DM is long-term delay averse for every discount function $\delta : \mathbb{N} \to (0, 1]$. We will see in the next section that the other notion that we introduced, namely short-term delay aversion, involves features of both weights and utility.

Remark 3.3.1. Construction of a long-term delay averse capacity. *While it is relatively straightforward to exhibit non σ-additive probabilities with the characteristics required in Proposition 3.3.2 and Corollary 3.3.1 (see for instance Example 3.3.1), it may seem hard to exhibit capacities which show the properties of Proposition 3.3.1. A simple way to bypass this difficulty is to consider probability distortion functions. A probability distortion function is an increasing mapping $f : [0, 1] \to [0, 1]$ with $f(0) = 0$ and $f(1) = 1$. It can be easily shown that, given a finitely additive probability P, the set function $v : 2^{\mathbb{N}} \to [0, 1]$ defined as $v(A) = f \circ P(A) = f(P(A))$ is a capacity. If the DM evaluates sequences in V through the functional $\int u(\mathbf{x}) d\, f \circ P$ we say that she is using the Rank Dependent Utility (RDU) model. Notice that the CEU model is a generalization of the RDU model.*

Consider now a strictly increasing and continuous distortion function f. Let $P : 2^{\mathbb{N}} \to (0, 1]$ be a simply additive probability s.t. $P(\{n\}) > 0\, \forall n \in \mathbb{N}$. It can easily be shown that the capacity v defined by $v(A) = f(P(A))\, \forall A \subset \mathbb{N}$ satisfies the properties of Proposition 3.3.1. The first part of the characterization follows from the fact that P is a probability and therefore $P(\{n\}) \to_n 0$. For the second part, it should be noted that $t \notin A$ implies $v(A \cup \{t\}) = f(P(A \cup \{t\})) = f(P(A) + P(\{t\})) > f(P(A)) = v(A)$ where the strict *inequality derives from the assumption that* $P(\{n\}) > 0\,\forall n \in \mathbb{N}$ *and that* f *is strictly increasing.*

It should be noted that we can obtain a capacity yielding long-term delay aversion with mild assumptions about the function f *and the probability* P. *Once again, we want to stress that this is due to the fact that long-term delay aversion is a very weak notion of preferences for advancing the time of future satisfaction. We need therefore to impose weak assumptions for a DM to exhibit such a behaviour.*

3.3.2 Long-Term Delay Aversion, strong monotonicity and (in)equalities among generations

An interesting feature of long-term delay aversion is its link with the strong monotonicity of preferences. This connection in turn implies some insights regarding the theory that studies the impossibility of treating all generations equally. In this Section, the results could be better understood if we bear in mind the preferences of a social planner rather than an agent. Moreover income or consumption in a certain period will be considered as the wealth of the generation living in that period. Obviously, this is just a matter of interpretation and everything could be restated in terms of preferences of an agent.

We consider the following partial orders over V : $\mathbf{x} \geq \mathbf{y}$ means $x_k \geq y_k\,\forall k$, $\mathbf{x} >> \mathbf{y}$ means $x_k > y_k\,\forall k$ and $\mathbf{x} > \mathbf{y}$ means $x_k \geq y_k\,\forall k$, with a strict inequality for at least one k. We recall that *monotonicity* is defined as $\mathbf{x} \geq \mathbf{y} \Rightarrow \mathbf{x} \succsim \mathbf{y}$ and *strong monotonicity* as $\mathbf{x} > \mathbf{y} \Rightarrow \mathbf{x} \succ \mathbf{y}$. The three results below set out the connection between the two notions precisely.

Proposition 3.3.3. *Let* $\succsim$ *be represented by the CEU model. Then* $(i) \Leftrightarrow (ii)$:
(i) $\succsim$ *is strongly monotonic ;*
(ii) $\forall A \in 2^{\mathbb{N}}$, $\forall t \notin A$ $v(A \cup \{t\}) > v(A)$.

Proposition 3.3.4. *Let* $\succsim$ *be represented by the MMEU model. Then* $(i) \Leftrightarrow (ii)$:
(i) $\succsim$ *is strongly monotonic ;*
(ii) $\forall P \in C$, $\forall n \in \mathbb{N}$, $P(\{n\}) > 0$.

Corollary 3.3.2. *Let* $\succsim$ *be represented by the EU model. Then* $(i) \Leftrightarrow (ii)$:
(i) $\succsim$ *is strongly monotonic ;*
(ii) $\forall n \in \mathbb{N}$, $P(\{n\}) > 0$.

Propositions 3.3.3, 3.3.4 and Corollary 3.3.2 together with our characterization show that strong monotonicity is equivalent to long-term delay aversion in the EU and MMEU models, and that it is implied by long-term delay aversion in the CEU framework. In other words, the set of long-term delay averse preference relations over V representable with the MMEU and EU models is equal to the set of strongly monotonic preference

relations over V which can be represented with these models. In addition, the set of long-term delay averse preference relations over V which can be represented with the CEU model is a subset of the set of strongly monotonic preference relations over V which can be represented by the CEU model.
We summarize the above discussion in the following proposition.

Proposition 3.3.5. *Let $\succsim$ be represented by either the MMEU model or the EU model. Then $(i) \Leftrightarrow (ii)$:*
(i) $\succsim$ is long-term delay averse;
(ii) $\succsim$ is strongly monotonic.

Let $\succsim$ be represented by the CEU model. Then $(i) \Rightarrow (ii)$:
(i) $\succsim$ is long-term delay averse;
(ii) $\succsim$ is strongly monotonic.

The research area that studies the link between preferences for advancing the time of future satisfaction and inter-generational equity dates back to 1965 with the seminal work of Diamond [1965]. This literature has gained much attention recently thanks to the paper of Basu and Mitra [2003]. The authors prove the impossibility of reconciling strong monotonicity and inter-generational equity when the preference relation of a social planner can be represented by a utility function. Basu and Mitra [2003] key axiom is called *anonymity*.

Definition 3.3.2. (Basu and Mitra [2003]) *A preference relation $\succsim$ satisfies the* anonymity axiom *if for all $\mathbf{x}, \mathbf{y} \in V$ s.t. there exists $i, j \in \mathbb{N}$ s.t. $x_i = y_j$ and $x_j = y_i$ and s.t. for $k \in \mathbb{N} \setminus \{i, j\}$, $x_k = y_k$, then $\mathbf{x} \sim \mathbf{y}$.*

Definition 3.3.2 says that, *ceteris paribus*, a permutation of the amount of income or consumption of two generations should not affect the preferences of the social planner.

We think that the rationale behind the incompatibility between anonymity and strong monotonicity is not obvious, and the results of Basu and Mitra [2003] pop out as a nice mathematical achievement. Our work could help interpreting Theorem 1 of Basu and Mitra [2003], in our (more limited) context.

It is easy to see that long-term delay aversion and anonymity are incompatible. Let us consider a social planner facing two extra payments a, made in period n_0 and b, made in period n, with $0 < a = b$ added to a constant stream of income $\mathbf{x}$. If she is long-term delay averse then whenever n is sufficiently big we will observe that $(x_{n_0} + a, \mathbf{x}_{-n_0}) \succ (x_n + a, \mathbf{x}_{-n})$. Since $\mathbf{x}$ is constant, the sequences $(x_{n_0} + a, \mathbf{x}_{-n_0})$ and $(x_n + a, \mathbf{x}_{-n})$ are the same, except for a permutation of the amount of income of two generations. The strict preference that derives from long-term delay aversion therefore contrasts with the anonymity axiom of Definition 3.3.2.

Since anonymity and long-term delay aversion are not compatible, and since long-term delay aversion is equivalent to strong monotonicity (in the EU and MMEU models), the impossibility result of Basu and Mitra [2003] is now explained.

From an economic point of view, Corollary 3.3.2 is the most explanatory. In the EU model, and hence in the discounting model, both strong monotonicity and long-term delay aversion are equivalent to assigning strictly positive weights to every period of time. Therefore, as soon as a social planner assigns a strictly positive weight to every generation, inequalities arise.

3.3.3 Long-Term Delay Aversion, impatience and myopia

As we said in the Introduction, one of the reasons that inspired us to define the concept of long-term delay aversion is the fact that the usual discounting model implies demanding notions related to the attitude towards time, namely myopia and impatience. We proceed reporting the formal definitions of these two concepts.

Definition 3.3.3. (Brown and Lewis [1981]) *A preference relation* $\succsim$ *is* myopic *if* $\forall \mathbf{x}, \mathbf{y} \in V$ *such that* $\mathbf{x} \succ \mathbf{y}$ *and* $\forall \epsilon > 0$*,* $\exists n_0(\mathbf{x}, \mathbf{y}, \epsilon) := n_0 \in \mathbb{N}$ *such that* $n \geq n_0 \Rightarrow \mathbf{x} \succ \mathbf{y} + \epsilon 1_{[n,+\infty)}$.

Myopia implies that the strict preference ordering between two sequences $\mathbf{x} \succ \mathbf{y}$ is not reversed when adding an (arbitrarily large) constant amount of income from period n to the second stream, provided that n is a sufficiently big number. Definition 3.3.3 represents a strong notion of future disliking behaviour : even when all but finitely many periods are improved, if this improvement starts far enough into the future, then the initial preferences are not changed.

Definition 3.3.4. (Chateauneuf and Ventura [2013]) *A preference relation* $\succsim$ *is* impatient *if* $\forall \mathbf{x} \in V$*,* $\forall A > 0$*,* $\exists N(\mathbf{x}, A) := N \in \mathbb{N}$ *such that* $n \geq N \Rightarrow (\mathbf{x} + A)1_{[0,n]} \succ \mathbf{x}$.

Definition 3.3.4 says that adding a fixed amount of income to a stream $\mathbf{x}$, to the first n periods and then getting zero from period $n+1$ on, is preferred to having $\mathbf{x}$ itself, provided that n is sufficiently big. Impatience, as myopia, depicts a strong notion of preferences for advancing the time of future satisfaction too. In this case, improving the income in a finite number of periods of time and then getting 0 is preferred, rather than sticking with the initial endowment of income or consumption.

It should be remarked that, if the DM is thought of as a social planner as in Section 3.3.2, and the income in period n represents the wealth of the n-th generation, then both myopia and impatience represent strong notions of generational inequality.

In Proposition 3.3.6 below, we can see that the discounted utility model implies both myopia and impatience (not allowing therefore for a distinction between these two concepts).

Proposition 3.3.6. *A preference relation $\succsim$ is represented by $\mathbb{E}_P[u(\cdot)]$ w.r.t. a σ-additive probability P if and only if $\succsim$ is myopic if and only if $\succsim$ is impatient.*

Démonstration. This follows easily from Proposition 3.2 and Proposition 3.4 of Chateauneuf and Ventura [2013]. □

We conclude this section by constructing a relatively simple example of long-term delay averse preferences which are neither myopic nor impatient. We therefore show that, using tools other than the usual discounted utility model, we can achieve a different (and milder) type of preferences for advancing the time of future satisfaction. In other words, a DM could be long-term delay averse without being neither impatient nor myopic..

Example 3.3.1. *Take $\succsim$ represented by EU w.r.t. a simply additive probability defined in the algebra $\mathcal{A}$ of finite and cofinite sets as : $P(\{n\}) = \left(\frac{1}{3}\right)^{n+1}$ $\forall n \in \mathbb{N}$, $P(\mathbb{N}) = 1$ and*

$$P(A) = \begin{cases} \sum_{n\in A} P(\{n\}) & \text{if } A \text{ is finite} \\ 1 - \sum_{n\in A^c} P(\{n\}) & \text{if } A \text{ is cofinite.} \end{cases}$$

By Lemma 3.6.1 in the Appendix, it is possible to extend the probability P to a simply additive probability Q over the power set $2^{\mathbb{N}}$ s.t. $Q|_{\mathcal{A}} = P$. It should be noted that Q is not σ-additive in fact

$$1 = Q(\mathbb{N}) = Q(\cup_n \{n\}) \neq \sum_{n=0}^{\infty} Q(\{n\}) = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{n+1} = \frac{1}{2}.$$

By Corollary 3.3.1 the DM is long-term delay averse. However by Proposition 3.3.6 she is neither myopic nor impatient. This example therefore shows that it is possible to construct preferences that are long-term delay averse, but that do not display the much stronger notions of impatience or myopia.

3.4 Short-Term Delay Aversion

In this section we study the concept of *short-term delay aversion*. We begin with the main definition.

Definition 3.4.1. *A preference relation $\succsim$ over V is* short-term delay averse *if for every $a > 0$, $k \in \mathbb{N}$ and every $\mathbf{x} \in V$, one has*

$$(x_k + a, \mathbf{x}_{-k}) \succsim (x_{k+1} + a, \mathbf{x}_{-(k+1)}).$$

Definition 3.4.1 says that a DM is short-term delay averse if, when facing two payments at two consecutive dates, she will always prefer the payment closer to the present.

As we said in the Introduction, in this section the sequences will be interpreted exclusively as streams of income. We apply this restriction in order to avoid some criticisms that could be raised when short-term delay aversion is also applied to streams of consumption goods. Consider the following example. Suppose that a DM is endowed with a consumption stream which gives her seven units of a consumption good on Mondays and zero for the rest of the week. Then short-term delay aversion says that she prefers to consume an additional unit of the consumption good on Monday rather than on Tuesday. From a behavioural point of view, this may seem too extreme.

In fact, we believe that it is natural to prefer some extra amount of money earlier rather than later, because money could be, for instance, invested.

Clearly, all the formal definitions and proofs could be applied to infinite streams of consumption good too, if one is willing to accept the implications of short-term delay aversion. Another way of seeing this is as follows. The properties of weights and utility that characterize short-term delay aversion should not be used if one is willing to work with streams of consumption and, at the same time, thinks that short-term delay aversion is not a plausible notion.

3.4.1 Short-Term Delay Aversion in CEU, MMEU and EU models

Now we characterize short-term delay aversion in the CEU and EU models. We give two partial characterizations for the MMEU model.

Our results in this section assume the utility function $u(\cdot)$ to be C^1, i.e. to be continuously differentiable, and such that $u'(x) > 0, \forall x \in \mathbb{R}_+$.

Proposition 3.4.1. *Let $\succsim$ be represented by the CEU model. Then $(i) \Leftrightarrow (ii)$:*
(i) $\succsim$ is short-term delay averse ;
(ii) The following holds :

1. *$\forall A \in 2^{\mathbb{N}}$, $\forall n \in \mathbb{N}$ s.t. $n, n+1 \notin A$, $v(A \cup \{n\}) \geq v(A \cup \{n+1\})$;*
2. *$\forall x, y \in \mathbb{R}_+$ s.t. $x > y$ $\forall n \in \mathbb{N}$, $\forall A, B \in 2^{\mathbb{N}}$ s.t. $A \subset B$, $n \in B$, $n \notin A$, $n+1 \notin B$,*

$$u'(x)(v(A \cup \{n\}) - v(A)) \geq u'(y)(v(B \cup \{n+1\}) - v(B))$$

3. *$\forall x, y \in \mathbb{R}_+$ s.t. $y > x$ $\forall n \in \mathbb{N}$, $\forall A, B \in 2^{\mathbb{N}}$ s.t. $B \subset A$, $n+1 \in A$, $n \notin A$, $n+1 \notin B$,*

$$u'(x)(v(A \cup \{n\}) - v(A)) \geq u'(y)(v(B \cup \{n+1\}) - v(B))$$

Corollary 3.4.1. *Let $\succsim$ be represented by the EU model. Then $(i) \Leftrightarrow (ii)$:*
(i) $\succsim$ is short-term delay averse ;
(ii) $\forall x, y \in \mathbb{R}_+$, $\forall n \in \mathbb{N}$, $u'(x)P(\{n\}) \geq u'(y)P(\{n+1\})$.

Proposition 3.4.2. *Let $\succsim$ be represented by the MMEU model. Then $(i) \Rightarrow (ii)$:*
(i) $\forall P \in C$, $\forall n \in \mathbb{N}$ and $\forall x, y \in \mathbb{R}_+$, $u'(x)P(\{n\}) \geq u'(y)P(\{n+1\})$;
(ii) $\succsim$ is short-term delay averse.

Proposition 3.4.3. *Let $\succsim$ be represented by the MMEU model. Then (i) $\Rightarrow$ (ii) :*
(i) $\succsim$ is short-term delay averse ;
(ii) $\forall P \in C$, $\forall n \in \mathbb{N}$, $P(\{n\}) \geq P(\{n+1\})$.

While long-term delay aversion involves only properties of the weights that the DM attaches to periods of time, short-term delay aversion requires features of both the capacity, or the probabilities, and the utility function. A necessary consequence of short-term delay aversion is that the DM attaches higher weights to earlier periods of time : we need $P(\{n\}) \geq P(\{n+1\})$ for the EU and MMEU models and $v(\{n\}) \geq v(\{n+1\})$ for the CEU model. The characterization of short-term delay aversion in the CEU model appears rather complicated but it is, in fact, just a generalization of part *(ii)* of Corollary 3.4.1. If the capacity v would have been additive, then, for instance, the condition $u'(x)(v(A \cup \{n\}) - v(A)) \geq u'(y)(v(B \cup \{n+1\}) - v(B))$ would read as $u'(x)v(\{n\}) \geq u'(y)v(\{n+1\})$. Since in general v is non-additive, instead of considering the weight $v(\{n\})$, we need to take into account the "marginal contribution" of period n when this period is considered together with another set of periods A (the same reasoning holds for period $n+1$). Moreover, part *(ii)* of Proposition 3.4.1 is split in three sub-parts since the Choquet integral depends on the ranking of the different outcomes. Therefore if x is income in period n and y is income in period $n+1$, then all the cases ($x < y$, $x = y$ and $x > y$) need to be considered.

In the EU and CEU frameworks, short-term delay aversion also entails substantial limitation of the shape of the (marginal) utility function. For example, if we consider a DM with preferences represented by the EU model, then a utility function $u(\cdot)$ satisfying the Inada conditions (e.g. $\lim_{x \to 0} u'(x) = +\infty$ and $\lim_{x \to +\infty} u'(x) = 0$) would *not* be suitable to obtaining short-term delay aversion. This result derives from the fact that short-term delay aversion is a strong requirement. Suppose in fact that a DM is facing an income stream which has a large amount of income in period n and 0 in period $n+1$. Short-term delay aversion says that the DM prefers to add an additional unit of income in period n rather than in period $n+1$. Roughly speaking, in order to exhibit such preferences, we need to make sure that the marginal utility at zero is "not too big", otherwise the preferences could be inverted.

The characterization of short-term delay averse preferences in the EU model has an appealing and simple interpretation. Let us focus on the case where $\succsim$ is strongly monotone ($P(\{n\}) > 0 \, \forall n \in \mathbb{N}$, see Corollary 3.3.2) and $u'(x) > 0 \, \forall x \in \mathbb{R}_+$. Rearranging quantities we get that $\forall x, y \in \mathbb{R}_+$ and $\forall n \in \mathbb{N}$:

$$\frac{P(\{n\})}{P(\{n+1\})} \frac{u'(x)}{u'(y)} \geq 1. \tag{3.1}$$

Considering x as the DM's income in period n and y as her income in period $n+1$, then the expression above states that : a DM is short-term delay averse if and only if she has

a marginal rate of intertemporal substitution that is always greater than 1.
For Fisher [1930] an economic agent has preferences for immediate utility compared to delayed utility if she has a marginal rate of substitution that is always greater than 1 (see footnote 18 p. 82 of Benoît and Ok [2007]). We call Fisher's definition F-impatience. Definition 3.4.1 of short-term delay aversion turns out therefore to be a behavioural characterization of the concept of F-impatience of Irving Fisher. In the following we will refer to (3.1) as *Fisher condition.* [4]

The particular case of convex preferences in the CEU model

A preference relation $\succsim$ over V is said to be *convex* if $\forall \mathbf{x}, \mathbf{y} \in V$, $\forall \alpha \in [0,1]$, $\mathbf{x} \succsim \mathbf{y} \Rightarrow \alpha\mathbf{x} + (1-\alpha)\mathbf{y} \succsim \mathbf{y}$. Schmeidler [1989] suggests that convex preferences can be interpreted as the natural inclination of a DM for smoothing income streams. See also Gilboa [1989] and Wakker [1990].

An extensive study of convex preferences in the CEU framework has been presented in the paper by Chateauneuf and Tallon [2002]. In their work, the authors prove that, in a finite dimensional setting, convexity of preferences is equivalent to the utility function $u(\cdot)$ being concave and the capacity v being convex. Under the mere assumption that $\exists A \in 2^{\mathbb{N}}$ s.t. $0 < v(A) < 1$, it is easy to see that in our infinite dimensional setting, convexity of preferences is still equivalent to $u(\cdot)$ concave and v convex.

As a result, we can obtain a characterization of short-term delay aversion in the CEU framework in terms of properties of the probabilities in the core of v. This result is presented as Proposition 3.4.4.

Proposition 3.4.4. *Let $\succsim$ be a convex preference relation represented by the CEU model. Then (i) $\Leftrightarrow$ (ii) :*
(i) $\succsim$ is short-term delay averse ;
(ii) $\forall P \in C(v)$, $\forall x, y \in \mathbb{R}_+$, $\forall n \in \mathbb{N}$, $u'(x)P(n) \geq u'(y)P(n+1)$.

The proposition above states that a CEU decision maker who has preferences for smoothing income streams is short-term delay averse if and only if the Fisher condition is satisfied for every simply additive probability in the core of v.

Remark 3.4.1. *The concept of* strict *short-term delay aversion is defined in an obvious way using a strict preference relation instead of a weak one. In this case, all the results of Sections 3.4.1 and 3.4.1 hold replacing the weak inequalities with strict ones under the additional assumption that the utility function $u(\cdot)$ is strictly concave.*

4. The Fisher condition is consistent by letting $\frac{P(\{n\})}{P(\{n+1\})}\frac{u'(x)}{u'(y)} = +\infty$ if $P(\{n\}) > 0$ and $P(\{n+1\}) = 0$, and using the convention $\frac{0}{0} = 1$ if $P(\{n\}) = 0$ since in this latter case $P(\{n\}) \geq P(\{n+1\})$ implies $P(\{n+1\}) = 0$.

3.4.2 Temporal Domination

In this section we present the notion of *temporal domination*, and we link it with short-term delay aversion. Temporal domination is an important tool for ranking streams of income. It is derived by condition (3b) of Foster and Mitra [2003]. In a finite dimensional setting, the authors show that whenever one stream of payments temporally dominates another, then the former has a higher present value than the latter for every possible choice of the interest rate. Intuitively, in our infinite dimensional setting, if a stream of income $\mathbf{x}$ temporally dominates $\mathbf{y}$, then a DM with preferences for advancing the time of future satisfaction should prefer $\mathbf{x}$ to $\mathbf{y}$. The aim of this section is to investigate precisely the connection between temporal domination and short-term delay aversion. We conclude our analysis by generalizing a result proved by Foster and Mitra [2003], in infinite dimension.

Definition 3.4.2. *Given* $\mathbf{x}, \mathbf{y} \in V$*, we say that* $\mathbf{x}$ temporally dominates $\mathbf{y}$*, denoted* $\mathbf{x} \succsim_T \mathbf{y}$*, if* $\sum_{i=0}^{k} x_i \geq \sum_{i=0}^{k} y_i \; \forall k \in \mathbb{N}$.

A flow of income $\mathbf{x}$ temporally dominates another flow of income $\mathbf{y}$ if the partial sum of the first k elements of $\mathbf{x}$ is higher or equal to the partial sum of the first k elements of $\mathbf{y}$, for every $k \in \mathbb{N}$. This definition extends condition (3b), of [Foster and Mitra, 2003, p. 474]. The interested reader can refer to their paper for a discussion of the relationship of Definition 3.4.2 to the criterion of stochastic dominance.

Definition 3.4.3. *We say that a preference relation* $\succsim$ *over* V agrees with temporal domination *if for every* $\mathbf{x}, \mathbf{y} \in V$*,* $\mathbf{x} \succsim_T \mathbf{y} \Rightarrow \mathbf{x} \succsim \mathbf{y}$.

Therefore a preference relation is consistent with temporal domination if, whenever a stream of income $\mathbf{x}$ temporally dominates a stream of income $\mathbf{y}$, then $\mathbf{x}$ is preferred to $\mathbf{y}$.

Before stating the main result of this section we recall the definitions of monotone convergence and continuity w.r.t. monotone convergence.

Definition 3.4.4. *A sequence (of sequences)* $\mathbf{x}^n \in V^{\mathbb{N}}$ monotonically converges *to a sequence* $\mathbf{x} \in V$*, denoted* $\mathbf{x}^n \uparrow_n \mathbf{x}$*, if* $\forall n \in \mathbb{N}, \forall p \in \mathbb{N}\, \mathbf{x}^n(p) \leq \mathbf{x}^{n+1}(p)$ *and* $\forall p \in \mathbb{N}$, $\mathbf{x}^n(p) \to_n \mathbf{x}(p)$.

Definition 3.4.5. *A preference relation* $\succsim$ *is continuous w.r.t.* monotone increasing convergence *if* $\mathbf{x}^n \uparrow_n \mathbf{x}$ *and* $\mathbf{y} \succsim \mathbf{x}^n \, \forall n \in \mathbb{N}$ *imply* $\mathbf{y} \succsim \mathbf{x}$.

Proposition 3.4.5. *Let* $\succsim$ *be a monotone and transitive preference relation over* V*, continuous w.r.t. monotone increasing convergence. Then* $(i) \Leftrightarrow (ii)$ *:*
(i) $\succsim$ *is short-term delay averse ;*
(ii) $\succsim$ *agrees with temporal domination.*

Under some monotonicity and continuity assumptions, the notion of short-term delay aversion is equivalent to the fact that, whenever $\mathbf{x}$ temporally dominates another flow of income $\mathbf{y}$, then the DM prefers $\mathbf{x}$ to $\mathbf{y}$. As we said above, in a finite dimension, if $\mathbf{x}$ temporally dominates $\mathbf{y}$, then $\mathbf{x}$ has an higher present value for every possible interest rate. It seems natural therefore for a DM to prefer $\mathbf{x}$ over $\mathbf{y}$. Therefore, such a result provides additional support to the definition of short-term delay aversion.

Remark 3.4.2. *If we want to focus on strict short-term delay aversion then we need some additional definitions. We say that* $\mathbf{x}$ strictly *temporally dominates* $\mathbf{y}$*, denoted* $\mathbf{x} \succ_T \mathbf{y}$*, if* $\mathbf{x} \succsim_T \mathbf{y}$ *and* $\exists k \in \mathbb{N}$ *s.t.* $\sum_{i=0}^{k} x_i > \sum_{i=0}^{k} y_i$*. We say that* $\succsim$ *is continuous w.r.t.* strict *monotone increasing convergence if* $\mathbf{x}^n \uparrow_n \mathbf{x}$*,* $\mathbf{x} \neq \mathbf{y}$ *and* $\mathbf{y} \succ \mathbf{x}^n \, \forall n \in \mathbb{N}$ *imply* $\mathbf{y} \succ \mathbf{x}$*. In this case Proposition 3.4.5 can be rewritten as follows : Let* $\succsim$ *be a monotone and transitive preference relation over* V*, continuous w.r.t. strict monotone increasing convergence. Then strict short-term delay aversion is equivalent to agreeing to strict temporal domination (e.g.* $\mathbf{x} \succ_T \mathbf{y} \Rightarrow \mathbf{x} \succ \mathbf{y}$*).*

We briefly study now the implication of continuity w.r.t. monotone increasing convergence on the three models considered in this paper.

The next proposition shows that for the CEU, MMEU and EU models, continuity w.r.t. monotone increasing convergence is nothing else than impatience as defined in Chateauneuf and Ventura [2013] (see Definition 3.3.4).

Proposition 3.4.6. *Let* $\succsim$ *be a preference relation represented by either the CEU, the MMEU or the EU model. Then* $(i) \Leftrightarrow (ii)$ *:*
(i) $\succsim$ *is continuous w.r.t. monotone increasing convergence ;*
(ii) $\succsim$ *is impatient.*

By means of Proposition 3.4.6, we can state the link between short-term delay aversion and temporal domination in the CEU, MMEU and EU models. The following corollary is obtained.

Corollary 3.4.2. *Let* $\succsim$ *be an* impatient *preference relation represented by either the CEU, the MMEU or the EU model. Then* $(i) \Leftrightarrow (ii)$ *:*
(i) $\succsim$ *is short-term delay averse ;*
(ii) $\succsim$ *agrees with temporal domination.*

For sake of completeness we characterize impatience and continuity w.r.t. monotone increasing convergence in the CEU, MMEU and EU models.

Proposition 3.4.7. *Let* $\succsim$ *be represented by the CEU model. Then* $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ *:*
(i) $\succsim$ *is impatient ;*
(ii) $\succsim$ *is continuous w.r.t. monotone increasing convergence ;*

(iii) v is inner continuous (i.e. $A_n \uparrow_n A \Rightarrow v(A_n) \uparrow_n v(A)$, where $A_n \uparrow_n A$ means $A_n \subseteq A_{n+1}$ and $\cup_n A_n = A$).

Proposition 3.4.8. *Let $\succsim$ be represented by the MMEU model. Then $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$:*
(i) $\succsim$ is impatient ;
(ii) $\succsim$ is continuous w.r.t. monotone increasing convergence ;
(iii) $\forall P \in C$, P is σ-additive.

Corollary 3.4.3. *Let $\succsim$ be represented by the EU model. Then $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$:*
(i) $\succsim$ is impatient ;
(ii) $\succsim$ is continuous w.r.t. monotone increasing convergence ;
(iii) P is σ-additive.

To the best of our knowledge, Proposition 3.4.8 appears to be a novel result. The proofs of Proposition 3.4.7 and Corollary 3.4.3 are well known, but we include them in the Appendix for the sake of completeness. Note that for Proposition 3.4.7 we show that v exact can be dispensed with, unlike in Proposition 3.2 of Chateauneuf and Ventura [2013].

The next proposition generalizes Theorem 7, p.487 of Foster and Mitra Foster and Mitra [2003].

Proposition 3.4.9. *For every $\mathbf{x}, \mathbf{y} \in V$, the following are equivalent :*
(i) $\mathbf{x} \succ_T \mathbf{y}$;
(ii) for every discount function β, $\sum_t \beta(t) x_t > \sum_t \beta(t) y_t$.

Corollary 3.4.4. *For every $\mathbf{x}, \mathbf{y} \in V$, the following are equivalent :*
(i) $\mathbf{x} \succsim_T \mathbf{y}$;
(ii) for every weakly decreasing discount function β, $\sum_t \beta(t) x_t \geq \sum_t \beta(t) y_t$.

How can we identify whether $\mathbf{x} \succsim_T \mathbf{y}$? Corollary 3.4.4 shows that a necessary and sufficient condition for $\mathbf{x}$ to dominate $\mathbf{y}$ temporally is that every (weak) time discounter (u, β) with constant marginal utility should prefer (weakly) $\mathbf{x}$ to $\mathbf{y}$.

Using the words of Foster and Mitra Foster and Mitra [2003], if we interpret the discount function $\beta(\cdot)$ of the DM as an interest rate, then the stream of income $\mathbf{x}$ dominates the stream of income $\mathbf{y}$ at all interest rates if and only if $\mathbf{x} \succsim_T \mathbf{y}$.

3.5 Conclusion

In this paper we define and characterize in three popular theoretical models of decision making two concepts that describe preferences for advancing the time of future satisfaction. We call them *long-term delay aversion* and *short-term delay aversion.*

The first important contribution lies in the use of an intertemporal version of the Expected Utility, the Choquet Expected Utility and the MaxMin Expected Utility models. While this has already been done in the literature, relatively few papers exploit these models as flexible tools to describe intertemporal choices. Our work therefore offers interesting and useful alternatives to the standard discounted utility model.

We show that our first notion, namely long-term delay aversion, is a very weak notion of preferences for anticipating future consumption : it only depends on the weights that the DM attaches to time periods (and subsets of time periods). We do not make any assumption about the shape of the (instantaneous) utility function. Moreover we find that it coincides with it coincides with strong monotonicity of preferences in the Expected Utility Model. This provides an insight about why strong monotonicity of preferences and treating all generations equally are incompatible.

Regarding short-term delay aversion, we prove that not only are the weights which a DM assigns to periods of time important, but her marginal utility plays a role too. We show that short-term delay aversion is equivalent to what Irving Fisher called impatience (here, F-impatience), and thus we provided a behavioural foundation for Fisher's notion.

We show that short-term delay aversion is equivalent to the concept of *temporal domination.* Temporal domination parallels the notion of second order stochastic dominance in decision making under risk. As stochastic dominance is a compelling notion in decision making under risk, temporal domination - its counterpart in decisions involving time - is equally sound. Thus, the equivalence between short-term delay aversion and temporal domination adds additional strength to the former concept.

Finally, as a dividend, we generalized one theorem of Foster and Mitra Foster and Mitra [2003] and we characterized the definition of impatience (as defined in Chateauneuf and Ventura [2013]) in the MaxMin Expected Utility model.

3.6 Proofs

We begin stating and proving a Lemma that allows us to extend a finitely additive probability from an algebra to a σ-algebra.

Lemma 3.6.1. *Let Ω be a set, $\mathcal{B}$ an algebra on Ω and $\mathcal{A}$ a σ-algebra such that $\mathcal{B} \subseteq \mathcal{A}$. Let Q be a simply additive probability on $\mathcal{B}$, then there exists a simply additive probability P on $\mathcal{A}$ such that $P|_{\mathcal{B}} = Q$.*

Démonstration. Define

$$v(A) = \sup\{Q(C) \text{ s.t. } C \in \mathcal{B},\, C \subseteq A\},\ A \in \mathcal{A}.$$

Claim : v is a convex normalized capacity on the σ-algebra $\mathcal{A}$.
Notice that $\forall A \in \mathcal{B}$, $v(A) = Q(A)$, and therefore $v(\emptyset) = 0$ and $v(\Omega) = 1$ since $\emptyset, \Omega \in \mathcal{B}$

and Q is a probability. Moreover if $A, B \in \mathcal{A}$ and $A \subseteq B$, $\{C \in \mathcal{B} | C \subseteq A\} \subseteq \{C \in \mathcal{B} | C \subseteq B\}$ and therefore $v(A) \leq v(B)$. Hence v is a capacity.
Let us prove now that v is convex. Given $A, B \in \mathcal{A}$, by definition of supremum, $\forall \epsilon > 0$, $\exists C \in \mathcal{B}$ and $\exists D \in \mathcal{B}$ such that $C \subseteq A$ and $D \subseteq B$ and

$$Q(C) > v(A) - \epsilon \text{ and } Q(D) > v(B) - \epsilon.$$

Therefore, by monotonicity of v, by the fact that $\mathcal{B}$ is an algebra and since, as noticed before, $Q(A) = v(A)$ if $A \in \mathcal{B}$, we have :

$$\begin{aligned} v(A \cup B) + v(A \cap B) &\geq v(C \cup D) + v(C \cap D) \\ &= Q(C \cup D) + Q(C \cap D) \\ &= Q(C) + Q(D) \\ &> v(A) + v(B) - 2\epsilon. \end{aligned}$$

Hence $v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$ since ϵ can be arbitrarily small.
Hence, we proved that v is a convex capacity.

It is well known that if v is convex then $C(v) \neq \emptyset$. Therefore taking $P \in C(v)$, we have that $\forall B \in \mathcal{B}$, $P(B) \geq v(B) = Q(B)$. Imagine $\exists B' \in \mathcal{B}$ s.t. $P(B') > Q(B')$, then $P(B'^c) < Q(B'^c)$, but $B'^c \in \mathcal{B}$ and so $Q(B'^c) = v(B'^c)$, contradiction. Hence $\forall B \in \mathcal{B}$, $P(B) = Q(B)$. Therefore we found a simply additive probability defined on $(\Omega, \mathcal{A})$ s.t. $P |_{\mathcal{B}} = Q$. □

3.6.1 Proofs of Section 3.3

We first state and prove a lemma that will help us in other proofs. Abusing notation, we will write $u(\mathbf{x}) \geq t \cup n := \{k \in \mathbb{N} | u(x_k) \geq t\} \cup \{n\}$.

Lemma 3.6.2. *For $\mathbf{x} \in V$, $u(\cdot)$ strictly increasing, $a > 0$ and $n \in \mathbb{N}$ we have*

$$\int u(\mathbf{x} + a1_n)\, dv = \int_0^{u(x_n)} v(u(\mathbf{x}) \geq t)\, dt + \\ + \int_{u(x_n)}^{u(x_n+a)} v(u(\mathbf{x}) \geq t \cup n)\, dt + \int_{u(x_n+a)}^{+\infty} v(u(\mathbf{x}) \geq t)\, dt.$$

Démonstration. Fix $\mathbf{x} \in V$, $n \in \mathbb{N}$ and $a > 0$. Notice that we can write $u(\mathbf{x} + a1_n) = u(\mathbf{x})1_{n^c} + u(x_n + a)1_n$ and therefore $\forall k \neq n$ $u(x_k + a1_n(k)) = u(x_k)$. Consider now the sets $A_t = \{u(\mathbf{x}) \geq t\}$ and $B_t = \{u(\mathbf{x} + a1_n) \geq t\}$ with $t \in [0, +\infty)$. We divide the interval $[0, +\infty)$ in 3 parts :

- Take $t \in [0, u(x_n)]$. We have that since $u(x_n + a) > u(x_n) \geq t$, $n \in A_t$ and $n \in B_t$. Moreover $\forall k \neq n$ we have that $k \in A_t \Leftrightarrow k \in B_t$. So $A_t = B_t$
- Take now $t \in (u(x_n), u(x_n + a)]$. We have that $n \notin A_t$ but $n \in B_t$. Also here $\forall k \neq n$, $k \in A_t \Leftrightarrow k \in B_t$. Therefore $B_t = A_t \cup n$.

– Finally if $t \in (u(x_n + a), +\infty)$, clearly $n \notin A_t$ and $n \notin B_t$. Again, $\forall k \neq n$, $k \in A_t \Leftrightarrow k \in B_t$. Hence $A_t = B_t$.

Therefore

$$\int u(\mathbf{x} + a1_n)\,dv = \int_0^{u(x_n)} v(u(\mathbf{x} + a1_n) \geq t)\,dt +$$
$$+ \int_{u(x_n)}^{u(x_n+a)} v(u(\mathbf{x} + a1_n) \geq t)\,dt + \int_{u(x_n+a)}^{+\infty} v(u(\mathbf{x} + a1_n) \geq t)\,dt =$$
$$= \int_0^{u(x_n)} v(u(\mathbf{x}) \geq t)\,dt + \int_{u(x_n)}^{u(x_n+a)} v(u(\mathbf{x}) \geq t \cup n)\,dt + \int_{u(x_n+a)}^{+\infty} v(u(\mathbf{x}) \geq t)\,dt.$$

□

Proof of Proposition 3.3.1. $(i) \Rightarrow (ii)$ a) Let us see first that $\forall A \in 2^{\mathbb{N}}$ $v(A \cup n) \to_n v(A)$. Take $A \in 2^{\mathbb{N}}$, $n_0 \in A$, and define $\mathbf{x} := 1_A$. For all $a > 0$, we have by definition of Choquet integral :

$$\int u(1_A + a1_{n_0})\,dv = \int_0^{+\infty} v(u(1_A + a1_{n_0}) \geq t)\,dt =$$
$$= u(1)v(A) + (u(1+a) - u(1))v(n_0).$$

Take a s.t. given $\epsilon > 0$, $\frac{u(1+a)-u(1)}{u(1)} v(n_0) < \epsilon$. Notice that it is possible since $u(\cdot)$ is continuous and therefore $\lim_{a\to 0} u(1+a) = u(1)$. Since $\succsim$ is long-term delay averse, $b \geq a$ (and $b \geq 1$ if necessary) $\exists N$ s.t. $\forall n \geq N$:

$$\int u(1_A + b1_n)\,dv < u(1)v(A) + (u(1+a) - u(1))v(n_0)$$

We can now have two cases. Either $n \notin A$, therefore $\int u(1_A + b1_n)\,dv = u(1)v(A \cup n) + (u(b) - u(1))v(n)$. Hence

$$\begin{aligned}
u(1)v(A \cup n) + (u(b) - u(1))v(n) &< u(1)v(A) + (u(1+a) - u(1))v(n_0) \\
u(1)(v(A \cup n) - v(A)) &< (u(1+a) - u(1))v(n_0) - (u(b) - u(1))v(n) \\
u(1)(v(A \cup n) - v(A)) &< (u(1+a) - u(1))v(n_0) \\
(v(A \cup n) - v(A)) &< \frac{(u(1+a) - u(1))}{u(1)} v(n_0) < \epsilon.
\end{aligned}$$

Or, $n \in A$, and in this case we simply get $v(A \cup n) - v(A) = v(A) - v(A) = 0 < \epsilon$.

And hence $v(A \cup n) \to_n v(A)$.

b) Here we want to show that $\forall A \in 2^{\mathbb{N}}$, $\forall t \notin A$ $v(A \cup t) > v(A)$. Take $A \in 2^{\mathbb{N}}$, $b \geq a = 1$ $t \notin A$ and notice that $\int u(1_A + 1_t)\,dv = u(1)v(A \cup t)$. Since $\succsim$ is long-term delay averse, $\exists N$ s.t. $\forall n \geq N$, $\int u(1_A + b1_n)\,dv < u(1)v(A \cup t)$. We can have two cases :

– $n \notin A$, and therefore $\int u(1_A + b1_n)\, dv = u(1)v(A \cup n) + (u(b) - u(1))v(n)$. Hence

$$u(1)v(A \cup t) > u(1)v(A \cup n) + (u(b) - u(1))v(n) \geq u(1)v(A \cup n) \geq u(1)v(A),$$

and hence $v(A \cup t) > v(A)$.

– $n \in A$ implies $\int u(1_A + b1_n)\, dv = u(1)v(A) + (u(b+1) - u(1))v(n)$, therefore

$$u(1)v(A \cup t) > u(1)v(A) + (u(b+1) - u(1))v(n) \geq u(1)v(A).$$

And again $v(A \cup t) > v(A)$.

$(ii) \Rightarrow (i)$ Fix $0 < a \leq b$, $n_0 \in \mathbb{N}$, $\mathbf{x} \in V$ and notice that we will be done as soon as we prove the following :

1) $\int u(\mathbf{x} + b1_n)\, dv \to \int u(\mathbf{x})\, dv$;
2) $\int u(\mathbf{x} + a1_{n_0})\, dv > \int u(\mathbf{x})\, dv$.

Suppose in fact that 1) and 2) are true, then $\int u(\mathbf{x} + a1_{n_0})\, dv - \int u(\mathbf{x} + b1_n)\, dv \to \int u(\mathbf{x} + a1_{n_0})\, dv - \int u(\mathbf{x})\, dv > 0$. And therefore $\exists N$ s.t. $\forall n \geq N$, $\int u(\mathbf{x} + a1_{n_0})\, dv - \int u(\mathbf{x} + b1_n)\, dv > 0$, proving that $\succsim$ is actually long-term delay averse.
1) Since $\mathbf{x}$ is bounded ($\exists M_{\mathbf{x}}$ s.t. $x_n \leq M_{\mathbf{x}}\, \forall n \in \mathbb{N}$), then

$$\int u(\mathbf{x} + b1_n)\, dv = \int_0^{+\infty} v(u(\mathbf{x} + b1_n) \geq t)\, dt = \int_0^{u(M_{\mathbf{x}}+b)} v(u(\mathbf{x} + b1_n) \geq t)\, dt.$$

Notice now that $\forall t \in [0, u(M_{\mathbf{x}} + b)]$, $A_t := \{k \text{ s.t. } u(x_k + b1_n(k)) \geq t\} \subseteq \{k \text{ s.t. } u(x_k) \geq t\} \cup \{n\} := B_t$. In fact $k \in A_t$ and $k \neq n$, then $u(x_k) \geq t$ and so $k \in B_t$. If $k \in A_t$ and $k = n$ then $k \in B_t$ since $n \in B_t\ \forall t$. Therefore since v is monotone, $\forall t \in [0, u(M_{\mathbf{x}} + b)]$ $v(u(\mathbf{x} + b1_n) \geq t) \leq v(u(\mathbf{x}) \geq t \cup n)$.
Define now two functions $f : [0, u(M_{\mathbf{x}} + b)] \to [0, 1]$ as $f(t) = v(u(\mathbf{x}) \geq t)$ and $f_n : [0, u(M_{\mathbf{x}} + b)] \to [0, 1]$ as $f_n(t) = v(u(\mathbf{x}) \geq t \cup n)$. We have that by hypothesis $\forall t \in [0, u(M_{\mathbf{x}} + b)]$, $f_n(t) \to_n f(t)$, and moreover $\forall t \in [0, u(M_{\mathbf{x}} + b)]$, $f_n(t) \leq 1$. We can therefore use the Dominated Convergence Theorem and hence

$$\int_0^{u(M_{\mathbf{x}}+b)} v(u(\mathbf{x}) \geq t \cup n)\, dt = \int_0^{u(M_{\mathbf{x}}+b)} f_n(t)\, dt \to_n$$
$$\to_n \int_0^{u(M_{\mathbf{x}}+b)} f(t)\, dt = \int_0^{u(M_{\mathbf{x}}+b)} v(u(\mathbf{x}) \geq t)\, dt.$$

We get therefore

$$\int u(\mathbf{x})\, dv \leq \int u(\mathbf{x} + b1_n)\, dv = \int_0^{u(M_{\mathbf{x}}+b)} v(u(\mathbf{x} + b1_n) \geq t)\, dt \leq$$
$$\leq \int_0^{u(M_{\mathbf{x}}+b)} v(u(\mathbf{x}) \geq t \cup n)\, dt \to_n \int_0^{u(M_{\mathbf{x}}+b)} v(u(\mathbf{x}) \geq t)\, dt = \int u(\mathbf{x})\, dv$$

So actually $\int u(\mathbf{x}+b1_n)\,dv \to_n \int u(\mathbf{x})\,dv$.
2) Consider now the difference $\int u(\mathbf{x}+a1_{n_0})\,dv - \int u(\mathbf{x})\,dv$, and notice that we can rewrite, by Lemma 3.6.2, $\int u(\mathbf{x}+a1_{n_0})\,dv = \int_0^{+\infty} v(u(\mathbf{x}+a1_{n_0}) \geq t)\,dt$ as :

$$\int_0^{+\infty} v(u(\mathbf{x}+a1_{n_0}) \geq t)\,dt = \int_0^{u(x_{n_0})} v(u(x) \geq t)\,dt + \\ + \int_{u(x_{n_0})}^{u(x_{n_0}+a)} v(u(\mathbf{x}) \geq t \cup n_0)\,dt + \int_{u(x_{n_0}+a)}^{+\infty} v(u(\mathbf{x}) \geq t)\,dt.$$

And since we can write

$$\int_0^{+\infty} v(u(\mathbf{x}) \geq t)\,dt = \int_0^{u(x_{n_0})} v(u(\mathbf{x}) \geq t)\,dt + \\ + \int_{u(x_{n_0})}^{u(x(n_0)+a)} v(u(\mathbf{x}) \geq t)\,dt + \int_{u(x_{n_0}+a)}^{+\infty} v(u(\mathbf{x}) \geq t)\,dt,$$

then the difference $\int u(\mathbf{x}+a1_{n_0})\,dv - \int u(\mathbf{x})\,dv$ will be just :

$$\int u(\mathbf{x}+a1_{n_0})\,dv - \int u(\mathbf{x})\,dv = \int_{u(x_{n_0})}^{u(x_{n_0}+a)} v(u(\mathbf{x}) \geq t \cup n_0) - v(u(\mathbf{x}) \geq t)\,dt.$$

Since we supposed that $\forall A \in \mathcal{B}$, $\forall t \notin A$ $v(A \cup t) > v(A)$, and for $t \in (u(x_{n_0}), u(x_{n_0}+a))$, $n_0 \notin \{u(\mathbf{x}) \geq t\}$, then $\int u(\mathbf{x}+a1_{n_0})\,dv > \int u(\mathbf{x})\,dv$. □

Proof of Proposition 3.3.2. $(i) \Rightarrow (ii)$ Take $\mathbf{x} = 0$ and suppose that $\exists \overline{P} \in C$ and $\exists n_0 \in \mathbb{N}$ s.t. $\overline{P}(n_0) = 0$. Then for $a > 0$

$$\min_{P \in C} \mathbb{E}_P[u(0+a1_{n_0})] = \min_{P \in C} u(a)P(n_0) = u(a)\overline{P}(n_0) = 0,$$

where $\overline{P}$ will be a solution of the minimization problem since $\forall P \in C$ and $a > 0$, $u(a)P(n_0) \geq 0$. Taking now $b \geq a$, we have that $\forall n \in \mathbb{N}$,

$$\min_{P \in C} \mathbb{E}_P[u(0+b1_n)] = \min_{P \in C} u(b)P(n) \geq 0 = \min_{P \in C} \mathbb{E}_P[u(0+a1_{n_0})].$$

Hence $\forall n \in \mathbb{N}$, $b1_n \succsim a1_{n_0}$, so $\succsim$ is not long-term delay averse, contradiction.

$(ii) \Rightarrow (i)$ We need to show that for $\mathbf{x} \in V$, $n_0 \in \mathbb{N}$, $0 < a \leq b$, $\exists N$ s.t. $\forall n \geq N$, $(x_{n_0}+a, \mathbf{x}_{-n_0}) \succ (x_n+b, \mathbf{x}_{-n})$ i.e. $\exists N$ s.t. $\forall n \geq N$, $\min_{P \in C} \mathbb{E}_P[u(\mathbf{x}+a1_{n_0})] > \min_{P \in C} \mathbb{E}_P[u(\mathbf{x}+b1_n)]$.
Notice that $u(\mathbf{x}+a1_{n_0}) = u(\mathbf{x})1_{n_0^c} + u(x_{n_0}+a)1_{n_0}$ and that $u(x_{n_0}+a) = u(x_{n_0}) + u(x_{n_0}+a) - u(x_{n_0})$. Therefore

$$\mathbb{E}_P[u(\mathbf{x}+a1_{n_0})] = \mathbb{E}_P[u(\mathbf{x})1_{n_0^c} + u(x_{n_0})1_{n_0} + (u(x_{n_0}+a) - u(x_{n_0}))1_{n_0}] = \\ = \mathbb{E}_P[u(\mathbf{x}) + (u(x_{n_0}+a) - u(x_{n_0}))1_{n_0}] = \mathbb{E}_P[u(\mathbf{x})] + (u(x_{n_0}+a) - u(x_{n_0}))P(n_0).$$

By the properties of the operator min we have :

$$\min_{P\in C}\mathbb{E}_P[u(\mathbf{x})]+(u(x_{n_0}+a)-u(x_{n_0}))P(n_0)\geq$$
$$\geq \min_{P\in C}\mathbb{E}_P[u(\mathbf{x})]+\min_{P\in C}(u(x_{n_0}+a)-u(x_{n_0}))P(n_0).$$

Call $k:=\min_{P\in C}(u(x_{n_0}+a)-u(x_{n_0}))P(n_0)$ and notice that $k>0$ since by hypothesis $P(n)>0\,\forall n\in\mathbb{N}$ and $u(\cdot)$ is strictly increasing. Let P^* be s.t. $P^*\in\arg\min\mathbb{E}_P[u(\mathbf{x})]$, then there is N_{P^*} s.t. $\forall n\geq N_{P^*}$ and $\forall b>0$, $(u(x_n+b)-u(x_n))P^*(n)<k$, since $P^*(n)\to_n 0$. So we have

$$\begin{aligned}\min_{P\in C}\mathbb{E}_P[u(\mathbf{x}+a1_{n_0})] &\geq \min_{P\in C}\mathbb{E}_P[u(\mathbf{x})]+k\\ &> \mathbb{E}_{P^*}[u(\mathbf{x})]+(u(x_n+b)-u(x_n))P^*(n)\\ &= \mathbb{E}_{P^*}[u(\mathbf{x}+b1_n)]\\ &\geq \min_{P\in C}\mathbb{E}_P[u(\mathbf{x}+b1_n)].\end{aligned}$$

Therefore $\exists N$ (i.e. in this case N_{P^*}) s.t. $\forall n\geq N$, $(x_{n_0}+a,\mathbf{x}_{-n_0})\succ(x_n+b,\mathbf{x}_{-n})$. □

Proof of Corollary 3.3.1. It follows immediately from either Proposition 3.3.1 or 3.3.2. □

Proof of Proposition 3.3.3. $(i)\Rightarrow(ii)$ Let us first prove that $u(\cdot):\mathbb{R}_+\to\mathbb{R}$ strictly increasing is a necessary condition for strong monotonicity. Fix $x,y\in\mathbb{R}_+$ s.t. $x>y$ and consider the sequences $\mathbf{x}:=x$ and $\mathbf{y}:=y$ $\forall n\in\mathbb{N}$. By strong monotonicity, $\mathbf{x}\succ\mathbf{y}$ and therefore $u(x)=\int u(\mathbf{x})\,dv>\int u(\mathbf{y})\,dv=u(y)$.
Fix now $A\in 2^{\mathbb{N}}$, $n\notin A$ and $\mathbf{y}:=1_A$ and $\mathbf{x}:=1_{A\cup n}$. We have that $\mathbf{x}\geq\mathbf{y}$ and $\mathbf{x}\neq\mathbf{y}$ therefore by strong monotonicity $\mathbf{x}\succ\mathbf{y}$, and hence $\int u(\mathbf{x})\,dv>\int u(\mathbf{y})\,dv$. Notice that

$$\begin{aligned}\int u(\mathbf{y})\,dv &= \int_0^{u(1)} v(A)\,dt=u(1)v(A)\\ \int u(\mathbf{x})\,dv &= \int_0^{u(1)} v(A\cup n)\,dt=u(1)v(A\cup n).\end{aligned}$$

Hence $\mathbf{x}\succ\mathbf{y}\Leftrightarrow u(1)v(A\cup n)>u(1)v(A)\Leftrightarrow v(A\cup n)>v(A)$.

$(ii)\Rightarrow(i)$ Take $\mathbf{x}\geq\mathbf{y}$, $\mathbf{x}\neq\mathbf{y}$ we want to show that $\mathbf{x}\succ\mathbf{y}$.
Since $\mathbf{x}\geq\mathbf{y}$, $\mathbf{x}\neq\mathbf{y}$ there exists an $n\in\mathbb{N}$, $\epsilon>0$ s.t. $x_n=y_n+\epsilon$. We already showed in the proof of Proposition 3.3.1 that $\forall A\subset\mathbb{N},\forall n\notin A$ $v(A\cup n)>v(A)$ implies $\int u(\mathbf{y}+\epsilon 1_n)\,dv>\int u(\mathbf{y})\,dv$. Therefore,

$$\int u(\mathbf{x})\,dv\geq\int u(\mathbf{y}+\epsilon 1_n)\,dv>\int u(\mathbf{y})\,dv,$$

i.e. $\mathbf{x}\succ\mathbf{y}$. □

Proof of Proposition 3.3.4. *(i)* $\Rightarrow$ *(ii)* Let us first prove that $u(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ strictly increasing is a necessary condition for strong monotonicity. Fix two numbers $x, y \in \mathbb{R}_+$ s.t. $x > y$ and consider the sequences $x1_n$ and $y1_n$ for some $n \in \mathbb{N}$. Clearly $x1_n \geq y1_n$ and $x1_n \neq y1_n$ and therefore by strong monotonicity $\min_{P\in C} \mathbb{E}_P[u(x1_n)] > \min_{P\in C} \mathbb{E}_P[u(y1_n)]$, hence $u(x) \min_{P\in C} P(n) > u(y) \min_{P\in C} P(n)$, i.e. $u(x) > u(y)$. Suppose now that $\exists P \in C$ and $\exists n_0 \in \mathbb{N}$ s.t. $P(n_0) = 0$. Take $\mathbf{x} := 1_{n_0}$ and $\mathbf{y} := 0$, we have $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$, but $\min_{P\in C} \mathbb{E}_P[u(\mathbf{x})] = 0 = \min_{P\in C} \mathbb{E}_P[u(\mathbf{y})]$, a contradiction.

(ii) $\Rightarrow$ *(i)* Fix $\mathbf{x}, \mathbf{y} \in V$ s.t. $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$. We have that there exists at least one $n \in \mathbb{N}$ s.t. $x_n > y_n$ and therefore $u(\mathbf{x}) \geq u(\mathbf{y}) + (u(x_n) - u(y_n))1_n$. But then $\forall P \in C$, we get :

$$\mathbb{E}_P[u(\mathbf{x})] \geq \mathbb{E}_P[u(\mathbf{y})] + (u(x_n) - u(y_n))P(n),$$

and considering the minimum :

$$\begin{aligned}
\min_{P\in C} \mathbb{E}_P[u(\mathbf{x})] &\geq \min_{P\in C} \mathbb{E}_P[u(\mathbf{y})] + (u(x_n) - u(y_n))P(n) \\
&\geq \min_{P\in C} \mathbb{E}_P[u(\mathbf{y})] + \min_{P\in C}(u(x_n) - u(y_n))P(n) \\
&> \min_{P\in C} \mathbb{E}_P[u(\mathbf{y})].
\end{aligned}$$

Where the last strict inequality comes form the fact that $\min_{P\in C}(u(x_n)-u(y_n))P(n) > 0$, since $u(\cdot)$ is strictly increasing and $\forall P \in C$, $\forall n \in \mathbb{N}$, $P(n) > 0$. Hence $\mathbf{x} \succ \mathbf{y}$. □

Proof of Corollary 3.3.2. The proof follows immediately from either Propositions 3.3.3 or 3.3.4. □

3.6.2 Proof of Section 3.4

Proof of Proposition 3.4.1. *(i)* $\Rightarrow$ *(ii)* **Point 1 :**
Let $A \in 2^{\mathbb{N}}$ and $n, n+1 \notin A$. Fix $\mathbf{x} := 1_A$ and $a = 1$. By short-term delay aversion, $\mathbf{x} + 1_n \succsim \mathbf{x} + 1_{n+1}$ and therefore $\int u(\mathbf{x} + 1_n)\, dv \geq \int u(\mathbf{x} + 1_{n+1})\, dv$. Notice that

$$\int u(\mathbf{x} + 1_n)\, dv = \int_0^{u(1)} v(A \cup n)\, dt,$$

and similarly

$$\int u(\mathbf{x} + 1_{n+1})\, dv = \int_0^{u(1)} v(A \cup n + 1)\, dt.$$

Therefore $v(A \cup n) \geq v(A \cup n + 1)$.

Point 2 :
Fix $x, y \in \mathbb{R}_+$ s.t. $x > y$ and fix $n \in \mathbb{N}$ and $A, B \in 2^{\mathbb{N}}$ s.t. $A \subset B$, $n \in B$, $n \notin A$, $n + 1 \notin B$.

Define the set $C := B \setminus (A \cup n)$ and notice that the sets A, C, $\{n\}$, $\{n+1\}$ are pairwise disjoint. Define now the following sequence :

$$\mathbf{x}_a := y1_{n+1} + c1_C + (x-a)1_n + d1_A$$

with a, c, d s.t. they satisfy $0 < y+a < c < x-a < d$. Let us consider :

$$\mathbf{x}_a + a1_n = y1_{n+1} + c1_C + x1_n + d1_A$$

and

$$\mathbf{x}_a + a1_{n+1} = (y+a)1_{n+1} + c1_C + (x-a)1_n + d1_A.$$

Integrating by mean of the Choquet integral these two sequences we have

$$\int u(\mathbf{x}_a + a1_n)\, dv = u(y)(v(n+1 \cup C \cup n \cup A) - v(C \cup n \cup A)) +$$
$$+ u(c)(v(C \cup n \cup A) - v(n \cup A)) + u(x)(v(n \cup A) - v(A)) + u(d)v(A) =$$
$$= u(y)(v(n+1 \cup B) - v(B)) + u(c)(v(B) - v(n \cup A)) + u(x)(v(n \cup A) - v(A)) + u(d)v(A),$$

and in the same way

$$\int u(\mathbf{x}_a + a1_{n+1})\, dv = u(y+a)(v(n+1 \cup B) - v(B)) +$$
$$+ u(c)(v(B) - v(n \cup A)) + u(x-a)(v(n \cup A) - v(A)) + u(d)v(A).$$

Hence by short-term delay aversion for every $a > 0$ small enough (so that it satisfies the inequalities above), $\mathbf{x}_a + a1_n \succsim \mathbf{x}_a + a1_{n+1}$, and therefore $\int u(\mathbf{x}_a + a1_n)\, dv - \int u(\mathbf{x}_a + a1_{n+1})\, dv \geq 0$. i.e.

$$(u(y) - u(y+a))(v(n+1 \cup B) - v(B)) + (u(x) - u(x-a))(v(n \cup A) - v(A)) \geq 0$$

Dividing both the left and right hand-side by a and letting $a \to 0$ we get the result.

Point 3 :
The proof is similar to the one of Point 2.

(ii) $\Rightarrow$ *(i)* Fix $\mathbf{x} \in V$, $n \in \mathbb{N}$ and $a > 0$. We need to prove the following inequality

$$\int u(\mathbf{x} + a1_n)\, dv \geq \int u(\mathbf{x} + a1_{n+1})\, dv.$$

To simplify notation call $x_n =: z$ and $x_{n+1} =: y$. We need to consider 5 possible cases summarized in the following table :

Case 0 : $z = y$	
$z > y$	Case 1 : $u(y) < u(y+a) \leq u(z) < u(z+a)$
	Case 2 : $u(y) < u(z) \leq u(y+a) < u(z+a)$
$z < y$	Case 3 : $u(z) < u(z+a) \leq u(y) < u(y+a)$
	Case 4 : $u(z) < u(y) \leq u(z+a) < u(y+a)$

Case 0 :

Since $z = y$ then $u(z) = u(y)$ and $u(z+a) = u(y+a)$, and therefore we can write, using Lemma 3.6.2 :

$$\int u(\mathbf{x} + a1_n)\, dv = \int_0^{u(z)} v(u(\mathbf{x}) \geq t)\, dt + \int_{u(z)}^{u(z+a)} v(u(\mathbf{x}) \geq t \cup n)\, dt + \\ + \int_{u(z+a)}^{+\infty} v(u(\mathbf{x}) \geq t)\, dt,$$

and

$$\int u(\mathbf{x} + a1_{n+1})\, dv = \int_0^{u(z)} v(u(\mathbf{x}) \geq t)\, dt + \int_{u(z)}^{u(z+a)} v(u(\mathbf{x}) \geq t \cup n+1)\, dt + \\ + \int_{u(z+a)}^{+\infty} v(u(\mathbf{x}) \geq t)\, dt.$$

We have therefore

$$\int u(\mathbf{x} + a1_n)\, dv - \int u(\mathbf{x} + a1_{n+1})\, dv = \\ = \int_{u(z)}^{u(z+a)} v(u(\mathbf{x}) \geq t \cup n) - v(u(\mathbf{x}) \geq t \cup n+1)\, dt \geq 0,$$

where the last inequality comes from the hypothesis. In fact, since $\forall t \in (u(z), u(z+a))$, $n, n+1 \notin \{u(\mathbf{x}) \geq t\} =: A_t$, we have $v(A_t \cup n) \geq v(A_t \cup n+1)$.

Case 1 :

We rewrite :

$$\int u(\mathbf{x} + a1_n)\, dv = \int_0^{u(y)} v(u(\mathbf{x}) \geq t)\, dt + \int_{u(y)}^{u(y+a)} v(u(\mathbf{x}) \geq t)\, dt + \\ + \int_{u(y+a)}^{u(z)} v(u(\mathbf{x}) \geq t)\, dt + \int_{u(z)}^{u(z+a)} v(u(\mathbf{x}) \geq t \cup n)\, dt + \int_{u(z+a)}^{+\infty} v(u(\mathbf{x}) \geq t)\, dt, \quad (3.2)$$

and

$$\int u(\mathbf{x} + a1_{n+1})\, dv = \int_0^{u(y)} v(u(\mathbf{x}) \geq t)\, dt + \int_{u(y)}^{u(y+a)} v(u(\mathbf{x}) \geq t \cup n+1)\, dt + \\ + \int_{u(y+a)}^{u(z)} v(u(\mathbf{x}) \geq t)\, dt + \int_{u(z)}^{u(z+a)} v(u(\mathbf{x}) \geq t)\, dt + \int_{u(z+a)}^{+\infty} v(u(\mathbf{x}) \geq t)\, dt. \quad (3.3)$$

Subtracting the equations (3.2) $-$ (3.3) we obtain :

$$\int_{u(z)}^{u(z+a)} v(u(\mathbf{x}) \geq t \cup n) - v(u(\mathbf{x}) \geq t)\, dt - \int_{u(y)}^{u(y+a)} v(u(\mathbf{x}) \geq t \cup n+1) - v(u(\mathbf{x}) \geq t)\, dt.$$

To simplify the notation we define two functions $f : (u(z), u(z+a)) \to [0,1]$ as $f(t) := v(u(\mathbf{x}) \geq t \cup n) - v(u(\mathbf{x}) \geq t)$ and $g : (u(y), u(y+a)) \to [0,1]$ as $g(t) := v(u(\mathbf{x}) \geq$

$t \cup n+1) - v(u(\mathbf{x}) \geq t)$.
Hence we have that :

$$\int_{u(z)}^{u(z+a)} f(t)\, dt \geq (u(z+a) - u(z)) \inf_{t \in (u(z), u(z+a))} f(t).$$

Notice that we can find a sequence $(t_i)_{i \in \mathbb{N}}$ s.t. $\forall i \in \mathbb{N}$ $t_i \in (u(z), u(z+a))$ s.t. $f(t_i) \to_i \inf_t f(t)$.
In fact suppose that it was not true and that for some $i \in \mathbb{N} \setminus \{0\}$, there is no $t_i \in (u(z), u(z+a))$ such that $\inf_t f(t) \leq f(t_i) < \inf_t f(t) + \frac{1}{i}$. Then this would imply that for all $t \in (u(z), u(z+a))$, $f(t) \geq \inf_t f(t) + \frac{1}{i} > \inf_t f(t)$, i.e. it would imply that there is a lower bound strictly greater then the infimum for the set $\{f(t) | t \in (u(z), u(z+a))\}$, which is not possible since the infimum is defined as the greatest lower bound.
So $\exists I \in \mathbb{N}$ s.t. $\forall i \geq I$,

$$\int_{u(z)}^{u(z+a)} f(t)\, dt \geq (u(z+a) - u(z)) f(t_i).$$

Fix one of such t_i and call it t^*.
We can do the same reasoning with g and we will get that

$$\int_{u(y)}^{u(y+a)} g(t)\, dt \leq (u(y+a) - u(y)) \sup_{t \in (u(y), u(y+a))} g(t).$$

And therefore we can find a sequence $(t_i)_{i \in \mathbb{N}}$ s.t. $\forall i \in \mathbb{N}$ $t_i \in (u(y), u(y+a))$ s.t. $g(t_i) \to_n \sup_t g(t)$. So $\exists I \in \mathbb{N}$ s.t. $\forall i \geq I$,

$$\int_{u(y)}^{u(y+a)} g(t)\, dt \leq (u(y+a) - u(y)) g(t_i).$$

Again, fix one of such t_i and call it t_*.
Notice that $t^* \in (u(z), u(z+a))$ and $t_* \in (u(y), u(y+a))$ and $u(y+a) \leq u(z)$. Therefore calling $A := \{u(\mathbf{x}) \geq t^*\}$ and $B := \{u(\mathbf{x}) \geq t_*\}$, we have that $A \subset B$, $n \in B$, $n \notin A$, $n+1 \notin B$.Therefore :

$$\begin{aligned} &\int_{u(z)}^{u(z+a)} f(t)\, dt - \int_{u(y)}^{u(y+a)} g(t)\, dt \geq \\ &\quad \geq (u(z+a) - u(z))(v(A \cup n) - v(A)) - (u(y+a) - u(y))(v(B \cup n+1) - v(B)), \end{aligned}$$

Now, since $u(\cdot)$ is C^1, we use the Mean Value Theorem and we have that $\exists c \in (z, z+a)$ and $\exists d \in (y, y+a)$ (so that $c > d$) s.t.

$$(u(z+a) - u(z)) = u'(c)a \text{ and } (u(y+a) - u(y)) = u'(d)a.$$

Since the hypothesis of Point 2 is satisfied we finally get :

$$\begin{aligned} &(u(z+a) - u(z))(v(A \cup n) - v(A)) - (u(y+a) - u(y))(v(B \cup n+1) - v(B)) = \\ &\quad = u'(c)a(v(A \cup n) - v(A)) - u'(d)a(v(B \cup n+1) - v(B)) \geq 0. \end{aligned}$$

This means $\int u(\mathbf{x}+a1_n)\,dv - \int u(\mathbf{x}+a1_{n+1})\,dv \geq 0$.

Case 2 :

We can write :

$$\int u(\mathbf{x}+a1_n)\,dv = \int_0^{u(y)} v(u(\mathbf{x}) \geq t)\,dt + \int_{u(y)}^{u(z)} v(u(\mathbf{x}) \geq t)\,dt+ \\ + \int_{u(z)}^{u(y+a)} v(u(\mathbf{x}) \geq t \cup n)\,dt + \int_{u(y+a)}^{u(z+a)} v(u(\mathbf{x}) \geq t \cup n)\,dt + \int_{u(z+a)}^{+\infty} v(u(\mathbf{x}) \geq t)\,dt, \tag{3.4}$$

and

$$\int u(\mathbf{x}+a1_{n+1})\,dv = \int_0^{u(y)} v(u(\mathbf{x}) \geq t)\,dt + \int_{u(y)}^{u(z)} v(u(\mathbf{x}) \geq t \cup n+1)\,dt \\ + \int_{u(z)}^{u(y+a)} v(u(\mathbf{x}) \geq t \cup n+1)\,dt + \int_{u(y+a)}^{u(z+a)} v(u(\mathbf{x}) \geq t)\,dt + \int_{u(z+a)}^{+\infty} v(u(\mathbf{x}) \geq t)\,dt. \tag{3.5}$$

Subtracting (3.4) – (3.5) we get :

$$\int u(\mathbf{x}+a1_n)\,dv - \int u(\mathbf{x}+a1_{n+1})\,dv = \int_{u(y)}^{u(z)} v(u(\mathbf{x}) \geq t) - v(u(x) \geq t \cup n+1)\,dt+ \\ + \int_{u(z)}^{u(y+a)} v(u(\mathbf{x}) \geq t \cup n) - v(u(\mathbf{x}) \geq t \cup n+1)\,dt + \int_{u(y+a)}^{u(z+a)} v(u(\mathbf{x}) \geq t \cup n) - v(u(\mathbf{x}) \geq t)\,dt.$$

Notice that the only difference between Case 1 and Case 2 is the term $\int_{u(z)}^{u(y+a)} v(u(\mathbf{x}) \geq t \cup n) - v(u(\mathbf{x}) \geq t \cup n+1)\,dt$. Since $\forall t \in (u(z), u(y+a))$, $n, n+1 \notin \{u(\mathbf{x}) \geq t\}$ we have, using Point 1 of the hypothesis :

$$v(u(\mathbf{x}) \geq t \cup n) \geq v(u(\mathbf{x}) \geq t \cup n+1) \text{ for every } t \in (u(z), u(y+a)).$$

Therefore

$$\int u(\mathbf{x}+a1_n)\,dv - \int u(\mathbf{x}+a1_{n+1})\,dv \geq \int_{u(y)}^{u(z)} v(u(\mathbf{x}) \geq t) - v(u(\mathbf{x}) \geq t \cup n+1)\,dt+ \\ + \int_{u(y+a)}^{u(z+a)} v(u(\mathbf{x}) \geq t \cup n) - v(u(\mathbf{x}) \geq t)\,dt. \tag{3.6}$$

Now, using the same argument as before we can find some $t^* \in (u(y+a), u(z+a))$ and $t_* \in (u(y), u(z))$ s.t., calling again $B := \{u(\mathbf{x}) \geq t_*\}$ and $A := \{u(\mathbf{x}) \geq t^*\}$ we get,

$$\int_{u(y+a)}^{u(z+a)} v(u(\mathbf{x}) \geq t \cup n) - v(u(\mathbf{x}) \geq t)\,dt - \int_{u(y)}^{u(z)} v(u(\mathbf{x}) \geq t \cup n+1) - v(u(\mathbf{x}) \geq t)\,dt \geq \\ \geq (u(z+a) - u(y+a))(v(A \cup n) - v(A)) - (u(z) - u(y))(v(B \cup n+1) - v(B)).$$

Moreover as in the previous case we have : $A \subset B$, $n \in B$, $n \notin A$, $n+1 \notin B$, as required by the hypothesis of Point 2. Therefore, through the Mean Value Theorem we can find some $c \in (y+a, z+a)$ and $d \in (y, z)$ (we remind that in Case 2, $(z \leq y+a)$ and hence $d < c$) s.t.

$$\begin{aligned}(u(z+a) - u(y+a))(v(A \cup n) - v(A)) - (u(z) - u(y))(v(B \cup n+1) - v(B)) = \\ = u'(c)(z-y)(v(A \cup n) - v(A)) - u'(d)(z-y)(v(B \cup n+1) - v(B)) \geq 0\end{aligned}$$

and the last inequality comes again from Point 2 of the hypothesis. Hence $\int u(\mathbf{x}+a1_n)\, dv - \int u(\mathbf{x} + a1_{n+1})\, dv \geq 0$.

Case 3 and **Case 4** are similar to the cases above. □

Proof of Corollary 3.4.1. The EU model is nothing else than the CEU model when the capacity v proves to be a simply additive probability P. Therefore $\succsim$ is short-term delay averse if and only if $P(n) \geq P(n+1)$, for $x > y$ $P(n)u'(x) \geq P(n+1)u'(y)$ and $x < y$ $P(n)u'(x) \geq P(n+1)u'(y)$. Therefore $\forall x, y \in \mathbb{R}_+$, $\forall n \in \mathbb{N}$, $P(n)u'(x) \geq P(n+1)u'(y)$. □

Proof of Proposition 3.4.2. Fix $\mathbf{x} \in V$, $n \in \mathbb{N}$ and $a > 0$. By the Mean Value Theorem there exist $c \in (x_n, x_n + a)$ and $d \in (x_{n+1}, x_{n+1} + a)$ s.t.

$$\begin{aligned} u'(c) &= \frac{u(x_n + a) - u(x_n)}{a} \\ u'(d) &= \frac{u(x_{n+1} + a) - u(x_{n+1})}{a}. \end{aligned}$$

And therefore using the fact that $\forall P \in C, \forall x, y \in \mathbb{R}_+, \forall n \in \mathbb{N}$, $u'(x)P(n) \geq u'(y)P(n+1)$,

$$\begin{aligned} \frac{u(x_n + a) - u(x_n)}{a} P(n) &\geq \frac{u(x_{n+1} + a) - u(x_{n+1})}{a} P(n+1) \\ \mathbb{E}_P[u(\mathbf{x})] + (u(x_n + a) - u(x_n))P(n) &\geq \mathbb{E}_P[u(\mathbf{x})] + (u(x_{n+1} + a) - u(x_{n+1}))P(n+1) \\ \mathbb{E}_P[u(\mathbf{x} + a1_n)] &\geq \mathbb{E}_P[u(\mathbf{x} + a1_{n+1})]. \end{aligned}$$

Since the last inequality is true $\forall P \in C$ then :

$$\min_{P \in C} \mathbb{E}_P[u(\mathbf{x} + a1_n)] \geq \min_{P \in C} \mathbb{E}_P[u(\mathbf{x} + a1_{n+1})],$$

i.e. $\succsim$ is short-term delay averse. □

Proof of Proposition 3.4.3. Fix $n \in \mathbb{N}$ and consider the following sequence :

$$\mathbf{x}_a := (c, c, \dots, \underbrace{c}_{n}, \underbrace{c-a}_{n+1}, c, \dots),$$

where $c > a > 0$. Clearly $\forall a < c$:

$$\min_{P \in C} \mathbb{E}_P[u(\mathbf{x}_a + a1_{n+1})] = u(c).$$

Moreover $\forall a < c$:

$$\min_{P\in C} \mathbb{E}_P[u(\mathbf{x}_a + a1_n)] =$$

$$= \min_{P\in C} \mathbb{E}_P[u(c)1_{\mathbb{N}} + (u(c+a) - u(c))1_n + (u(c-a) - u(c))1_{n+1}] =$$

$$= u(c) + \min_{P\in C}(u(c+a) - u(c))P(n) + (u(c-a) - u(c))P(n+1)$$

Then by short-term delay aversion we will get for every a s.t. $0 < a < c$,

$$u(c) + \min_{P\in C}(u(c+a) - u(c))P(n) + (u(c-a) - u(c))P(n+1) \geq u(c).$$

Therefore $\forall P \in C$ and $\forall a < c$, we get $(u(c+a)-u(c))P(n)+(u(c-a)-u(c))P(n+1) \geq 0$. Dividing both sides times $a > 0$ and letting $a \to 0$ we get

$$u'(c)P(n) \geq u'(c)P(n+1),$$

and since we supposed $u'(\cdot) > 0$, we obtain the result. □

Lemma 3.6.3. *If v is convex then condition 2 of Proposition 3.4.1, is equivalent to the following :*

$$\forall x, y \in \mathbb{R}_+ \text{ s.t. } x > y,\ \forall n \in \mathbb{N},\ \forall P \in C(v),\ u'(x)P(\{n\}) \geq u'(y)P(\{n+1\})$$

Proof of Lemma 3.6.3. We first prove that if v is convex then condition 2 of Proposition 3.4.1 is equivalent to the following :

$$\forall x, y \in \mathbb{R}_+, \text{ s.t. } x > y,\ \forall n \in \mathbb{N},\ u'(x)v(n) \geq u'(y)(1 - v(\{n+1\}^c)) \qquad (3.7)$$

If condition 2 of Proposition 3.4.1 is satisfied, then choosing $A = \emptyset$ ans $B = \{n+1\}^c$ we get condition (3.7).
Let us suppose that condition (3.7) is satisfied let us consider $A, B \in 2^{\mathbb{N}}$ s.t. $A \subset B$, $n \in B$, $n \notin A$, $n+1 \notin B$. Recall that v is convex if $\forall C, D \in 2^{\mathbb{N}}$, $v(C\cup D)+v(C\cap D) \geq v(C)+v(D)$. Consider now $C := B \cup n+1$ and $D := \{n+1\}^c$, we have :

$$(v(A\cup n) - v(A))u'(x) \geq u'(x)v(n) \geq u'(y)(1 - v(\{n+1\}^c)) \geq u'(y)(v(B\cup n+1) - v(B)).$$

It remains to prove that condition (3.7) is equivalent to $\forall x, y \in \mathbb{R}_+$ s.t. $x > y$, $\forall n \in \mathbb{N}$, $\forall P \in C(v)$, $u'(x)P(n) \geq u'(y)P(n+1)$.
If (3.7) is satisfied then, since $P \in C(v)$,

$$u'(x)P(n) \geq u'(x)v(n) \geq u'(y)(1 - v(\{n+1\}^c)) \geq u'(y)P(n+1).$$

Conversely, since v is convex and $\{n\} \subset \{n+1\}^c$, then $\exists P_0 \in C(v)$ s.t. $P_0(n) = v(n)$ and $P_0(\{n+1\}^c) = v(\{n+1\}^c)$ [5]. Therefore $u'(x)v(n) \geq u'(y)(1 - v(\{n+1\}^c))$.

□

5. This comes for instance from Lemma 2 of Delbaen [1974]

Proof of Proposition 3.4.4. $(i) \Rightarrow (ii)$ Suppose the DM is short-term delay averse. Since $\succsim$ is convex then v is convex and therefore by Lemma 3.6.3, $\forall x, y \in R_+$ s.t. $x > y$, $\forall n \in \mathbb{N}$ and $\forall P \in C(v)$, $u'(x)P(n) \geq u'(y)P(n+1)$. Letting x decrease towards y we see that $\forall n \in \mathbb{N}$, $\forall P \in C(v)$, $P(n) \geq P(n+1)$. Take now $x \leq y$ and notice that, by convexity of $\succsim$, $u(\cdot)$ is concave and hence $u'(x) \geq u'(y)$. Since $\forall n \in \mathbb{N}$, $P(n) \geq P(n+1)$, we conclude that $u'(x)P(n) \geq u'(y)P(n+1)$.
Hence $\forall P \in C(v)$, $\forall x, y \in \mathbb{R}_+$, $\forall n \in \mathbb{N}$, $u'(x)P(n) \geq u'(y)P(n+1)$.

$(ii) \Rightarrow (i)$ Suppose that $\forall P \in C(v)$, $\forall x, y \in \mathbb{R}_+$, $\forall n \in \mathbb{N}$, $u'(x)P(n) \geq u'(y)P(n+1)$. Since when v is convex the CEU model is the MMEU model, Proposition 3.4.2 entails that the DM is short-term delay averse. □

We state and prove now a Lemma which will be used in the proof of Proposition 3.4.5. We first need a definition that will simplify notation.

Definition 3.6.1. *Given $x, y \in \mathbb{R}^n_+$, we say that x* elementary dominates *y, denoted $x \succsim_{eT} y$, if $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$ and $\exists \epsilon > 0$ and $i_0, i_1 \in [1, n]$ with $i_0 < i_1$ such that $x_{i_0} = y_{i_0} + \epsilon$, $x_{i_1} = y_{i_1} - \epsilon$ and $x_i = y_i$ for $i \neq i_0$ and $i \neq i_1$.*

Lemma 3.6.4. *Let x, y be vectors in $\mathbb{R}^n_+$. Then $(i) \Leftrightarrow (ii)$:*
(i) $\sum_{i=1}^k x_i \geq \sum_{i=1}^k y_i$ $\forall k \in [1, n]$ with $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$;
(ii) there exists a finite sequence of vectors $y^{(p)} \in \mathbb{R}^n_+$ with $p \in [0, m]$ s.t. $y^{(0)} = y$, $y^{(p+1)} \succsim_{eT} y^{(p)}$ and $y^{(m)} = x$. Moreover the number of steps m is at most $n-1$.

Démonstration. $(i) \Rightarrow (ii)$ If for every $k = 1, 2, \ldots, n$ $\sum_{i=1}^k x_i = \sum_{i=1}^k y_i$ then $x = y$. Suppose therefore that for at least one k we have a strict inequality. Denote $\hat{k}$ the smallest k s.t. $\sum_{i=1}^k x_i > \sum_{i=1}^k y_i$. Since for $k \in [1, \hat{k}-1]$, $\sum_{i=1}^k x_i = \sum_{i=1}^k y_i$, then $x_k = y_k$ for such ks, whereas $x_{\hat{k}} - y_{\hat{k}} > 0$.
Denote $\epsilon_{\hat{k}} := x_{\hat{k}} - y_{\hat{k}}$ and $d_k := y_k - x_k$ for $k \in [\hat{k}+1, n]$. Notice that, since $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, we have $\sum_{i=\hat{k}+1}^n d_k = \epsilon_{\hat{k}} > 0$. Therefore there exists $k^{(1)} := \min\{k \in [\hat{k}+1, n] | d_k > 0\}$.
Then the algorithm to construct the sequence is the following :

- $\epsilon_{\hat{k}} \leq d_{k^{(1)}}$. Then, take the amount $\epsilon_{\hat{k}}$ from $y_{k^{(1)}}$ and add it to $y_{\hat{k}}$ so that $y_{\hat{k}} + \epsilon_{\hat{k}} = x_{\hat{k}}$. We will have :

$$\begin{array}{rccccccc} y^{(1)} = & (y_1, & \ldots, & y_{\hat{k}} + \epsilon_{\hat{k}}, & \ldots, & y_{k^{(1)}} - \epsilon_{\hat{k}}, & \ldots, & y_n) \\ & \shortparallel & & \shortparallel & & \geq & & \\ x = & (x_1, & \ldots, & x_{\hat{k}}, & \ldots, & x_{k^{(1)}}, & \ldots, & x_n) \end{array}$$

Clearly, $y^{(1)} \succsim_{eT} y^{(0)} = y$, and $y^{(1)}_{\hat{k}} = x_{\hat{k}}$. Restart the algorithm considering now $y^{(1)}$, and finding the next smallest k s.t. $x_k > y^{(1)}_k$.

– $\epsilon_{\hat{k}} > d_{k^{(1)}}$. Then, take the amount $d_{k^{(1)}}$ from $y_{k^{(1)}}$ so that $y_{k^{(1)}} - d_{k^{(1)}} = x_{k^{(1)}}$, and add it to $y_{\hat{k}}$. We will have :

$$\begin{array}{rcccccccccc} y^{(1)} = & (y_1, & \dots, & y_{\hat{k}} + d_{k^{(1)}}, & \dots, & y_{k^{(1)}} - d_{k^{(1)}}, & \dots, & y_n) \\ & \shortparallel & & < & & \shortparallel & & \\ x = & (x_1, & \dots, & x_{\hat{k}}, & \dots, & x_{k^{(1)}}, & \dots, & x_n) \end{array}$$

Start again the algorithm finding $k^{(2)} := \min\{k \in [\hat{k}+1, n] | d_k > 0\}$, and so on. Since at every step at least one coordinate of the vector $y^{(\cdot)}$ becomes equal to one coordinate of the vector x, and at the very last step, necessarily we have that two coordinates will become equal, then the number of steps needed is at most $n-1$.

$(ii) \Rightarrow (i)$ It follows by the transitivity of $\succsim_{eT}$ and by the fact that $\forall x, y \in \mathbb{R}^n_+$, $x \succsim_{eT} y$ implies (i). $\square$

Proof of Proposition 3.4.5. $(i) \Rightarrow (ii)$ Consider $\mathbf{x}, \mathbf{y} \in V$ s.t. $\mathbf{x} \succsim_T \mathbf{y}$ and let us show that $\mathbf{x} \succsim \mathbf{y}$.
Let $n \in \mathbb{N}$, $n \geq 2$ and consider the sequences $\mathbf{x}1_{[0,n]} := (x_0, x_1, \dots, x_n, 0, 0, \dots)$ and $\mathbf{y}1_{[0,n]} := (y_0, y_1, \dots, y_n, 0, 0, \dots)$. Since $\mathbf{x} \succsim_T \mathbf{y}$, $\sum_{i=0}^k x_i \geq \sum_{i=0}^k y_i$, $\forall k \in [0, n]$. Let $\epsilon_n \geq 0$ be s.t. $\sum_{i=0}^n x_i = \sum_{i=0}^{n-1} y_i + (y_n + \epsilon_n)$ and let $\tilde{\mathbf{y}}1_{[0,n]} := (y_0, y_1, \dots, y_n + \epsilon_n, 0, 0, \dots)$. Then, considering the first n components of the two sequences as n-dimensional vector, thanks to Lemma 3.6.4 we can find a sequence $y^{(p)} \in \mathbb{R}^n_+$ with $p \in [0, m]$ (with $m \leq n-1$) s.t.

$$y^{(0)} = \tilde{\mathbf{y}}1_{[0,n]}, \ y^{(p+1)} \succsim_{eT} y^{(p)}, \text{ and } y^{(m)} = \mathbf{x}1_{[0,n]}.$$

We can notice that by short-term delay aversion, $y^{(p+1)} \succsim_{eT} y^{(p)}$ implies $\mathbf{y}^{(p+1)}1_{[0,n]} \succsim \mathbf{y}^{(p)}1_{[0,n]}$ and therefore by transitivity of $\succsim$, $\mathbf{x}1_{[0,n]} \succsim \tilde{\mathbf{y}}1_{[0,n]}$. Monotonicity implies therefore $\mathbf{x}1_{[0,n]} \succsim \mathbf{y}1_{[0,n]}$ and again by monotonicity $\mathbf{x} \succsim \mathbf{y}1_{[0,n]}$. Since by hypothesis $\succsim$ is continuous w.r.t. monotone increasing convergence, $\mathbf{y}1_{[0,n]} \uparrow_n \mathbf{y}$ gives $\mathbf{x} \succsim \mathbf{y}$, which completes the proof.

$(ii) \Rightarrow (i)$ Since $(x_k + a, \mathbf{x}_{-k}) \succsim_T (x_{k+1} + a, \mathbf{x}_{-(k+1)})$, then by hypothesis $(x_k + a, \mathbf{x}_{-k}) \succsim (x_{k+1} + a, \mathbf{x}_{-(k+1)})$.

$\square$

Proof of Proposition 3.4.6. First, it is straightforward to see that if $\succsim$ is represented by a utility index $I : V \to \mathbb{R}$ (where $I(\cdot)$ can be the functional of the CEU, MMEU or EU model) then $\succsim$ is continuous w.r.t. monotone increasing convergence if and only if $\mathbf{x}^n \uparrow_n \mathbf{x} \Rightarrow I(\mathbf{x}^n) \uparrow_n I(\mathbf{x})$.

$(i) \Rightarrow (ii)$ Fix $\mathbf{x} \in V$ and $A > 0$. We have clearly $(\mathbf{x} + A)1_{[0,n]} \uparrow_n \mathbf{x} + A$ and therefore $I((\mathbf{x} + A)1_{[0,n]}) \uparrow_n I(\mathbf{x} + A)$ by hypothesis. Since in CEU, MMEU and EU models, $I(\mathbf{x} + A) > I(\mathbf{x})$, $\exists N(\mathbf{x}, A) := N \in \mathbb{N}$ such that $n \geq N \Rightarrow I((\mathbf{x} + A)1_{[0,n]}) > I(\mathbf{x})$ and hence $(\mathbf{x} + A)1_{[0,n]} \succ \mathbf{x}$.

(ii) $\Rightarrow$ *(i)* Let $\mathbf{x}^n \uparrow_n \mathbf{x}$, we need to show that $I(\mathbf{x}^n) \uparrow_n I(\mathbf{x})$.

Case 1 : $\mathbf{x} \in V$ is such that there exists a real number $k > 0$ s.t. $x_p \geq k \, \forall p \in \mathbb{N}$.
Fix $\epsilon > 0$, we are going to show that $\exists N(\epsilon) \in \mathbb{N}$ s.t. $n \geq N(\epsilon) \Rightarrow I(\mathbf{x}^n) \geq I(\mathbf{x}) - \epsilon$. Notice that since $\mathbf{x}$ is bounded and $u(\cdot)$ is continuous, $u(\cdot)$ is uniformly continuous and hence $\exists \beta > 0$ s.t. $\beta < k$ and s.t. $\forall n \in \mathbb{N}$, $u(x_n - \beta) \geq u(x_n) - \epsilon$. Therefore from the definition of the utility index $I(\cdot)$ in CEU, MMEU and EU, it turns out that $I(\mathbf{x} - \beta) \geq I(\mathbf{x}) - \epsilon$. Fix now $\gamma \in \mathbb{R}$ s.t. $0 < \gamma < \beta$, impatience implies that $\exists n_0$ s.t. $n \geq n_0 \Rightarrow (\mathbf{x} - \beta + \gamma) 1_{[0,n]} \succ \mathbf{x} - \beta$. Since $\gamma < \beta$, we have $\mathbf{x} >> \mathbf{x} - \beta + \gamma$. $\mathbf{x}^n \uparrow_n \mathbf{x}$ implies that $\forall k \in [0, n_0]$, $\exists N_k$ s.t. $n \geq N_k \Rightarrow \mathbf{x}^n(k) > (\mathbf{x} - \beta + \gamma) 1_{[0,n_0]}(k)$. Take $N(\epsilon) := \max\{N_0, \ldots, N_{n_0}\}$, we have $n \geq N(\epsilon) \Rightarrow \mathbf{x}^n \geq (\mathbf{x} - \beta + \gamma) 1_{[0,n_0]}$, and hence in the CEU, MMEU and EU models, $\mathbf{x}^n \succsim (\mathbf{x} - \beta + \gamma) 1_{[0,n_0]}$. But then we get

$$I(\mathbf{x}^n) \geq I((\mathbf{x} - \beta + \gamma) 1_{[0,n_0]}) > I(\mathbf{x} - \beta) \geq I(\mathbf{x}) - \epsilon$$

and therefore Case 1 is proved.

Case 2 : We consider a general $\mathbf{x} \in V$.
Since $\mathbf{x}^n \uparrow_n \mathbf{x}$ by monotonicity of $\succsim$ in CEU, MMEU and EU models we have $I(\mathbf{x}^n) \uparrow_n a$. Let us suppose by contradiction that $a < I(\mathbf{x})$. Notice that the set of sequences $\{\mathbf{x}^n : n \in \mathbb{N}\}$ is uniformly bounded since $\mathbf{x}^n(p) \leq \mathbf{x}(p)$, $\forall p, n \in \mathbb{N}$ and $\mathbf{x} \in V$. Fix $\epsilon \in \mathbb{R}$ s.t. $0 < \epsilon < I(\mathbf{x}) - a$ and notice that by uniform continuity of $u(\cdot)$, $\exists k > 0$ s.t. $u(\mathbf{x}^n(p) + k) - u(\mathbf{x}^n(p)) \leq \epsilon \; \forall n, p \in \mathbb{N}$. This implies again by monotonicity that $\forall n \in \mathbb{N}$, $I(\mathbf{x}^n + k) \leq I(\mathbf{x}^n) + \epsilon$. But then

$$\lim_n I(\mathbf{x}^n + k) \leq \lim_n I(\mathbf{x}^n) + \epsilon = a + \epsilon < I(\mathbf{x}) \leq I(\mathbf{x} + k).$$

Since $\mathbf{x}^n + k \uparrow_n \mathbf{x} + k$ and $x_p + k \geq k > 0$ for every $p \in \mathbb{N}$ this contradicts the proof of Case 1. Hence $I(\mathbf{x}^n) \uparrow_n I(\mathbf{x})$.

□

Proof of Proposition 3.4.7. *(i)* $\Leftrightarrow$ *(ii)* It follows from Proposition 3.4.6. Moreover as noticed in Proposition 3.4.6 if $\succsim$ is represented by the CEU model, continuity w.r.t. monotone increasing convergence is equivalent to $\mathbf{x}^n \uparrow_n \mathbf{x} \Rightarrow \int u(\mathbf{x}^n)\, dv \uparrow_n \int u(\mathbf{x})\, dv$.

(ii) $\Rightarrow$ *(iii)* Consider $A_n \uparrow_n A$ and define $\mathbf{x}^n := 1_{A_n}$. Clearly $1_{A_n} \uparrow_n 1_A$ and by hypothesis $\int u(1_{A_n})\, dv \uparrow_n \int u(1_A)\, dv$. Since $\int u(1_{A_n})\, dv = u(1) v(A_n)$ and $\int u(1_A)\, dv = u(1) v(A)$, we can conclude that $v(A_n) \uparrow_n v(A)$.

(iii) $\Rightarrow$ *(ii)* Let $\mathbf{x}^n \uparrow_n \mathbf{x}$. Since the utility function $u(\cdot)$ is continuous and strictly increasing, $u(\mathbf{x}^n) \uparrow_n u(\mathbf{x})$. By definition of the Choquet integral, $\int u(\mathbf{x}^n)\, dv = \int_0^\infty v(u(\mathbf{x}^n) \geq t)\, dt$ and $\int u(\mathbf{x})\, dv = \int_0^\infty v(u(\mathbf{x}) \geq t)\, dt$. Let us define $A_n(t) := \{k | u(\mathbf{x}^n(k)) \geq t\}$ and $A(t) := \{k | u(\mathbf{x}(k)) \geq t\}$. Clearly $\forall t$, $A_n(t) \uparrow_n A(t)$ and by hypothesis $v(A_n(t)) \uparrow_n v(A(t))$. By the Monotone Convergence Theorem, $\int_0^\infty v(A_n(t))\, dt \uparrow_n \int_0^\infty v(A(t))\, dt$, i.e. $\int u(\mathbf{x}^n)\, dv \uparrow_n \int u(\mathbf{x})\, dv$. □

Proof of Proposition 3.4.8. *(i)* $\Leftrightarrow$ *(ii)* It follows from Proposition 3.4.6. Again, as noticed in Proposition 3.4.6 if $\succsim$ is represented by the MMEU model, continuity w.r.t. monotone increasing convergence is equivalent to $\mathbf{x}^n \uparrow_n \mathbf{x} \Rightarrow \min_{P\in C} \mathbb{E}_P[u(\mathbf{x}^n)] \uparrow_n \min_{P\in C} \mathbb{E}_P[u(\mathbf{x})]$.

(ii) $\Rightarrow$ *(iii)* Fix $Q \in C$ and consider a sequence of sets $A_n \uparrow_n \mathbb{N}$. Define $\mathbf{x}^n := 1_{A_n}$. Clearly $1_{A_n} \uparrow_n 1_{\mathbb{N}} = 1$ and therefore $u(1)1_{A_n} = u(1_{A_n}) \uparrow_n u(1)$ since $u(\cdot)$ is continuous and strictly increasing (with $u(0) = 0$). Moreover $\mathbb{E}_P[u(1)1_{A_n}] = u(1)P(A_n)$. Therefore :

$$u(1) \geq u(1)Q(A_n) \geq u(1)\min_{P\in C} P(A_n) = \min_{P\in C} \mathbb{E}_P[u(1)1_{A_n}] \uparrow_n \min_{P\in C} \mathbb{E}_P[u(1)] = u(1).$$

Dividing everything by $u(1)$ we can see that $Q(A_n) \uparrow_n 1$.

(iii) $\Rightarrow$ *(ii)* To simplify notation, let us write $\forall n \in \mathbb{N}$, $\mathbb{E}_{P_n}[u(\mathbf{x}^n)] := \min_{P\in C} \mathbb{E}_P[u(\mathbf{x}^n)]$ and $\mathbb{E}_{P^*}[u(\mathbf{x})] := \min_{P\in C} \mathbb{E}_P[u(\mathbf{x})]$ (i.e. for every $n \in \mathbb{N}$ $P_n \in \arg\min_{P\in C} \mathbb{E}_P[u(\mathbf{x}^n)]$ and $P^* \in \arg\min_{P\in C} \mathbb{E}_P[u(\mathbf{x})]$). In words, P_n is a probability in the convex and compact set C that minimizes the expectation of the sequence $u(\mathbf{x}^n)$ and P^* is a probability in C that minimizes the expectation of the sequence $u(\mathbf{x})$.

Claim 1. $\mathbb{E}_{P_n}[u(\mathbf{x}^n)] \uparrow_n a \in \mathbb{R}$.

Proof of Claim 1. We can notice that

$$\mathbb{E}_{P_{n+1}}[u(\mathbf{x}^{n+1})] \geq \mathbb{E}_{P_{n+1}}[u(\mathbf{x}^n)] \geq \mathbb{E}_{P_n}[u(\mathbf{x}^n)],$$

where the first inequality comes from the fact that $\mathbf{x}^n \uparrow_n \mathbf{x}$ and the second from the fact that P_n is the probability that minimize the expectation of $u(\mathbf{x}^n)$.
Moreover

$$\forall P \in C \;\; \mathbb{E}_{P_n}[u(\mathbf{x}^n)] \leq \mathbb{E}_P[u(\mathbf{x}^n)] \leq \mathbb{E}_P[u(\mathbf{x})].$$

Therefore $\mathbb{E}_{P_n}[u(\mathbf{x}^n)] \leq \mathbb{E}_{P^*}[u(\mathbf{x})] < u(M_{\mathbf{x}}) < +\infty$, where $M_{\mathbf{x}}$ is an upper bound of $\mathbf{x}$. Hence $\mathbb{E}_{P_n}[u(\mathbf{x}^n)]$ is increasing and bounded and therefore it converges to $a \in \mathbb{R}$. □

Let us consider the sequence of probabilities $(P_n)_{n\in\mathbb{N}}$. Since $\forall n \in \mathbb{N}$, $P_n \in C$ and C is a compact set, then $(P_n)_{n\in\mathbb{N}}$ has a cluster point $\bar{P} \in C$. Recall that C is compact for the weak*-topology, where the convergence of a sequence is defined as $P_n \xrightarrow{w*} P$ if and only if $\forall \mathbf{y} \in V \; \mathbb{E}_{P_n}[u(\mathbf{y})] \rightarrow_n \mathbb{E}_P[u(\mathbf{y})]$.

Claim 2. $\mathbb{E}_{P_n}[u(\mathbf{x}^n)] \uparrow_n \mathbb{E}_{\bar{P}}[u(\mathbf{x})]$.

Proof of Claim 2. Suppose by contradiction $\mathbb{E}_{P_n}[u(\mathbf{x}^n)] \uparrow_n a < \mathbb{E}_{\bar{P}}[u(\mathbf{x})]$. Since $\bar{P} \in C$ is σ-additive by hypothesis, we are allowed to use the Monotone Convergence Theorem and hence $\mathbb{E}_{\bar{P}}[u(\mathbf{x}^n)] \uparrow_n \mathbb{E}_{\bar{P}}[u(\mathbf{x})]$. Therefore $\exists N$ s.t. $\forall n \geq N$, $\mathbb{E}_{\bar{P}}[u(\mathbf{x}^n)] > a$. If we define $\epsilon := \mathbb{E}_{\bar{P}}[u(\mathbf{x}^N)] - a > 0$, since $\bar{P}$ is a cluster point for $(P_n)_{n\in\mathbb{N}}$, $\exists m > N$ s.t.

$|\mathbb{E}_{P_m}[u(\mathbf{x}^N)] - \mathbb{E}_{\bar{P}}[u(\mathbf{x}^N)]| < \frac{\epsilon}{2}$. This implies that $\mathbb{E}_{P_m}[u(\mathbf{x}^N)] > \mathbb{E}_{\bar{P}}[u(\mathbf{x}^N)] - \frac{\epsilon}{2}$. Now, since $m > N$ we have that $u(\mathbf{x}^m) \geq u(\mathbf{x}^N)$ and therefore :

$$\mathbb{E}_{P_m}[u(\mathbf{x}^m)] \geq \mathbb{E}_{P_m}[u(\mathbf{x}^N)] > \mathbb{E}_{\bar{P}}[u(\mathbf{x}^N)] - \frac{\epsilon}{2} = \frac{\mathbb{E}_{\bar{P}}[u(\mathbf{x}^N)] + a}{2} > \frac{a+a}{2} = a.$$

This means that $\exists m \in \mathbb{N}$ s.t. $\mathbb{E}_{P_m}[u(\mathbf{x}^m)] > a$. Since we supposed $\mathbb{E}_{P_n}[u(\mathbf{x}^n)] \uparrow_n a$, we obtain a contradiction. □

Notice that $\mathbb{E}_{\bar{P}}[u(\mathbf{x})] \geq \mathbb{E}_{P^*}[u(\mathbf{x})]$ since P^* is the probability that minimizes the expectation of $u(\mathbf{x})$. Moreover, as we saw in the proof of Claim 1, $\forall n \in \mathbb{N}$ $\mathbb{E}_{P_n}[u(\mathbf{x}^n)] \leq \mathbb{E}_{P^*}[u(\mathbf{x})]$, and hence if $\mathbb{E}_{\bar{P}}[u(\mathbf{x})]$ is the limit, necessarily $\mathbb{E}_{\bar{P}}[u(\mathbf{x})] \leq \mathbb{E}_{P^*}[u(\mathbf{x})]$, and the result follows.

□

Proof of Corollary 3.4.3. The proof follows easily either from Proposition 3.4.7 or Proposition 3.4.8. □

Proof of Proposition 3.4.9. $(i) \Rightarrow (ii)$ Let us consider a strictly decreasing discount function β and define $\mathbf{z} := \mathbf{x} - \mathbf{y}$. For every $n \in \mathbb{N}$ we can use the decomposition :

$$\sum_{i=0}^{n} \beta(i) z_i = \beta(n) \sum_{i=0}^{n} z_i + \sum_{j=0}^{n-1} (\beta(j) - \beta(j+1)) \sum_{i=0}^{j} z_i.$$

Taking the limit for $n \to \infty$ we have that, since the limit on the left side of the equation exists, also the one on the right side of the equation exists. Moreover, using the hypothesis, we know that there exists at least one $\hat{k} \in \mathbb{N}$ s.t. $\sum_{i=0}^{\hat{k}} z_i > 0$. Defining $c := (\beta(\hat{k}) - \beta(\hat{k}+1)) \sum_{i=0}^{\hat{k}} z_i > 0$ then $\forall n > \hat{k}$, we have that $\beta(n) \sum_{i=0}^{n} z_i + \sum_{j=0}^{n-1} (\beta(j) - \beta(j+1)) \sum_{i=0}^{j} z_i \geq c > 0$ since $\beta(n) \sum_{i=0}^{n} z_i \geq 0$ $\forall n \in \mathbb{N}$ and $\sum_{j=0}^{n-1} (\beta(j) - \beta(j+1)) \sum_{i=0}^{j} z_i \geq c$ for every $n > \hat{k}$. Therefore

$$\lim_{n \to \infty} \beta(n) \sum_{i=0}^{n} z_i + \sum_{j=0}^{n-1} (\beta(j) - \beta(j+1)) \sum_{i=0}^{j} z_i \geq c > 0,$$

and hence $\sum_{i=0}^{\infty} \beta(i) z_i > 0$.

$(ii) \Rightarrow (i)$ Define $\mathbf{z} := \mathbf{x} - \mathbf{y}$ and suppose that for all β strictly decreasing, $\sum_t \beta(i) z_i > 0$, but that $\exists k$ s.t. $\sum_{i=0}^{k} z_i < 0$. Thanks to the decomposition already used we can write :

$$\sum_{i=0}^{\infty} \beta(i) z_i = \beta(k) \sum_{i=0}^{k} z_i + \sum_{j=0}^{k-1} (\beta(j) - \beta(j+1)) \sum_{i=0}^{j} z_i + \sum_{i=k+1}^{\infty} \beta(i) z_i.$$

Consider now the strictly decreasing discount function β s.t. for a sufficiently small $\delta > 0$, $\beta(0) = 1$, $\beta(1) = 1 - \delta$, $\beta(2) = 1 - 2\delta$ and so on until $\beta(k) = 1 - k\delta$; and $\beta(k+1), \beta(k+$

2), … strictly decreasing s.t. $\sum_{i=0}^{\infty}\beta(i) = k+1$. We can notice that

$$\sum_{i=0}^{k}\beta(i) = 1 + (1-\delta) + \cdots + (1-k\delta) = (k+1) - \delta\sum_{i=0}^{k} i.$$

and therefore

$$\sum_{i=k+1}^{\infty}\beta(i) = (k+1) - \sum_{i=0}^{k}\beta(i) = \delta\sum_{i=0}^{k} i.$$

Then since the sequence z_i, $i = 0, 1, 2, \dots$ is bounded (since $0 \le x_i \le M_{\mathbf{x}}\, \forall i \in \mathbb{N}$ and $0 \le y_i \le M_{\mathbf{y}}\, \forall i \in \mathbb{N}$, then $|z_i| \le \max\{M_{\mathbf{x}}, M_{\mathbf{y}}\} =: M\, \forall i \in \mathbb{N}$) we have, substituting for β defined above :

$$(1-k\delta)\sum_{i=0}^{k} z_i + \delta\sum_{j=0}^{k-1}\sum_{i=0}^{j} z_i + \sum_{i=k+1}^{\infty}\beta(i)z_i \le$$

$$\le (1-k\delta)\sum_{i=0}^{k} z_i + \delta M\sum_{j=0}^{k-1} j + M\delta\sum_{i=0}^{k} i.$$

And hence for δ sufficiently close to 0 we will have

$$0 < \sum_{i=0}^{\infty}\beta(i)z_i \le (1-k\delta)\sum_{i=0}^{k} z_i + 2M\delta\sum_{i=0}^{k} i =$$

$$= \sum_{i=0}^{k} z_i + \delta(-k\sum_{i=0}^{k} z_i + 2M\sum_{i=0}^{k} i) < 0$$

a contradiction.

Suppose now that $\sum_{i=0}^{k} z_i = 0\, \forall k \in \mathbb{N}$. We will have that for every strictly decreasing discount function and $\forall n \in \mathbb{N}$, $\beta(n)\sum_{i=0}^{n} z_i + \sum_{j=0}^{n-1}(\beta(j) - \beta(j+1))\sum_{i=0}^{j} z_i = 0$ and hence $\lim_{n\to\infty}\beta(n)\sum_{i=0}^{n} z_i + \sum_{j=0}^{n-1}(\beta(j) - \beta(j+1))\sum_{i=0}^{j} z_i = 0$, a contradiction.

□

Proof of Corollary 3.4.4. The proof is similar to the one of Proposition 3.4.9. □

Chapitre 4

A topological approach to delay aversion

Ce chapitre est issu d'un travail en cours.

Abstract. A decision maker is to choose between two different amounts of money, with the smaller one available at an earlier period. Then she is *delay averse* if she chooses the smaller and earlier extra amount whenever the bigger one is delivered sufficiently far in the future. In this paper we study new topologies on l^∞ which "discount" the future consistently with the notion of delay aversion. We compare these topologies with other topologies that have the property of representing impatient, or patient, preferences. Our results bear relevance on the theory of infinite-dimensional general equilibrium and with the works that consider bubbles as the pathological (not countably additive) part of a charge. Finally we show that the definition of delay aversion is consistent with the notion of more delay aversion in Benoît and Ok [2007].

4.1 Introduction

One of the standard assumptions made in most economic models is that agents have preferences for advancing the time of future satisfaction. This behaviour is commonly known with the term of impatience.

This paper studies the preferences of a Decision Maker (DM) over infinite flows of income. An alternative interpretation is to think not of an agent but of different generations living in different ages. In this case we will talk about a Social Planner who has preferences over infinite streams of wealth (each period represents the wealth of a generation). In both situations, the natural framework to model this infinite horizon problem is the study of preferences over the set l^∞ of bounded, real-valued sequences.

The classical way of describing impatient preferences is to use a discounted sum of utilities. If the DM is facing a monetary flow $(x_0, x_1, \dots)$ then she evaluates it through the functional :

$$U(x_0, x_1, \dots) = \sum_{t=0}^{\infty} \delta(t) u(x_t).$$

The function $u : \mathbb{R} \to \mathbb{R}$ is an instantaneous utility function that represents the utility derived by using a certain amount of income. The function $\delta : \mathbb{N} \to (0, 1]$, is called discount function and it represents the willingness of the DM to anticipate future consumption. Often, the function $\delta(t) = \delta^t$ (where δ is a constant in the interval $(0, 1)$) and the model is called the exponential discounted utility model. The exponential discounted utility model was first proposed back in the thirties in a seminal paper of Samuelson [1937]. A further boost to its popularity was given by Koopmans [1960] who showed that the model could be derived from a set of plausible axioms. Since then, it has become the standard treatment of impatient behaviours in the economic field.

Departures from the classical discounted utility model are present in the literature of mathematical economics. These deviations consider functional different from the discounted sum of utilities presented above. For instance, Chateauneuf and Ventura [2013] use the Choquet integral in order to analyze different kind of impatient behavior. Marinacci [1998] characterizes complete patience through the MaxMin model and Rébillé [2007] considers patience in the Choquet model. Finally in Bastianello and Chateauneuf [2016] a concept called long-term delay aversion was introduced and analyzed in both the Choquet and MaxMin models.

This paper does not follow any of the approaches aforementioned but we consider a topological approach. We do not specify any utility function for the DM, and rather focus on the continuity of her preferences with respect to a suitable topology. The topology considered makes the DM "discount" the future in a way consistent with the notion of long-term delay aversion proposed in Bastianello and Chateauneuf [2016]. As noted by Koopmans [1960] and Brown and Lewis [1981] the choice of the topology over the infinite dimensional space l^∞ is relevant for its behavioural implication. Continuity is not a technical requirement, it is, in fact, a behavioural assumption.

As we mentioned, in most economic papers preferences for advancing the time of future satisfaction are just taken into account through the discount factor. In this work we start from the description of delay averse preferences (we drop here the adjective long-term, used in Bastianello and Chateauneuf [2016]). Delay aversion means the following. Suppose that an agent has to choose between two extra payments of, say, 1000$ and 10000$. The 1000$ are paid on a fixed date whereas the 10000$ will be payed later. We believe that, if the second and bigger payment is done sufficiently far in the future, then she will chose the first one. More formally, let us consider a DM who is supposed to receive two additional amounts of income or consumption good, a and b, with $a \leq b$, delivered respectively in periods n_0 and n with $n_0 < n$. Then she is delay averse if she prefers a over b if n is sufficiently big.

After presenting the main definition, we consider two Hausdorff locally convex topologies that represent a future-disliking behavior consistent with delay aversion. The key idea is that a suitable topology should make a cash flow which pays one unit of income in the n-th period very close to the cash flow paying zero at all periods, provided that n is big enough. Such a property could be rephrased as "the far future is negligible".

Endowed with such topologies we proceed comparing them with the strong and weak myopic topologies introduced by Brown and Lewis [1981]. These topologies are fundamental in economics and specifically in the theory of general equilibrium in infinite dimension, see Mas-Colell and Zame [1991]. Roughly speaking, we find that the delay averse topologies are finer than the myopic topologies. This implies that it is easier to be delay averse rather than myopic and therefore, it is possible to have preference for advancing the time future satisfaction and still an equilibrium may fail to exist. Such a result clarifies the famous paper of Araujo [1985], where the author shows that impatience is a necessary condition to insure the existence of an equilibrium in an infinite dimensional setting. Our results show that DMs should be *enough* impatient to get an equilibrium.

Next, we study the property of the topological dual of l^∞ when paired with the delay averse topologies. Dual spaces play a major role in general equilibrium since the equilibrium prices are functionals belonging to the dual space. Interestingly, we find that the dual space is bigger than the one obtained with the topologies usually considered. This entails the possibility of having bubbles (in the sense of Gilles and LeRoy [1992]) even when agents show a form of impatience.
As a dividend, we obtain a new characterization of the space ba of bounded charges. This space is the dual of l^∞ when paired with a particular delay averse topology.

We conclude the paper with a justification of the use of delay aversion as a notion of impatience. In Bastianello and Chateauneuf [2016] it was argued, but not proved, that delay aversion is coherent with the concept of *more delay aversion* given by Benoît and Ok [2007]. In the last section of the paper we prove formally how, from delay aversion, we can recover the main theorem in Benoît and Ok [2007].

The paper is organized as follows. Section 2 presents some preliminary notions. Section 3 and Section 4 present and analyze the delay averse topology and the delay averse topology with a monotone base respectively. Section 5 links delay aversion with more delay aversion.

4.2 Preliminary notions

We study the preferences of a DM over the space l^∞ of real-valued bounded sequences. The generic elements of l^∞ are denoted as $\mathbf{x}, \mathbf{y}$, etc. and are considered as infinite streams of income. The p-th element of sequence $\mathbf{x}$ is denoted equivalently x_p or $\mathbf{x}(p)$. Clearly, the set $\mathbb{N}$ of natural numbers represents time.

Given a sequence $\mathbf{x} = (x_0, x_1, \dots)$, $(x_k + a, \mathbf{x}_{-k})$ denotes the sequence $\mathbf{y}$ s.t. $y_k = x_k + a$ and $y_n = x_n$ for all $n \neq k$. Sum between two sequences and the multiplication by a scalar correspond to the pointwise sum and multiplication, meaning that if $\mathbf{x}, \mathbf{y} \in l^\infty$ and $\lambda \in \mathbb{R}$ then $\mathbf{x}+\mathbf{y} = (x_0+y_0, x_1+y_1, \dots)$ and $\lambda\mathbf{x} = (\lambda x_0, \lambda x_1, \dots)$. We also write $(x_k+a, \mathbf{x}_{-k})$ as $\mathbf{x}+a1_k$, where 1_A is the indicator function of the set $A \subseteq \mathbb{N}$, i.e. $1_A(n) := \begin{cases} 1 & \text{if } n \in A \\ 0 & \text{if } n \in A^c. \end{cases}$
Therefore 1_A denotes the sequence with $1_A(p) = 1$ if $p \in A$ and $1_A(p) = 0$ if $p \notin A$ and 1_k the sequence with all the elements equal to 0, but the element k which is equal to 1.

A vector space X is an ordered vector space with an order $\geq$ if X is partially ordered by $\geq$ and if for every $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X$ and every real number $\lambda \geq 0$, $\mathbf{x} \geq \mathbf{y}$ implies $\mathbf{x}+\mathbf{z} \geq \mathbf{y}+\mathbf{z}$ and $\mathbf{x} \geq 0$ implies $\lambda\mathbf{x} \geq 0$. The space we are considering, l^∞, comes equipped with a natural order. We write $\mathbf{x} \geq \mathbf{y}$ when $x_k \geq y_k \, \forall k$, $\mathbf{x} \gg \mathbf{y}$ when $x_k > y_k \, \forall k$ and $\mathbf{x} > \mathbf{y}$ when $x_k \geq y_k \, \forall k$ with a strict inequality for at least one k. A sequence is non-negative if $\mathbf{x} \geq 0$ and l^∞_+ denotes the positive orthant of l^∞ i.e. $l^\infty_+ := \{\mathbf{x} \in l^\infty : \mathbf{x} \geq 0\}$.

Let X be an ordered vector space. A *seminorm* on X is a function $p : X \to \mathbb{R}$ s.t. $\forall \mathbf{x}, \mathbf{y} \in X$ and $\forall \alpha \in \mathbb{R}$, (i) $p(\mathbf{x}+\mathbf{y}) \leq p(\mathbf{x}) + p(\mathbf{y})$ and (ii) $p(\alpha\mathbf{x}) = |\alpha| p(\mathbf{x})$. If moreover $p(\mathbf{x}) = 0$ if and only if $\mathbf{x} = 0$ then p is called a *norm*. A *locally convex topology* is a topology generated by a family of seminorms. We say that p is a *monotone seminorm* if $0 \leq \mathbf{y} \leq \mathbf{x} \Rightarrow p(\mathbf{y}) \leq p(\mathbf{x})$. A subfamily of seminorms $\mathbb{Q}$ is said to be a *base* for a family of seminorms $\mathcal{P}$ if for every $p \in \mathcal{P}$ there is $q \in \mathbb{Q}$ and $c > 0$ s.t. $p(\mathbf{x}) < cq(\mathbf{x})$ for every $\mathbf{x}$. In this case we say that every seminorm p in $\mathcal{P}$ is dominated by a seminorm from $\mathbb{Q}$. A topology is said to be a *locally convex topology with a monotone base* if the associated family of seminorms has a monotone base.

Regarding convergence of sequences or nets we use the following notation. If $\{a_n\}_{n \in \mathbb{N}}$

is a sequence of real numbers, $a_n \to_n l$ means that the sequence converges to the real number $l \in \mathbb{R}$. If $\{\mathbf{x}_\lambda\}_{\lambda\in\Lambda}$ is a net of elements of a set X endowed with a topology $\mathcal{T}$, then $\mathbf{x}_\lambda \xrightarrow{\mathcal{T}}_\lambda \mathbf{x}$ means that the net converges to the element $\mathbf{x}$ in the topology $\mathcal{T}$. If $\mathcal{T}$ is a locally convex topology generated by a family of seminorms $\{p_\alpha, \alpha \in A\}$ then a net $\mathbf{x}_\lambda \xrightarrow{\mathcal{T}}_\lambda \mathbf{x}$ if and only if $p_\alpha(\mathbf{x}_\lambda - \mathbf{x}) \to_\lambda 0$ for every $\alpha \in A$ (see Aliprantis and Border [2006], Lemma 5.76). Sometimes we may write only $\mathbf{x}_n \to \mathbf{x}$ for convergence of sequences (or $\mathbf{x}_\lambda \to \mathbf{x}$ for convergence of nets) when no confusion can arise about the index and the topology that we are considering.

The symbol $\mathcal{T}_\infty$ designates the sup-norm topology on l^∞, that is the topology generated by the supremum norm $\|\mathbf{x}\|_\infty = \sup_k |x_k|$. When l^∞ is endowed with a particular topology $\mathcal{T}$, its *(topological) dual* w.r.t. $\mathcal{T}$ is the set of $\mathcal{T}$-continuous linear function on l^∞ and it is denoted $(l^\infty, \mathcal{T})^*$. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on l^∞. If $\mathcal{T}_1 \subseteq \mathcal{T}_2$ we say that $\mathcal{T}_1$ is *weaker (or coarser)* than $\mathcal{T}_2$ or that $\mathcal{T}_2$ is *stronger (or finer)* that $\mathcal{T}_1$. If additionally $\mathcal{T}_1 \neq \mathcal{T}_2$, we write $\mathcal{T}_1 \subset \mathcal{T}_2$ and we say that $\mathcal{T}_1$ is strictly weaker (or strictly coarser) than $\mathcal{T}_2$ or that $\mathcal{T}_2$ is strictly stronger (or strictly finer) that $\mathcal{T}_1$.

A preference relation $\succsim$ over l^∞ is a complete and transitive binary relation, i.e. a weak order. Given a preference relation $\succsim$ we denote its symmetric and asymmetric parts by $\sim$ and $\succ$ respectively. We say that a preference relation over l^∞ is *monotone* if $\mathbf{x} \geq \mathbf{y}$ implies $\mathbf{x} \succsim \mathbf{y}$ and *strongly monotone* if $\mathbf{x} > \mathbf{y}$ implies $\mathbf{x} \succ \mathbf{y}$. A preference relation $\succsim$ over l^∞ is *continuous w.r.t. a topology* $\mathcal{T}$ if the sets of the form $\{\mathbf{x}|\mathbf{x} \succ \mathbf{y}\}$ and $\{\mathbf{x}|\mathbf{y} \succ \mathbf{x}\}$ are $\mathcal{T}$-open for every $\mathbf{y} \in l^\infty$.

Given a set X and a field $\mathcal{F}$ of its subsets, a set function $\mu : \mathcal{F} \to \mathbb{R}$ is called a *charge* if (i) $\mu(\emptyset) = 0$ and (ii) if $A, B \in \mathcal{F}$ and $A \cap B = \emptyset$ then $\mu(A \cup B) = \mu(A) + \mu(B)$. A charge μ is said to be *bounded* if $\sup\{|\mu(F)| : F \in \mathcal{F}\} < +\infty$. If $\mu(F) \geq 0$ for every $F \in \mathcal{F}$ then μ is said to be *positive*. If, given a sequence of sets $\{A_n\}_n$ s.t. $\cup_n A_n \in \mathcal{F}$ and $A_i \cap A_j = \emptyset$ for $i \neq j$, implies that $\mu(\cup_n A_n) = \sum_n \mu(A_n)$ then μ is said to be a *countably additive charge* or, more simply, a *measure*. As it is standard, we denote by $ba(X, \mathcal{F})$ and $ca(X, \mathcal{F})$ the set of all bounded charges and all bounded measures respectively. For $X = \mathbb{N}$ and $\mathcal{F} = 2^{\mathbb{N}}$ we denote $ba = ba(\mathbb{N}, 2^{\mathbb{N}})$ and $ca = ca(\mathbb{N}, 2^{\mathbb{N}})$. A positive charge μ is called a pure charge if there is no non-zero positive measure λ such that $\lambda \leq \mu$.

We recall that the space ba is isomorphic to the dual space $(l^\infty, \mathcal{T}_\infty)^*$ via the mapping $\mathbf{x} \to \int \mathbf{x} d\mu$ for every $\mathbf{x} \in l^\infty$. The latter integral is known as the Dunford integral, see Chapter 4 of Rao and Rao [1983]. The set l^1 is the subspace of l^∞ such that $l^1 = \{\mathbf{x} \in l^\infty : \sum |x_i| < \infty\}$. It is known that l^1 can be put in a one to one correspondence with ca. The *Mackey topology*, $\mathcal{T}_{ma}$, is the strongest topology on l^∞ for which the dual is l^1.

We conclude this section with a terminological caveat. In the sequel we use the word *impatience* to denote the generic attitude of a DM who enjoys more present consumption than future consumption. Impatience specializes in different nuances. Every specific behavioural shade of this generic concept will be formally defined in numbered Definitions.

4.3 Delay Averse Topology

4.3.1 Existence

We recall the definition of delay aversion given in Bastianello and Chateauneuf [2016].

Definition 4.3.1. (Bastianello and Chateauneuf [2016]) *A preference relation* $\succsim$ *over* l^∞ *is* delay averse *if for* $0 < a \leq b$, $n_0 \in \mathbb{N}$ *and* $\mathbf{x} \in l^\infty$ $\exists N := N(\mathbf{x}, n_0, a, b) \geq n_0$ *s.t.* $\forall n \geq N$,

$$(x_{n_0} + a, \mathbf{x}_{-n_0}) \succ (x_n + b, \mathbf{x}_{-n}).$$

Definition 4.3.1 says the following. Suppose that a DM with a given endowment is to choose between two extra payments : $a > 0$ done in a fixed period n_0 and $b \geq a$ done in a period n. The DM will be delay averse if she prefers the lower payment done at the early date if the bigger one is done sufficiently far in the future (namely after the period N).

In the subsequent analysis, we will confine in the case of strongly monotone preferences. This assumption may seem strong, but in some sense strong monotonicity places us in the good framework for two reasons. First, it places us in the same setting used by Benoît and Ok [2007] (see p. 75 of their paper), whose work inspired the definition of delay aversion. Second, if we are ready to accept delay aversion and simple monotonicity of preferences, then strong monotonicity follows. This last assertion is proved in Lemma 4.3.1 below.

Lemma 4.3.1. *Let* $\succsim$ *be a delay averse preference relation. Then* $\succsim$ *is monotone iff* $\succsim$ *is strongly monotone.*

Démonstration. $\Rightarrow$ Let $\mathbf{x} > \mathbf{y}$, we need to prove $\mathbf{x} \succ \mathbf{y}$. Since $\mathbf{x} > \mathbf{y}$, there exists n_0 s.t. $x_{n_0} = y_{n_0} + \epsilon$ for some $\epsilon > 0$. Clearly $\mathbf{x} \geq \mathbf{y}$ and hence by monotonicity $\mathbf{x} \succsim \mathbf{y} + \epsilon 1_{n_0}$ and by delay aversion $\exists N$ s.t. $\forall n \geq N$, $\mathbf{y} + \epsilon 1_{n_0} \succ \mathbf{y} + \epsilon 1_n$. Again by monotonicity we get $\mathbf{y} + \epsilon 1_n \succsim \mathbf{y}$ and hence $\mathbf{x} \succ \mathbf{y}$.

$\Leftarrow$ Obvious. □

Since the aim of this paper is to study delay aversion, and since monotonicity is a natural requirement, we impose strong monotonicity from the start.

As we said in the Introduction our specific purpose is to present some topologies over l^∞ which are linked with the concept of delay aversion. We define therefore what we mean for a delay averse topology.

Definition 4.3.2. *A topology* $\mathcal{T}$ *on* l^∞ *is said to be* delay averse *if every strongly monotone,* $\mathcal{T}$*-continuous preference relation is delay averse.*

We focus locally convex topologies and we are interested in finding which properties such topologies should satisfy in order to represent delay averse preferences. The two propositions below provide the starting point. In the statements of the propositions, 1_n denotes the sequence $1_n := (\underbrace{0,\dots,0}_{n-1},1,0,\dots)$ and 0 denotes the sequence $(0,0,\dots)$.

Proposition 4.3.1. *Every locally convex topology $\mathcal{T}$ for which $1_n \xrightarrow{\mathcal{T}}_n 0$ is a delay averse topology.*

Démonstration. We need to show that every strongly monotone and $\mathcal{T}$-continuous preference relation is delay averse. Fix $\mathbf{x} \in l^\infty$, $n_0 \in \mathbb{N}$, $0 < a \leq b$ and consider a $\mathcal{T}$-continuous and strongly monotone preference relation $\succsim$.
Since $\mathcal{T}$ is a locally convex topology, it is generated by a family of seminorms of the form $\{p_\alpha | \alpha \in A\}$ (see Aliprantis and Border [2006], Theorem 5.73). The fact that $1_n \xrightarrow{\mathcal{T}}_n 0$ implies that for all $\alpha \in A$

$$0 \leq p_\alpha(\mathbf{x} + b1_n - \mathbf{x}) = p_\alpha(b1_n) = bp_\alpha(1_n) \to_n 0.$$

This shows that $\mathbf{x} + b1_n \xrightarrow{\mathcal{T}}_n \mathbf{x}$. Moreover since $\succsim$ is strongly monotone, $\mathbf{x} + a1_{n_0} \succ \mathbf{x}$. Therefore by $\mathcal{T}$-continuity of the preference relation $\succsim$, there exists N s.t. $\forall n \geq N$ $\mathbf{x} + a1_{n_0} \succ \mathbf{x} + b1_n$, i.e. the $\succsim$ is delay averse. $\square$

We now prove a kind of converse implication of Proposition 4.3.1.

Proposition 4.3.2. *Given a locally convex topology $\mathcal{T}$, if every $\mathcal{T}$-continuous, preference relation is delay averse then $1_n \xrightarrow{\mathcal{T}}_n 0$.*

Démonstration. We first need this auxiliary Lemma. A proof is given for sake of completeness.

Lemma 4.3.2. *Let $\mathcal{T}$ be generated by a family of seminorms $\{p_i | i \in I\}$. Define a preference relation $\succsim_i$ as $\mathbf{x} \succsim_i \mathbf{y}$ iff $p_i(\mathbf{x}) \geq p_i(\mathbf{y})$. Then $\succsim_i$ is a $\mathcal{T}$-continuous preference relation for each $i \in I$.*

Démonstration. To see that $\succsim_i$ is a preference relation, we need to show that it is complete and transitive. For completeness notice that $\forall \mathbf{x}, \mathbf{y} \in l^\infty$ $p_i(\mathbf{x}), p_i(\mathbf{y}) \in \mathbb{R}$ and therefore either $p_i(\mathbf{x}) \geq p_i(\mathbf{y})$ or $p_i(\mathbf{y}) \leq p_i(\mathbf{x})$, i.e. either $\mathbf{x} \succsim_i \mathbf{y}$ or $\mathbf{y} \succsim_i \mathbf{x}$, hence $\succsim_i$ is complete. For transitiveness, take $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ s.t. $\mathbf{x} \succsim_i \mathbf{y}$ and $\mathbf{y} \succsim_i \mathbf{z}$. Then $p_i(\mathbf{x}) \geq p_i(\mathbf{y})$ and $p_i(\mathbf{y}) \geq p_i(\mathbf{z})$. By transitivity of $\geq$ for the real numbers, $p_i(\mathbf{x}) \geq p_i(\mathbf{z})$ and hence $\mathbf{x} \succsim_i \mathbf{z}$. We conclude that $\succsim_i$ is a preference relation.
Regarding $\mathcal{T}$-continuity we need to show that the set $O_\mathbf{y} := \{\mathbf{x} | \mathbf{x} \succ_i \mathbf{y}\} = \{\mathbf{x} | p_i(\mathbf{x}) > p_i(\mathbf{y})\}$ is open for each $\mathbf{y} \in l^\infty$. Take $\overline{\mathbf{x}} \in O_\mathbf{y}$ then $p_i(\overline{\mathbf{x}}) > p_i(\mathbf{y})$. Choose $\epsilon > 0$ s.t. $\epsilon < p_i(\overline{\mathbf{x}}) - p_i(\mathbf{y})$, we will show that the open ball $B_{\overline{\mathbf{x}},\epsilon} = \{\mathbf{z} | p_i(\overline{\mathbf{x}} - \mathbf{z}) < \epsilon\} \subset O_\mathbf{y}$. Notice

that by the triangular inequality we have that $|p_i(\overline{\mathbf{x}})-p_i(\mathbf{z})| \leq p_i(\overline{\mathbf{x}}-\mathbf{z})$ and so if $\mathbf{z} \in B_{\overline{\mathbf{x}},\epsilon}$ then $|p_i(\overline{\mathbf{x}}) - p_i(\mathbf{z})| < \epsilon$. If $p_i(\mathbf{z}) \geq p_i(\overline{\mathbf{x}})$ then $p_i(\mathbf{z}) \geq p_i(\overline{\mathbf{x}}) > p_i(\mathbf{y})$ so $\mathbf{z} \in O_{\mathbf{y}}$. Else if $p_i(\mathbf{z}) < p_i(\overline{\mathbf{x}})$ then $p_i(\overline{\mathbf{x}}) - p_i(\mathbf{z}) < \epsilon < p_i(\overline{\mathbf{x}}) - p_i(\mathbf{y})$ and hence $p_i(\mathbf{z}) > p_i(\mathbf{y})$ i.e. $\mathbf{z} \in O_{\mathbf{y}}$. A similar argument shows that the set $O^{\mathbf{y}} := \{\mathbf{x}|\mathbf{y} \succ_i \mathbf{x}\}$ is open. □

Suppose now that $\mathcal{T}$ is generated by a family of seminorms $\{p_\alpha|\alpha \in A\}$ and that every $\mathcal{T}$-continuous preference relation $\succsim$ over l^∞ is delay averse. First notice that $\forall\alpha \in A$ and $\forall n \in \mathbb{N}$, $p_\alpha(1_n) \neq 0$. In fact, suppose that this is not true and that there exist $\bar{\alpha} \in A$ and $n_0 \in \mathbb{N}$ such that $p_{\bar{\alpha}}(1_{n_0}) = 0$. Then choosing $\mathbf{x} = 0$ ($\mathbf{x}$ is the constant sequence equal to 0), and $0 < a \leq b$ we get $p_{\bar{\alpha}}(\mathbf{x} + a1_{n_0}) = 0 \leq p_{\bar{\alpha}}(\mathbf{x} + b1_n)$ $\forall n \in \mathbb{N}$. Now, defining $\succsim_{\bar{\alpha}}$ as in Lemma 4.3.2, $\succsim_{\bar{\alpha}}$ is $\mathcal{T}$-continuous but not delay averse since $\mathbf{x} + a1_{n_0} \succsim_{\bar{\alpha}} \mathbf{x} + b1_n$ for all $n \in \mathbb{N}$, a contradiction.
Fix now $\alpha \in A$ and $n_0 \in \mathbb{N}$ and consider $\succsim_\alpha$ related to p_α. By Lemma 4.3.2 $\succsim_\alpha$ is $\mathcal{T}$-continuous and since by hypothesis every $\mathcal{T}$-continuous preference relation is delay averse, for $0 < \epsilon < 1$, we have that $\exists N$ s.t. $\forall n \geq N$ $\epsilon 1_{n_0} \succ_\alpha 1_n$. Hence, $\epsilon p_\alpha(1_{n_0}) > p_\alpha(1_n)$, and since ϵ can be arbitrarily small, $p_\alpha(1_n) \to_n 0$. Since this is true for every $\alpha \in A$, $1_n \xrightarrow{\mathcal{T}}_n 0$. □

We were not able to completely describe all the locally convex delay averse topology. This is due to the fact that for proving Proposition 4.3.1 strong monotonicity is needed, whereas for Proposition 4.3.2, the delay averse preferences constructed using seminorms may not be strongly monotone. Anyway, even if the two propositions above fall short of a complete characterization of locally convex, delay averse topology, they underline that a salient feature is the converge of $1_n \to 0$. Taking into account this fact, we define the topology $\mathcal{T}_{DA}$, which will play a central role in the subsequent analysis.

Definition 4.3.3. *We denote $\mathcal{T}_{DA}$ the finest Hausdorff locally convex topology on l^∞ for which we have $1_n \xrightarrow{\mathcal{T}_{DA}}_n 0$.*

The idea underlying $\mathcal{T}_{DA}$ is the following. Suppose that one is endowed with a unit of income or consumption good at a certain date n. Continuity of a preference relation with respect to $\mathcal{T}_{DA}$ says that postponing this unit of income or consumption in the future eventually will make the value of it arbitrarily close to the sequence $(0, 0, \dots)$. Notice that this property is not enjoyed by the sup-norm topology : the "weight" of one unit of endowment is the same whatever is the date at which the DM receives it. It should be said that there are locally convex topologies insuring the convergence of $1_n \to 0$, e.g. the product or the Mackey topology. However, we will see in the following sections that these topologies are different from $\mathcal{T}_{DA}$. The fact that we impose $\mathcal{T}_{DA}$ to be the finest topology enjoying this property is a crucial requirement. We think that it is also a natural one since we want to impose as few restrictions as possible on agents' preferences. Remember that the finer is a topology, the more there are open sets and the "easier" is for a preference relation to be continuous (for, continuity of a preference relation with respect to some topology is defined in term of openness of the upper and lower contour sets).

We can easily prove the following corollary.

Corollary 4.3.1. *$\mathcal{T}_{DA}$ is a delay averse topology*

Démonstration. It follows from the definition of $\mathcal{T}_{DA}$ and from Proposition 4.3.1. □

Corollary 4.3.1 says that every strongly monotone preference relation which is continuous with respect to $\mathcal{T}_{DA}$ is delay averse. We would like to underline that, Corollary 1 is silent about preferences that are not strongly monotone. In other words, it is possible to have non-strongly monotone preferences continuous with respect to $\mathcal{T}_{DA}$, which are *not* delay averse.

Finally, we show that the topology $\mathcal{T}_{DA}$ exists.

Proposition 4.3.3. *There exists a finest Hausdorff locally convex topology, $\mathcal{T}_{DA}$, over l^∞ such that $1_n \xrightarrow{\mathcal{T}_{DA}}_n 0$.*

Démonstration. Let $\mathcal{P} = \{p_\alpha | \alpha \in A\}$ be the family of seminorms over l^∞ s.t. $\forall \alpha \in A$, $p_\alpha(1_n) \to_n 0$. Below it is shown that $\mathcal{P}$ is non-empty.
By definition, a locally convex topology is a topology generated by a family of seminorms. Moreover this topology is Hausdorff if and only if $p_\alpha(\mathbf{x}) = 0 \, \forall \alpha \in A \Rightarrow \mathbf{x} = \mathbf{0}$ (see Aliprantis and Border [2006], Lemma 5.76). We will show that $\{p_\alpha | \alpha \in A\}$ separates points and that it generates therefore an Hausdorff locally convex topology.
Consider the family of seminorms $\{q_k | k \in \mathbb{N}\}$ defined as $q_k(\mathbf{x}) = |x_k|$. (Notice that actually q_k is a seminorm in fact : $q_k(\mathbf{x}) = |x_k| \geq 0$, $q_k(\lambda \mathbf{x}) = |\lambda x_k| = |\lambda||x_k|$ and $q_k(\mathbf{x} + \mathbf{y}) = |x_k + y_k| \leq |x_k| + |y_k| = q_k(\mathbf{x}) + q_k(\mathbf{y})$).
We will show now that $\{q_k | k \in \mathbb{N}\} \subset \{p_\alpha | \alpha \in A\}$. Consider the sequence 1_n. We have that $\forall k \in \mathbb{N}$,

$$q_k(1_n) = \begin{cases} 0 & \text{if } n \neq k \\ 1 & \text{if } n = k \end{cases}$$

So $\forall n > k$, $q_k(1_n) = 0$ and hence $q_k(1_n - 0) = q_k(1_n) \to_n 0 \; \forall k \in \mathbb{N}$.
Suppose now that $p_\alpha(\mathbf{x}) = 0 \, \forall \alpha \in A$. We will have also that $q_k(\mathbf{x}) = 0 \; \forall k \in \mathbb{N}$ since $\{q_k | k \in \mathbb{N}\} \subset \{p_\alpha | \alpha \in A\}$. But this means that $|x_k| = 0 \, \forall k \in \mathbb{N}$, i.e. $\mathbf{x} = 0$. Hence $\{p_\alpha | \alpha \in A\}$ separates points.
Therefore $\{p_\alpha | \alpha \in A\}$ generates an Hausdorff locally convex topology on l^∞. It is the finest because of the definition of $\mathcal{P}$. □

4.3.2 Dual space of $(l^\infty, \mathcal{T}_{DA})$

The space of continuous linear functions is important in economics and especially for the theory of general equilibrium since price vectors are elements of the dual space. See

Mas-Colell and Zame [1991] for a detailed exposition on the subject.

We are able to describe all the continuous linear functional in the topological dual space $(l^\infty, \mathcal{T}_{DA})^*$. It is done in the next proposition.

Proposition 4.3.4. *A linear functional L is $\mathcal{T}_{DA}$-continuous iff $L(1_n) \to_n 0$.*

Démonstration. $\Rightarrow$ $\mathcal{P} = \{p_\alpha | \alpha \in A\}$ generates $\mathcal{T}_{DA}$. By Conway [2013], Theorem 3.1(f), p. 108, L is $\mathcal{T}_{DA}$-continuous iff there are $p_1, \ldots p_l \in \mathcal{P}$ and positive scalars $\alpha_1 \ldots, \alpha_l$ s.t. $|L(\mathbf{x})| \leq \sum_{k=1}^{l} \alpha_k p_k(\mathbf{x})$ $\forall \mathbf{x} \in l^\infty$. Hence $|L(1_n)| \leq \alpha_1 p_1(1_n) + \ldots \alpha_l p_l(1_n)$ and since $p_i(1_n) \to_n 0$ for $i = 1 \ldots l$ and l is a finite number then $0 \leq |L(1_n)| \to_n 0$ and the result follows.

$\Leftarrow$ Consider a linear functional L s.t. $L(1_n) \to_n 0$ and consider the seminorm $p(\mathbf{x}) = |L(\mathbf{x})|$. It follows that p is a continuous seminorm of $\mathcal{T}_{DA}$. Hence by Conway [2013], Theorem 3.1(e) p. 108, L is continuous. □

The characterization of the $\mathcal{T}_{DA}$-continuous functional given in Proposition 4.3.4 allows us to study the usual spaces considered in the literature of general equilibrium in infinite dimensional spaces.

Recall that a linear functional L on l^∞ is a *purely finitely additive integral* if for every $\mathbf{x} \in l^\infty$ such that $\mathbf{x}$ has at most finitely many values different from zero, $L(\mathbf{x}) = 0$.

Corollary 4.3.2. *Every purely finitely additive integral is $\mathcal{T}_{DA}$-continuous.*

Démonstration. Since in the sequence 1_n just the n-th value is different from zero, if L is a purely finitely additive integral then $L(1_n) = 0$. Therefore $L(1_n) \to_n 0$, and the assertion follows from Proposition 4.3.4. □

Corollary 4.3.3. $l^1 \subseteq (l^\infty, \mathcal{T}_{DA})^*$

Démonstration. Fix $\mathbf{y} \in l^1$ and consider the linear function on l^∞ associated with $\mathbf{y}$: $T_y(\mathbf{x}) = \sum y_i x_i$. We need to show that $T_{\mathbf{y}}(\cdot)$ is $\mathcal{T}_{DA}$ continuous. To do this it is enough to prove that $p(\mathbf{x}) = |T_{\mathbf{y}}(\mathbf{x})|$ is a continuous seminorm of $\mathcal{T}_{DA}$. This is the case indeed

$$|T_{\mathbf{y}}(1_n)| = |\sum_i y_i 1_n(i)| = |y_n|$$

and since $\mathbf{y} \in l^1$, $|y_n| \to_n 0$ and the result follows from Proposition 4.3.4. □

Recall that the dual of l^∞, when equipped with the sup-norm, is isomorphic to the space *ba* of bounded charge. When l^∞ is paired with the topology $\mathcal{T}_{DA}$ the dual space is actually larger.

Proposition 4.3.5. $ba \subseteq (l^\infty, \mathcal{T}_{DA})^*$

Démonstration. Consider $\mu \in ba$. Define the functional L on l^∞ by $L(\mathbf{x}) = \int \mathbf{x} d\mu$. By Proposition 4.3.4, L is $\mathcal{T}_{DA}$-continuous iff $L(1_n) \to_n 0$. Notice that $L(1_n) = \int 1_n d\mu = \mu(n)$. Since $\mu \in ba$, by the Yoshida-Hewitt Theorem (see Rao and Rao [1983], Theorem 10.2.1) we can uniquely decompose $\mu = \mu_c + \mu_p$ with μ_c countably additive and μ_p pure. Then $\mu(n) = \mu_c(n) + \mu_p(n)$ but $\mu_p(n) = 0$ by Aliprantis and Border [2006], Lemma 16.29, and hence $\mu(n) = \mu_c(n)$. But then $L(1_n) = \mu(n) = \mu_c(n) \to_n 0$, since μ_c is bounded and countably additive. Therefore L is $\mathcal{T}_{DA}$-continuous. □

The next question is whether ba is a strict subset of $(l^\infty, \mathcal{T}_{DA})^*$. We see in the proof of Proposition 4.3.7 in the next section that the answer is actually yes.

4.3.3 Comparison with others topologies on l^∞

As we said in the introduction, imposing continuity on preferences with respect to a particular topology has behavioral implications. Several topologies share the properties of providing preferences for advancing the time of future satisfaction, for instance the aforementioned product and Mackey topologies. In this section we are going to consider a topology introduced by Brown and Lewis [1981] which imposes myopic tastes. The formal definitions are given below.

Definition 4.3.4. (Brown and Lewis [1981]) $\succsim$ *is* weakly myopic *if* $\forall \mathbf{x}, \mathbf{y} \in l^\infty$ *such that* $\mathbf{x} \succ \mathbf{y}$ *and* $\forall A > 0$, $\exists n_1(\mathbf{x}, \mathbf{y}, A) := n_1 \in \mathbb{N}$ *such that* $n \geq n_1 \Rightarrow \mathbf{x} \succ \mathbf{y} + A1_{[n,+\infty)}$

Definition 4.3.4 says the following. Let $\mathbf{x}$ and $\mathbf{y}$ be two streams of income such that $\mathbf{x}$ is strictly preferred to $\mathbf{y}$. Suppose now that $\mathbf{y}$ is increased by an extra payment of a fixed amount $A > 0$ which will be paid for all the period after a certain date n_1 (notice that the extra payment A can be as big as one wishes). A weakly myopic DM will still prefer the stream $\mathbf{x}$ to the stream $\mathbf{y}$ improved by the amount A from period n_1 onwards, provided that n_1 is sufficiently far in the future.

Endowed with the notion of a weakly myopic preference relation, we can proceed with the definition of weakly myopic topology as given in Brown and Lewis [1981].

Definition 4.3.5. *The topology $\mathcal{T}_{WM}$ on l^∞ is the finest topology such that every $\mathcal{T}_{WM}$-continuous (not necessarily monotone) preference relation is weakly myopic.*

The proof that such a topology exists and the study of its basic properties can be found in Brown and Lewis [1981]. Both weak myopia and delay aversion describe preferences for advancing the time of future satisfaction. Intuitively, we feel that the former definition implies a stronger form of impatient tastes when compared to the latter. Consider the example of a social planner with preferences over all the possible distribution of wealth of different generations (identified with l^∞). Then weak myopia implies that such a social planner has a strong taste for inequality among generations since improving the wealth of *all* but finitely many generations does not reverse her preferences. We propose delay aversion as a weaker notion of impatience.

In Proposition 4.3.6 below we write down formally the intuition that delay aversion is less demanding (in terms of continuity of preferences) than weak myopia and therefore a DM will be more prone to be delay averse. Roughly speaking, Proposition 4.3.6 says that all the open sets of the weakly myopic topology $\mathcal{T}_{WM}$ are also open set for the delay averse topology $\mathcal{T}_{DA}$ and therefore it is "easier" for a DM to be delay averse rather than weakly myopic.

Proposition 4.3.6. $\mathcal{T}_{WM} \subset \mathcal{T}_{DA}$.

Démonstration. By Brown and Lewis [1981], Lemma 1, an Hausdorff locally convex topology $\mathcal{T}$ is weakly myopic iff $1^{(n)} := (\underbrace{0,\ldots,0}_{n-1}, 1, 1, \ldots) \xrightarrow{\mathcal{T}}_n 0$ ($1^{(n)}$ is the sequence that has 0 in the first $n-1$ terms and then 1 from the n-th term on). Let $\{q_\beta | \beta \in B\}$ be the family of all seminorms s.t. $\forall \beta \in B\ q_\beta(1^{(n)}) \to 0$. Then by Brown and Lewis [1981], Theorem 1, $\{q_\beta | \beta \in B\}$ generates the finest Hausdorff locally convex topology $\mathcal{T}_{WM}$ which is weakly myopic. Consider now the family of all seminorms $\{p_\alpha | \alpha \in A\}$ s.t. $\forall \alpha \in A$, $p_\alpha(1_n) \to_n 0$. We proved in Proposition 4.3.3 that it generates the finest Hausdorff locally convex topology $\mathcal{T}_{DA}$ which is delay averse.
Since $1_n = 1^{(n)} - 1^{(n+1)}$, by the triangular inequality we get $\forall \beta \in B$:

$$0 \leq q_\beta(1_n) = q_\beta(1^{(n)} - 1^{(n+1)}) \leq q_\beta(1^{(n)}) + q_\beta(1^{(n+1)}) \to_n 0,$$

and therefore $q_\beta \in \{p_\alpha | \alpha \in A\}$. So the family of seminorms that generates $\mathcal{T}_{WM}$ is included in the family of seminorms that generates $\mathcal{T}_{DA}$ and therefore $\mathcal{T}_{WM} \subseteq \mathcal{T}_{DA}$, i.e. $\mathcal{T}_{DA}$ is finer than $\mathcal{T}_{WM}$. Also consider the seminorm $\bar{p}$ defined as $\bar{p}(\mathbf{x}) = \limsup_i |x_i|$. The seminorm $\bar{p}$ belongs to $\{p_\alpha | \alpha \in A\}$, but not to the family generating $\mathcal{T}_{WM}$. For, one can easily see that $\bar{p}(1_n) = \limsup_i |1_n(i)| = 0$ and $\bar{p}(1^{(n)}) = \limsup_i |1^{(n)}(i)| = 1$. Therefore $\bar{p}(1^{(n)}) \not\to_n 0$ and hence $\mathcal{T}_{WM} \subset \mathcal{T}_{DA}$, i.e. $\mathcal{T}_{DA}$ is strictly finer than $\mathcal{T}_{WM}$. □

The set l^∞ of real valued, bounded sequences comes equipped with a natural topology, $\mathcal{T}_\infty$, the topology generated by the sup-norm. It is natural therefore to confront the topologies $\mathcal{T}_{DA}$ and $\mathcal{T}_\infty$. From an economic point of view, such a comparison is interesting because the topology $\mathcal{T}_\infty$ has the property of not "discounting" the value of one unit of consumption.

Proposition 4.3.7. $\mathcal{T}_{DA} \cap \mathcal{T}_\infty \subset \mathcal{T}_{DA}$.

Démonstration. It is clear that $\mathcal{T}_{DA} \cap \mathcal{T}_\infty \subseteq \mathcal{T}_{DA}$. Let us show now that $\mathcal{T}_{DA} \cap \mathcal{T}_\infty \neq \mathcal{T}_{DA}$. We follow the proof of Theorem 4 of Brown and Lewis [1981]. The set $\{1_k\}_{k=1}^\infty$ is a linearly independent subset of l^∞ and therefore can be extended to a Hamel basis B using Zorn's lemma (see Aliprantis and Border [2006], Theorem 1.8). Hence every $\mathbf{x} \in l^\infty$ can be uniquely written as

$$\mathbf{x} = \sum_{i \in F} \alpha_i b_i$$

whit F finite, $\alpha_i \in \mathbb{R}$ and $b_i \in B$ for every $i \in F$. Choosing images of basis vector uniquely determines a linear function, i.e., if B is a basis of V then for any vector space W and any map $g : B \to \mathbb{R}$ there exists exactly one linear map $f : V \to \mathbb{R}$ such that $f|_B = g$. Define the function $g(1_n) = 0$ for every n. There are at least a countable infinity of vectors in the sets $B \setminus \{1_k\}_{k=1}^\infty$. Take the countable set with a fixed enumeration $\{b_k\}_{k=1}^\infty$ in $B \setminus \{1_k\}_{k=1}^\infty$. Define $g(b_n) = n$. There is a linear functional f such that $f|_B = g$. By Proposition 4.3.4 this function is $\mathcal{T}_{DA}$-continuous since $f(1_n) = g(1_n) \to_n 0$. But it is not bounded with respect to $\mathcal{T}_\infty$ and hence not $\mathcal{T}_\infty$-continuous. Hence $\mathcal{T}_{DA} \cap \mathcal{T}_\infty \subset \mathcal{T}_{DA}$. □

We are going to see in the next section that under some additional assumptions of monotonicity of the seminorms generating $\mathcal{T}_{DA}$, it is possible to define a topology $\mathcal{T}$, different from $\mathcal{T}_{DA}$, such that $\mathcal{T} \cap \mathcal{T}_\infty = \mathcal{T}$.

4.4 Delay averse topology with a monotone base

4.4.1 Existence of a delay averse topology with a monotone base

We look now, like Raut [1986], at the finest Hausdorff locally convex topology *with a monotone base* which is delay averse. The restriction to topologies with a monotone base yields interesting results in terms of dual spaces and comparison with other topologies. We are going to provide several examples of monotone seminorms in the proofs of the propositions in this section. Clearly, we maintain the general idea of convergence that we studied in Section 4.3. Formally, the topology that we intend to study is the one defined below.

Definition 4.4.1. *We denote $\mathcal{T}_{DA}^{mon}$ the finest Hausdorff locally convex topology on l^∞ with a monotone base for which we have $1_n \xrightarrow{\mathcal{T}_{DA}^{mon}}_n 0$.*

The justifications for considering the topology $\mathcal{T}_{DA}^{mon}$ as a good candidate for the study of delay aversion are the same as in Section 4.3.1. Moreover we have that the proof of Proposition 4.3.3 remains true (it suffices to check that $\{q_k : q_k(\mathbf{x}) := |x_k|, k \in \mathbb{N}\}$ are actually monotone seminorms). We formalize this result in the following proposition.

Proposition 4.4.1. *There exists a finest Hausdorff locally convex topology with a monotone base over l^∞, $\mathcal{T}_{DA}^{mon}$, such that $1_n \xrightarrow{\mathcal{T}_{DA}^{mon}}_n 0$.*

4.4.2 Comparison with others topologies on l^∞ and dual space

We noticed in the beginning of Section 4.3.3 that continuity of preferences with respect to the Mackey topology induces an impatience behaviour of the DM. This topology is particularly relevant to our analysis because of its extensive use in the theory of general equilibrium in infinite dimensional spaces. Part of its glory is due to the work of Brown and Lewis [1981]. These authors show that every preference relation which is continuous with respect to the Mackey topology is impatient in the precise sense described below.

Definition 4.4.2. (Brown and Lewis [1981]) $\succsim$ *is* strongly myopic *if* $\forall \mathbf{x}, \mathbf{y} \in l^\infty$ *such that* $\mathbf{x} \succ \mathbf{y}$ *and* $\forall \mathbf{z} \in l^\infty$, $\exists n_1(\mathbf{x}, \mathbf{y}, \mathbf{z}) := n_1 \in \mathbb{N}$ *such that* $n \geq n_1 \Rightarrow \mathbf{x} \succ \mathbf{y} + \mathbf{z}\mathbf{1}_{[n,+\infty)}$

The interpretation of strong myopia is similar to the one of weak myopia given in Definition 4.3.4. Clearly, strong myopia implies weak myopia. Moreover if the preferences are monotone then weak myopia and strong myopia are equivalent as noted in footnotes 4 and 7 of Brown and Lewis [1981] (an explicit proof of this result can be found in Chateauneuf and Ventura [2013]).

Definition 4.4.3. *The topology $\mathcal{T}_{SM}$ on l^∞ is the finest topology such that every $\mathcal{T}_{SM}$-continuous (not necessarily monotone) preference relation is strongly myopic.*

The proof that such a topology exists and the study of its basic properties can be found in Brown and Lewis [1981]. Let us introduce now the *strict topology* on l^∞. The idea of the strict topology comes from the seminal paper of Buck [1958].

Definition 4.4.4. *The* strict topology $\mathcal{T}_S$ *on* l^∞ *is defined by the family of seminorms* $q_a(\mathbf{x}) = \sup_n |a_n \mathbf{x}(n)|$ *with* $(a_n)_n$ *a sequence of real numbers converging to 0.*

Conway [1967] proved that the strict topology $\mathcal{T}_S$ and the Mackey topology $\mathcal{T}_{ma}$ coincide on l^∞. Brown and Lewis [1981] show that in fact we have $\mathcal{T}_S = \mathcal{T}_{ma} = \mathcal{T}_{SM}$.

As we did when studying $\mathcal{T}_{DA}$, we proceed by comparing $\mathcal{T}_{DA}^{mon}$ with other topologies over l^∞. Namely we are interested to look at the relation between $\mathcal{T}_{DA}^{mon}$ and the topologies $\mathcal{T}_{SM}$, which discounts the future, and $\mathcal{T}_\infty$, which does not.

Proposition 4.4.2. $\mathcal{T}_{SM} \subset \mathcal{T}_{DA}^{mon} \subset \mathcal{T}_\infty$

Démonstration. As we mentioned earlier, Brown and Lewis [1981] proved that $\mathcal{T}_{SM} = \mathcal{T}_S$, therefore we will work with the topology $\mathcal{T}_S$.
We prove that $\mathcal{T}_S \subset \mathcal{T}_{DA}^{mon}$. Consider the family of seminorms generating the strict topology. Notice that this family of seminorms is monotone. In fact for a sequence $(a_n)_n$ converging to 0 and $\mathbf{x}, \mathbf{y}$ s.t. $|\mathbf{y}| \leq |\mathbf{x}|$ we have $\forall n \in \mathbb{N}$, $|a_n||y_n| \leq |a_n||x_n|$ and therefore $p_a(\mathbf{y}) = \sup_n |a_n \mathbf{y}(n)| \leq \sup_n |a_n \mathbf{x}(n)| = p_a(\mathbf{x})$. Moreover for every decreasing sequence $(a_n)_n$ generating a member p_a of the strict topology we have

$$p_a(1_n) = \sup_k |a_k 1_n(k)| = |a_n| \to_n 0.$$

This means that every seminorm of $\mathcal{T}_S$ is also a seminorm of $\mathcal{T}_{DA}^{mon}$, and therefore $\mathcal{T}_S \subseteq \mathcal{T}_{DA}^{mon}$. To show that $\mathcal{T}_S \neq \mathcal{T}_{DA}^{mon}$, let us consider the seminorm $q(\mathbf{x}) = \limsup_k |x_k|$. This seminorm is monotone and moreover, $q(1_n) = 0 \, \forall n \in \mathbb{N}$ and hence q is a seminorm of $\mathcal{T}_{DA}^{mon}$. Recall that $1^{(n)}$ is the sequence with 0 in the first $n-1$ terms and 1 after. Clearly $1^{(n)} \to_n 0$ in the strict topology $\mathcal{T}_S$, but since $\forall n \in \mathbb{N}$, $\limsup_k |1^{(n)}(k)| = 1$, $1^{(n)} \not\to_n 0$ for the topology $\mathcal{T}_{DA}^{mon}$, we have $\mathcal{T}_S \neq \mathcal{T}_{DA}^{mon}$.
We prove that $\mathcal{T}_{DA}^{mon} \subset \mathcal{T}_\infty$. We follow Raut [1986] (Lemma 5.4) argument. Let p be a monotonic seminorm generating $\mathcal{T}_{DA}^{mon}$. We need to show that there exists $c > 0$ s.t. $p(\mathbf{x}) < c$ for all $\mathbf{x} \in l^\infty$ with $\|\mathbf{x}\|_\infty = 1$. Suppose by contradiction that $p(\mathbf{x}) > c$ for all $c > 0$. Then for all $m > 0$ there exists $\mathbf{x}^m$ with $\|\mathbf{x}^m\|_\infty = 1$ such that $p(\mathbf{x}^m) > m$. Notice that, since $\|\mathbf{x}^m\|_\infty = 1$, $|\mathbf{x}^m| \leq \mathbf{u}$ where $\mathbf{u}$ is the unit vector of l^∞, i.e. $\mathbf{u} = (1, 1, \dots)$. But then since p is monotone for every $m > 0$, $p(\mathbf{u}) \geq p(\mathbf{x}^m) > m$, which contradicts the fact that p is real valued. As all seminorms generating $\mathcal{T}_{DA}^{mon}$ are dominated by monotone seminorms (by definition of locally convex topology with a monotone base), we have that all $\mathcal{T}_{DA}^{mon}$-continuous seminorms are $\mathcal{T}_\infty$-continuous, i.e. $\mathcal{T}_{DA}^{mon} \subseteq \mathcal{T}_\infty$. Notice now that $\forall n \in \mathbb{N} \sup_k |1_n(k)| = 1$ and hence $1_n \not\to_n 0$ in $\mathcal{T}_\infty$. Since $1_n \to_n 0$ in $\mathcal{T}_{DA}^{mon}$ then $\mathcal{T}_{DA}^{mon} \subset \mathcal{T}_\infty$. □

It turns out that $\mathcal{T}_{DA}^{mon}$ is strictly finer than the Strict topology $\mathcal{T}_S$ and strictly coarser than the sup-norm topology $\mathcal{T}_\infty$. Remember that $\mathcal{T}_S = \mathcal{T}_{ma}$ i.e. the Strict and the Mackey topology coincides. Hence saying that $\mathcal{T}_{DA}^{mon}$ is strictly finer than $\mathcal{T}_S$ means that in the dual space $(l^\infty, \mathcal{T}_{DA}^{mon})^*$ there will be continuous linear functionals that are not in l^1. This is because the Mackey topology is the finest topology over l^∞ such that the dual space consists of the set of countably additive probabilities. On the other end $\mathcal{T}_{DA}^{mon} \subset \mathcal{T}_\infty$ implies that we know that the dual will be a subset of the space of charges ba.

From Proposition 4.4.2, the following corollary is immediate and summarizes the discussion above.

Corollary 4.4.1. $l^1 \subset (l^\infty, \mathcal{T}_{DA}^{mon})^* \subseteq ba$.

One question now arises : is the dual space $(l^\infty, \mathcal{T}_{DA}^{mon})^*$ equal to ba or is it strictly smaller ? The following proposition shows that the former statement holds. The topology $\mathcal{T}_{DA}^{mon}$ is therefore strictly weaker than the sup-norm topology, but it preserves the dual space. From a mathematical point of view, this is an interesting result, since it yields a new characterization of the space ba. Section 4.4.2 is devoted to analyse its economic implications.

Proposition 4.4.3. $(l^\infty, \mathcal{T}_{DA}^{mon})^* = ba$

Démonstration. The inclusion $(l^\infty, \mathcal{T}_{DA}^{mon})^* \subseteq ba$ follows from Corollary 4.4.1.

We will prove now the inclusion $ba \subseteq (l^\infty, \mathcal{T}_{DA}^{mon})^*$. Take $\mu \in ba$. By the Yoshida-Hewitt Theorem (see Rao and Rao [1983], Theorem 10.2.1), we can decompose $\mu = \mu_c + \mu_p$ with μ_c

countably additive and μ_p pure. Since every countably additive measure can be identified with an element of l^1, by Corollary 4.4.1 we have that $\mu_c \in (l^\infty, \mathcal{T}_{DA}^{mon})^*$.
It is left to show that μ_p is also an element of $(l^\infty, \mathcal{T}_{DA}^{mon})^*$. In order to prove it we consider the topology generated by a family of seminorms $\mathbb{Q}$ defined as follows. Consider all the sequences $\mathbf{a} \in l^1$ s.t. $a_n \neq 0$ for all but finitely many n. Notice that $\forall N$, $\sum_{k \geq N} |a_k| > 0$. A seminorm q is in the family $\mathbb{Q}$ if it is of the type :

$$q_{\mathbf{a}}(\mathbf{x}) = \limsup_{N \to \infty} \frac{\sum_{k \geq N} |a_k x_k|}{\sum_{k \geq N} |a_k|}.$$

This family of seminorms was introduced in Orrillo and Bazán [2014]. The seminorms in $\mathbb{Q}$ define a Hausdorff locally convex topology topology, $\mathcal{T}_h$, on l^∞. The authors prove that $\mathcal{T}_h$ is weaker than the sup-norm topology $\mathcal{T}_\infty$ and they show that the dual $(l^\infty, \mathcal{T}_h)^*$ consists of the set of pure charges.
We claim that the family of seminorm $\mathbb{Q}$ is included in the family of seminorms generating $\mathcal{T}_{DA}^{mon}$. In fact for all sequences $\mathbf{a}$ generating the topology $\mathcal{T}_h$ and for every $n \in \mathbb{N}$,

$$q_{\mathbf{a}}(1_n) = \limsup_{N} \frac{\sum_{k \geq N} |a_k 1_n(k)|}{\sum_{k \geq N} |a_k|} = 0.$$

Moreover we can easily check that they are monotone. Therefore since $\mathbb{Q}$ is contained in the family of seminorms generating $\mathcal{T}_{DA}^{mon}$, we have that $\mathcal{T}_h \subseteq \mathcal{T}_{DA}^{mon}$. Hence $(l^\infty, \mathcal{T}_h)^* \subseteq (l^\infty, \mathcal{T}_{DA}^{mon})^*$ which implies $\mu_p \in (l^\infty, \mathcal{T}_{DA}^{mon})^*$. □

Finally we compare $\mathcal{T}_{DA}^{mon}$ and $\mathcal{T}_{DA}$ in the following corollary.

Corollary 4.4.2. $\mathcal{T}_{DA}^{mon} \subset \mathcal{T}_{DA}$

Démonstration. We have by definition, $\mathcal{T}_{DA}^{mon} \subseteq \mathcal{T}_{DA}$. This implies

$$\mathcal{T}_{DA}^{mon} \cap \mathcal{T}_\infty \subseteq \mathcal{T}_{DA} \cap \mathcal{T}_\infty.$$

Now notice that $\mathcal{T}_{DA}^{mon} \cap \mathcal{T}_\infty = \mathcal{T}_{DA}^{mon}$ by Proposition 4.4.2 and $\mathcal{T}_{DA} \cap \mathcal{T}_\infty \subset \mathcal{T}_{DA}$ by Proposition 4.3.7. Therefore $\mathcal{T}_{DA}^{mon} \subset \mathcal{T}_{DA}$. □

Link with general equilibrium and bubbles

Proposition 4.4.2 and Proposition 4.4.3 have interesting implications when linked with general equilibrium theory in infinite dimension and the study of bubbles.

Concerning the general general equilibrium literature, Proposition 4.4.2 can be considered as a refinement of a result of Araujo [1985]. In that paper it is shown that in an economy where consumers have preferences continuous with respect to a topology stronger than the Mackey topology, $\mathcal{T}_{ma}$, an equilibrium may fail to exist. Even worse, it is possible to construct an economy without individually rational Pareto optimal allocations

(Theorem 3 of Araujo [1985]). Hence the need to consider preferences continuous with respect to the Mackey topology.

As we said in the previous section, Brown and Lewis [1981] prove that the Mackey topology is equivalent to the strongly myopic topology, and therefore its use in economics is justified whenever agents are impatient. Proposition 4.4.2 clarifies this interpretation. When preferences are continuous with respect to $\mathcal{T}_{DA}$ then it is clear that the DM exhibits some kind of impatience. Nevertheless, an equilibrium in an economy with such agents may fail to exist. Therefore the need for impatience is in reality the need for *enough* impatience : discounting the future just as a delay averse DM may lead to non existence of equilibria.

On the other hand, Proposition 4.4.3 allows us to link our work with the theory that studies bubbles as lack of countable additivity in prices, as first presented in the paper of Gilles and LeRoy [1992]. For the authors a price function is a positive continuous linear functional over the set l^∞, view as the set of all bounded cash flows. They use the Yoshida–Hewitt theorem to decompose the price function in a countably additive part and in a purely finitely additive part. The first part is defined to be the fundamental value of a cash flow, the second one characterizes the bubble. They justify at length this definition, both formally and with examples. In a nutshell their argument is that if the price functional is in l^1, i.e. it consists just of the countably additive part, then the price of an element of l^∞ is just the sum of the discounted cash flows.

Interestingly, they underline the fact that if a DM discounts the future, then a bubble cannot occur (see [Gilles and LeRoy, 1992, p. 332]). Proposition 4.4.3 shows that the dual of l^∞ when paired with $\mathcal{T}_{DA}^{mon}$ is exactly *ba*, which includes both countably additive and purely finitely additive charges. Such a result provides a counterexample to the claim of the authors in the following sense : it proves that in the set of prices we can have bubbles (i.e. purely finitely additive charges) even when the DMs discount the future. Again, the point is that in order to avoid bubbles the DMs should discount the future *enough*.

Recent related work linking bubble with some kind of impatience behaviour was done by Araujo et al. [2011]. More precisely the authors study *wary* agents. Formally, an agent is wary if her preferences are upper but not lower Mackey semi-continuous. In this case DMs are "semi-impatient, in the sense of overlooking what they earn but not what they lose at far away dates" (see [Araujo et al., 2011, p. 786]). Therefore their approach differs from the one of this paper since they stick with the definition of myopia of Brown and Lewis [1981] and relax the definition of continuity of preferences. They do not propose an alternative definition of impatience, as done here with delay aversion. The authors then focus on how lack of countably additivity, implied by wariness, influences equilibria in a sequential market. It would be interesting, even though outside the scope of this paper, to study what happens if we replace wariness with delay aversion in the class of economies where an equilibrium actually exists.

A $\mathcal{T}_{DA}^{mon}$-continuous utility function

We conclude this section with an elementary example of a $\mathcal{T}_{DA}^{mon}$-continuous utility function.

Example 4.4.1. *Let $u : \mathbb{R} \to \mathbb{R}$ be a strictly positive affine function (i.e. $u(t) = at + b$ with $a > 0$ and $b \in \mathbb{R}$). Let $P : 2^{\mathbb{N}} \to [0,1]$ be a simply additive probability (i.e. a positive charge with $P(\mathbb{N}) = 1$) with $P(n) > 0 \, \forall n \in \mathbb{N}$, clearly $P \in ba$. Consider now the following functional $U : l^\infty \to \mathbb{R}$*

$$U(\mathbf{x}) = \int_{\mathbb{N}} u(\mathbf{x}) dP,$$

where $u(\mathbf{x})$ denotes the sequence $(u(x_1), u(x_2), \dots)$. We are going to show that $U(\cdot)$ is $\mathcal{T}_{DA}^{mon}$-continuous. First notice that since $P(n) > 0 \, \forall n \in \mathbb{N}$, and $u(\cdot)$ is strictly increasing and continuous then $U(\cdot)$ represents strongly monotone preferences (see Bastianello and Chateauneuf [2016]). The simply additive probability P should be interpreted as weights that the DM puts on (subsets of) periods of time.
Take now a net $\mathbf{x}_\lambda \xrightarrow{\mathcal{T}_{DA}^{mon}}_\lambda \mathbf{x}$. We need to show that $U(\mathbf{x}_\lambda) \to_\lambda U(\mathbf{x})$. Notice that since $u(\cdot)$ is affine for every seminorm p of $\mathcal{T}_{DA}^{mon}$,

$$p(u(\mathbf{x}_\lambda) - u(\mathbf{x})) = p(a(\mathbf{x}_\lambda - \mathbf{x})) = ap(\mathbf{x}_\lambda - \mathbf{x}) \to_\lambda 0$$

and therefore $u(\mathbf{x}_\lambda) \xrightarrow{\mathcal{T}_{DA}^{mon}}_\lambda u(\mathbf{x})$, i.e. $u(\mathbf{x}_\lambda)$ converges to $u(\mathbf{x})$ in the $\mathcal{T}_{DA}^{mon}$ topology.
Remark that the functional $J(\mathbf{x}) = \int_{\mathbb{N}} \mathbf{x} dP$ is a continuous linear functional in $(l^\infty, \mathcal{T}_{DA}^{mon})^*$. To see this, we will prove that $|J(\mathbf{x})|$ is a monotone seminorm of $\mathcal{T}_{DA}^{mon}$. Clearly $|J(\mathbf{x})|$ is a seminorm. That is monotone come from Theorem 4.4.13(vi) of Rao and Rao [1983] and the fact that P is positive. Moreover $|J(1_n)| = \int 1_n dP = P(n) \to_n 0$.
Now since $u(\mathbf{x}_\lambda) \xrightarrow{\mathcal{T}_{DA}^{mon}}_\lambda u(\mathbf{x})$, and the functional J is $\mathcal{T}_{DA}^{mon}$-continuous, $\int_{\mathbb{N}} u(\mathbf{x}_\lambda) dP \to_\lambda \int_{\mathbb{N}} u(\mathbf{x}) dP$, which means

$$U(\mathbf{x}_\lambda) \to_\lambda U(\mathbf{x}),$$

i.e. $U(\cdot)$ represent a $\mathcal{T}_{DA}^{mon}$-continuous preference relation.

4.5 More delay aversion

As we said in the introduction, the paper of Benoît and Ok [2007] inspired the notion of delay aversion. The aim of this section is to show that Definition 4.3.1 is the correct one, in the sense that it entails the same notion of more delay aversion developed by these authors.

More precisely, we assume that agents are delay averse and we develop a natural notion of more delay aversion. We show that this notion is in fact equivalent to the one proposed by Benoît and Ok [2007] and we prove their main result, stated below as Proposition 4.5.2, in a more general framework. Notice that Benoît and Ok [2007] do not assume that the agents are delay averse. Anyway, in their main results the class of utility functions that they study implies not only delay aversion (Corollary 3.1 of Bastianello and Chateauneuf [2016]) but also myopia (Proposition 3.6 of Bastianello and Chateauneuf [2016]).

In the sequel we will restrict our attention to the set l^∞_+ (which is the same set considered by Benoît and Ok [2007]).

Let us consider now two delay averse DMs. From Definition 4.3.1, we believe that a natural way of developing a concept of more delay aversion is the following. Let us define the two quantities

$$N_1(\mathbf{x}, a, b, n_0) := \min\{n \geq n_0 \,|\, (x_{n_0} + a, \mathbf{x}_{-n_0}) \succ_1 (x_n + b, \mathbf{x}_{-n})\} \tag{4.1}$$

$$N_2(\mathbf{x}, a, b, n_0) := \min\{n \geq n_0 \,|\, (x_{n_0} + a, \mathbf{x}_{-n_0}) \succ_2 (x_n + b, \mathbf{x}_{-n})\} \tag{4.2}$$

To simplify notation we will refer to $N_i(\mathbf{x}, a, b, n_0)$ as N_i for $i = 1, 2$. The natural number N_i, $i = 1, 2$, represents the first dates at which the DM i is willing to choose a payment done at a sooner date rather than one done later, holding fixed $\mathbf{x}$, a, b, and n_0. Since we assumed that agents are delay averse, N_1 and N_2 are finite. Notice that, following Benoît and Ok [2007], we do not require $b \geq a$. Anyway by strong monotonicity of preferences, this relaxation does not pose any problem (notice that also in Benoît and Ok [2007] strong monotonicity is assumed). To see it, fix $\mathbf{x} \in l^\infty_+$, $a, b \in \mathbb{R}_+$ and $n_0 \in \mathbb{N}$ and consider $b' \geq \max\{a, b\}$. Obviously $b' \geq a$ and hence by delay aversion there exists n_i such that $n \geq n_i$ implies $(x_{n_0} + a, _{-n_0}) \succ_i (x_n + b', \mathbf{x}_{-n})$. By monotonicity $(x_n + b', \mathbf{x}_{-n}) \succsim_i (x_n + b, \mathbf{x}_{-n})$ and by transitivity we get the result.

Once the numbers N_1 and N_2 are introduced, the natural definition of more delay aversion is the following.

Definition 4.5.1. *A preference relation* $\succsim_1$ *is said to be* more delay averse *than a preference relation* $\succsim_2$ *if* $N_1 \leq N_2$ *for all* $\mathbf{x} \in l^\infty$, $n_0 \in \mathbb{N}$ *and* $a, b > 0$.

The definition above states that given $\mathbf{x} \in l^\infty$, $n_0 \in \mathbb{N}$ and $a, b > 0$, for DM 1 it is enough to wait $N_1 - n_0$ periods in order to prefer the stream $(x_{n_0} + a, \mathbf{x}_{-n_0})$ (i.e. the stream with the earlier payment) to the stream $(x_n + b, \mathbf{x}_{-n})$. On the contrary, DM 2 would be ready to wait more (since $N_2 - n_0 \geq N_1 - n_0$) to switch from the first to the second stream.

We state now the definition of more delay aversion proposed by Benoît and Ok [2007].

Definition 4.5.2. (Benoît and Ok [2007]) $\succsim_1$ *is* more delay averse *than* $\succsim_2$ *if for any* $\mathbf{x} \in l^\infty_+$, $n_0 \in \mathbb{N}$, $a, b \geq 0$, *and* $n > n_0$ *we have :*

$$(x_{n_0} + a, \mathbf{x}_{-n_0}) \succsim_2 (\succ_2)(x_n + b, \mathbf{x}_{-n}) \Rightarrow (x_{n_0} + a, \mathbf{x}_{-n_0}) \succsim_1 (\succ_1)(x_n + b, \mathbf{x}_{-n}),$$

and for all $0 \leq b \leq x_{n_0}$ *and* $0 \leq a \leq x_n$,

$$(x_{n_0} - b, \mathbf{x}_{-n_0}) \succsim_2 (\succ_2)(x_n - a, \mathbf{x}_{-n}) \Rightarrow (x_{n_0} - b, \mathbf{x}_{-n_0}) \succsim_1 (\succ_1)(x_n - a, \mathbf{x}_{-n}).$$

Definition 4.5.2 says that agent 1 is more delay averse than agent 2 if, whenever 1 prefers to receive an earlier payment rather that a later one, then 2 does. Notice that the second part of Definition 4.5.2 is redundant (see footnote 12 p. 79 of Benoît and Ok [2007], therefore in the sequel we will concentrate just on the first one.

Below, we show the equivalence of Definition 4.5.2 and Definition 4.5.1.

Proposition 4.5.1. $N_1 \leq N_2$ *if and only if* $\forall \mathbf{x} \in l^\infty_+$, $n_0 \in \mathbb{N}$, $a, b > 0$, *and* $n \geq n_0$ *we have :*

$$(x_{n_0} + a, \mathbf{x}_{-n_0}) \succ_2 (x_n + b, \mathbf{x}_{-n}) \Rightarrow (x_{n_0} + a, \mathbf{x}_{-n_0}) \succ_1 (x_n + b, \mathbf{x}_{-n}).$$

Démonstration. $\Rightarrow$ Fix $\mathbf{x} \in l^\infty_+$, $n_0 \in \mathbb{N}$, $a, b > 0$ and $n \geq n_0$ and suppose $(x_{n_0}+b, \mathbf{x}_{-n_0}) \succ_2 (x_n + b, \mathbf{x}_{-n})$. Then $n \geq N_2$ because of the definition of N_2 given in (4.2). Since $\succsim_1$ is more delay averse than $\succsim_2$ then $N_2 \geq N_1$ and hence $n \geq N_1$. Therefore because of the definition of N_1 in (4.1), $(x_{n_0} + a, \mathbf{x}_{-n_0}) \succ_1 (x_n + b, \mathbf{x}_{-n})$.

$\Leftarrow$ Fix $\mathbf{x} \in l^\infty_+$, $n_0 \in \mathbb{N}$, $a, b > 0$ and suppose that $(x_{n_0} + a, \mathbf{x}_{-n_0}) \succ_2 (x_n + b, \mathbf{x}_{-n})$ for some $n \geq n_0$. Then $(x_{n_0} + a, \mathbf{x}_{-n_0}) \succ_2 (x_{n_2} + b, \mathbf{x}_{-n_2}) \Rightarrow (x_{n_0} + a, \mathbf{x}_{-n_0}) \succ_1 (x_{n_2} + b, \mathbf{x}_{-n_2})$. Therefore $N_1 \leq N_2$. □

We will prove now the main theorem of Benoît and Ok [2007] in a more general framework. The authors, after introducing their main definition in terms of preferences, focus on the class of intertemporal separable utility functions. An intertemporal separable utility function, consist on a pair (u, δ), where u is an instantaneous utility and δ a discount factor. The utility $u : \mathbb{R} \to \mathbb{R}$ is strictly increasing and continuous with $u(0) = 0$ and $u(\infty) = \infty$, the discount factor is represented by a strictly positive, strictly decreasing sequence in l^1 with $\delta_0 = 1$. The sequences are ranked through the functional $U : l^\infty_+ \to \mathbb{R}$

$$U(\mathbf{x}) = \sum_{t=0}^{\infty} \delta_t u(x_t).$$

We are going to work in a framework that subsumes the one described above. We endow the DMs with a preference relation represented by the following functional

$$U(\mathbf{x}) = \int u(\mathbf{x}) d\mu. \tag{4.3}$$

In the integral in (4.3), u in an instantaneous utility function, whereas μ is a positive charge in *ba* with $\mu(n) > 0$ $\forall n \in \mathbb{N}$. The charge μ plays the role of the discounting function. Notice that if $\mu \in l^1$ then we are back to the model of Benoît and Ok [2007].

Such a generalization might seem a minor one, but in fact it has a relevant link with the sections above. It can be shown that the utility function studied by Benoît and Ok [2007] entails that the preferences relation are strongly myopic. The functional considered here, instead, allows one to disentangle different nuances of impatience. Namely, Bastianello and Chateauneuf [2016] prove in Example 3.1 that this particular functional form can represent preferences which are delay averse but not myopic.

Notice that the sequence of instantaneous utilities can be rewritten as :

$$u_i(\mathbf{x} + a1_{n_0}) = u_i(\mathbf{x})1_{(n \cup n_0)^c} + u_i(x_{n_0} + a)1_{n_0} + u_i(x_n)1_n$$

and

$$u_i(\mathbf{x} + b1_n) = u_i(\mathbf{x})1_{(n \cup n_0)^c} + u_i(x_{n_0})1_{n_0} + u_i(x_n + b)1_n.$$

For $i = 1, 2$, applying Theorem 4.4.13(ii) of Rao and Rao [1983], we can define the function $d^i_{\mathbf{x},b,n_0,n} : \mathbb{R}_+ \to \mathbb{R}$ as

$$\begin{aligned} d^i_{\mathbf{x},b,n_0,n}(a) = \int u_i(\mathbf{x} + a1_{n_0}) d\mu_i - \int u_i(\mathbf{x} + b1_n) d\mu_i = \\ = (u_i(x_{n_0} + a) - u_i(x_{n_0}))\mu_i(n_0) - (u_i(x_n + b) - u_i(x_n))\mu_i(n). \end{aligned}$$

To simplify notation, we will write $d^1(a) = d^i_{\mathbf{x},b,n_0,n}(a)$ for $a \in \mathbb{R}_+$. We can notice that since $u_i(\cdot)$ is strictly increasing and continuous then also d^i is strictly increasing and continuous and therefore :

$$\lim_{a \to 0^+} d^i(a) = -(u_i(x_n + b) - u_i(x_n))\mu_i(n) < 0$$

and

$$\lim_{a \to \infty} d^i(a) = \infty.$$

Therefore the image $Im(d^i(a)) = (-(u_i(x_n + b) - u_i(x_n))\mu_i(n), +\infty)$ and hence, by the Intermediate Value Theorem there exists a number $a_i^* < +\infty$ s.t. $d^i(a_i^*) = 0$, i.e. such that $(u_i(x_{n_0} + a_i^*) - u_i(x_{n_0}))\mu_i(n_0) = (u_i(x_n + b) - u_i(x_n))\mu_i(n)$.

We are ready now to prove the following lemma :

Lemma 4.5.1. *$N_1 \leq N_2$ if and only if $a_1^* \leq a_2^*$ $\forall \mathbf{x} \in l^\infty_+, n \geq n_0$ and $b > 0$.*

Démonstration. $\Rightarrow$ Fix $\mathbf{x} \in l^\infty_+$, $n \geq n_0$ and $b > 0$, since $N_1 \leq N_2$, using the characterization proved in Proposition 4.5.1 we have that if $d^2(a) > 0$ then $d^1(a) > 0$. Being $d^i(\cdot)$ strictly increasing for $i = 1, 2$, and since $d^i(a_i^*) = 0$, we have that $\forall \epsilon > 0$:

$$d^2(a_2^* + \epsilon) > 0 \Rightarrow d^1(a_2^* + \epsilon) > 0.$$

But $d^1(a_2^* + \epsilon) > 0$ iff $(u_1(x_{n_0} + a_2^* + \epsilon) - u_1(x_{n_0}))\mu_1(n_0) > (u_1(x_n + b) - u_1(x_n))\mu_1(n) = (u_1(x_{n_0} + a_1^*) - u_1(x_{n_0}))\mu_1(n_0)$ and therefore it follows that $\forall \epsilon > 0$:

$$(u_1(x_{n_0} + a_2^* + \epsilon) - (u_1(x_{n_0} + a_1^*))\mu_1(n_0) > 0$$

and being $u(\cdot)$ strictly increasing this will imply $a_2^* + \epsilon > a_1^*$ and since ϵ can be arbitrarily small $a_2^* \geq a_1^*$.

$\Leftarrow$ Let $a_1^* \leq a_2^*$ and fix an $\mathbf{x} \in l_+^\infty$, $n \geq n_0$ and $a, b > 0$. If $(x_{n_0} + a, x_{-n_0}) \succ_2 (x_n + b, x_{-n})$ then $d^2 > 0 = d^2(a_2^*)$ and being $d^i(\cdot)$ strictly increasing we have $a > a_2^* \geq a_1^*$ (where we have the last inequality by hypothesis). Since a_1^* is s.t. $d^1(a_1^*) = 0$ we have that $d^1(a) > d^1(a_1^*) = 0$ i.e. $\int u_1(x + a1_{n_0})d\mu_1 - \int u_1(x + b1_n)d\mu_1 > 0$ which means $(x_{n_0} + a, x_{-n_0}) \succ_1 (x_n + b, x_{-n})$. And so by Proposition 4.5.1, $N_1 \leq N_2$. $\square$

We present the main result of Benoît and Ok [2007] as Proposition 4.5.2. Since Proposition 4.5.2 follows applying Lemma 4.5.1 and repeating the same steps of Benoît and Ok [2007], its proof will be omitted.

Proposition 4.5.2. (Benoît and Ok [2007], Theorem 2) *Suppose that $\succsim_i$ is represented by the function $U_i(\cdot) = \int u_i(\cdot)d\mu_i$ given in (4.3). Then the following are equivalent.*

(a) $U_1(\cdot)$ is more delay averse than $U_2(\cdot)$.

(b) There exists a map $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $u_1 = h \circ u_2$ and

$$h\left(x + \frac{\mu_2(n)}{\mu_2(n_0)}y\right) \geq h(x) + \frac{\mu_1(n)}{\mu_1(n_0)}(h(y+z) - h(z))$$

for all $n_0, n \in \mathbb{N}$ with $n_0 < n$ and $x, y, z \geq 0$.

moreover if u_1 and u_2 are continuously differentiable on $\mathbb{R}_{++}$ then either of the above statements is equivalent to either of the following statements.

(c) There exists a continuously differentiable map $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $u_1 = h \circ u_2$ and

$$\inf\{h'(x) : x > 0\} \geq \frac{\mu_1(n)/\mu_1(n_0)}{\mu_2(n)/\mu_2(n_0)} \sup\{h'(x) : x > 0\} \text{ whenever } n_0 < n$$

(d)

$$\frac{\mu_1(n_0)u_1'(x)}{\mu_1(n)u_1'(y)} \geq \frac{\mu_2(n_0)u_2'(x)}{\mu_2(n_0)u_2'(y)}$$

for all $n_0, n \in \mathbb{N}$ with $n_0 < n$ and $x, y \geq 0$.

4.6 Conclusion

In this paper we analysed the notion of *delay aversion* following a topological approach. We defined two Hausdorff locally convex topologies that discount the future in a way that is consistent with delay aversion. We compared this two topologies with other topologies usually considered in the theory of general equilibrium in infinite dimensional spaces. We proved formally that the notion of delay aversion is a weak notion representing the concept of advancing the time of future satisfaction. Namely, it is easier for a DM to show a delay aversion behaviour rather than a myopic one. Next, we analysed the topological dual spaces of l^∞ when paired with the delay averse topologies. We found that the duals are larger than the space l^1 of summable sequences. As a dividend, we gave an alternative characterization of the space ba of bounded charges.

These results have implications for the theory of general equilibrium in infinite dimensions and for the theory that study bubbles as lack of countable additivity of prices. For the former theory, our results imply that an equilibrium may fail to exist. For the latter, we showed that bubbles may occur. Both situations could happen even when the economy is composed by impatient agents. The key point in both cases is that the agents should be enough impatient in order to guarantee equilibrium and no bubbles.

We concluded our analysis by proving that the concept of delay aversion is consistent with the notion of more delay aversion proposed by Benoît and Ok [2007]. Finally, we proved their main theorem in a more general setting.

Chapitre 5

Target-based solutions for Nash bargaining

Ce chapitre est issu de l'article "Target-based solutions for Nash bargaining", en collaboration avec Marco LiCalzi [1].

Abstract. We revisit the Nash model for two-person bargaining. A mediator knows agents' *ordinal* preferences over feasible proposals, but has incomplete information about their acceptance thresholds. We provide a behavioural characterisation under which the mediator recommends a proposal that maximises the probability that bargainers strike an agreement. Some major solutions are recovered as special cases ; in particular, we offer a straightforward interpretation for the product operator underlying the Nash solution.

5.1 Introduction

The Nash model for two-person bargaining pivots on the assumption that agents are expected utility maximisers. The underlying feasible alternatives are abstracted away by mapping any proposal into a pair of von Neumann-Morgenstern utilities (u_1, u_2). A Nash bargaining problem (S, d) consists of a compact and convex set $S \subset \mathbb{R}^2$ of feasible utility pairs, and a disagreement point d in S representing the utilities associated with bargaining breakdown. Moreover, the convexity of S is frequently justified by including lotteries among the feasible proposals.

A bargaining solution assigns to any Nash problem (S, d) a single pair of utility values in S. Nash [1950] proved that the unique solution satisfying four axioms is defined as the

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maximiser of the product $(u_1 - d_1)(u_2 - d_2)$ for (u_1, u_2) in S and $u_i \geq d_i$ for $i = 1, 2$. These four axioms are usually known as symmetry, Pareto optimality, invariance to positive affine transformations, and independence of irrelevant alternatives.

The Nash model is a cornerstone of two-person bargaining theory. Its simplicity and robustness have fostered both its widespread application and its theoretical prominence. A vast body of literature has adopted it, proposing different axiomatizations for the Nash solution as well as several alternative solutions ; see f.i. Thomson et al. [1994]. Along its many glories, however, not all is well with the model : the Nash solution cannot stake a claim for being intuitively appealing. Rubinstein et al. [1992] put it very sharply : *"the solution lacks a straightforward interpretation since the meaning of the product of two von Neumann–Morgenstern utility numbers is unclear"* ; see Section 5.4.2.

Motivated by this, they review the foundations of the Nash model and offer a very lucid account of its interpretive limitations. Restating the classical utility-based Nash model in terms of agents' preferences, they offer a more attractive definition of the Nash solution. This conceptual switch to a preference-based language is a key step for reinterpreting the logic underlying the axioms and the solution. However, while their focus on preferences brings substantial theoretical insights, it does not yet uncover an intuitive meaning neither for the product operator nor for the Nash solution.

This paper revisits the Nash model from a related viewpoint. We switch from a utility-based language to a probability-based language. (Specifically, we dispense with most of the formalities of expected utility.) This unlocks several theoretical dividends. We offer a behavioural characterisation for a general class of solutions, equivalent to maximising the probability that the bargainers strike an agreement. This provides a sound underpinning for giving prescriptive advice to a mediator. We also characterise a few major solutions as special cases of this approach, where the single feature separating them is the nature of the stochastic dependence between the bargainers' stance.

Our probability-based approach suggests a straightforward interpretation for the *product of two von Neumann–Morgenstern utility numbers* advocated by the Nash solution. This is revealed as the product of two probabilities, and corresponds to an implicit assumption of stochastic independence between the bargainers' positions. We then show how relaxing this assumption generates other well-known but less frequently used alternatives, namely the egalitarian and the (truncated) utilitarian solutions.

A simple example may be useful to elucidate our interpretation of the Nash solution, leaving generalisations and details to the rest of the paper. Two agents are bargaining over a set A of feasible alternatives, described in physical terms. (The Nash model ignores A and focuses on the space of utilities.) Assume that A is a nonempty, compact and connected subset of $\mathbb{R}^n$. Each agent $i = 1, 2$ has an ordinal continuous preference $\succsim_i$ over A. The two agents hire a mediator to suggest a solution and help them strike an agreement. The mediator knows agents' ordinal preferences, but she is not sure what it takes for an agent to accept a proposal x from A.

More formally, we postulate that i accepts a proposal x if and only if $x \succsim_i t_i$, where t_i in A is i's acceptance threshold (for short, his *target*). The mediator has incomplete information about the bargainers' targets : she believes that each target is a random variable T_i, with a compact and convex support in A. Under her beliefs, she maps each proposal x to a pair of individual acceptance probabilities (p_1, p_2) in $[0,1]^2$, where $p_i = P(x \succsim_i T_i)$. If the bargainers' targets are stochastically independent, the probability that both accept x is given by the product $p_1 \cdot p_2$ of the individual acceptance probabilities.

The mediator can recommend any feasible alternative, but she cannot impose it : if she suggests x, it is left to the bargainers to accept it. Her goal is to find a proposal x that maximises the probability that agents strike an agreement. Assuming that targets are stochastically independent, she should advance a proposal x that maximises the product $p_1 \cdot p_2$. Hence, the Nash solution may be interpreted as the rule that recommends to maximise the probability to strike an agreement when agents' targets are private information and independently distributed.

This is not merely an analogy. Section 5.4.2 shows that the distribution function $P(x \succsim_i T_i)$ for T_i is formally equivalent to the Bernoulli index function $U_i(x)$ and thus we can set $P(x \succsim_i T_i) = U_i(x)$. The Nash solution requires to maximise the product of two numbers, and they can be equivalently interpreted as utilities or probabilities. The existing axiomatizations are framed in a utility-based language for which the product operator is a puzzle. Switching to a probability-based language uncovers a straightforward interpretation.

The rest of the paper is organised as follows. Section 5.2 provides preliminary information on bivariate copulas. Section 5.3 describes our model, offers a general characterisation for preferences over bargaining solutions, as well as an axiomatization for the Nash solution, the egalitarian solution, and the (truncated) utilitarian solution. Section 5.4 reviews related literature, and offers a commentary on our results. Section 5.5 illustrates applications and extensions, as well as some testable restrictions on the model.

5.2 Preliminaries

Copulas are functions that link multivariate distributions to their one-dimensional marginal distributions; see Nelsen [2006]. They are used to model different forms of statistical dependence and construct families of distributions exhibiting them. We focus on the prominent case of bivariate copulas.

A (bivariate) copula is a function $C : [0,1]^2 \to [0,1]$ that satisfies two properties :

C1) for any p, q in $[0,1]$, $C(p,0) = C(0,q) = 0$, $C(p,1) = p$, and $C(1,q) = q$;

C2) for any $p_1 > q_1$ and $p_2 > q_2$ in $[0,1]$, $C(p_1,p_2) + C(q_1,q_2) \geq C(p_1,q_2) + C(q_1,p_2)$.

Property C2 is usually called 2-increasingness in the literature, but is of course equivalent to supermodularity. We use this latter name because it is presumably more familiar for our readers. The combination of C1 and C2 implies that $C(p,q)$ is increasing in each argument ; see Lemma 2.1.4 in Nelsen [2006]. When the weak inequality in C2 is replaced by a strict one, the copula C is said to be strictly supermodular and it is strictly increasing in each argument, provided that the other one is not zero.

The following result is in Sklar [1959] and characterizes how a copula links the bivariate distribution to its univariate marginals.

Theorem 1. *Let (X,Y) be a random vector with marginal distributions $F(x)$ and $G(y)$. The following are equivalent :*

i) $H(x,y)$ is the joint distribution function of (X,Y) ;

ii) there exists a copula $C(p,q)$ such that $H(x,y) = C[F(x),G(y)]$ for all x,y.

If $F(x)$ and $G(y)$ are continuous, then $C(p,q)$ is unique. Otherwise, $C(p,q)$ is uniquely defined on the cartesian product $Ran(F) \times Ran(G)$ of the ranges of the two marginal distributions. Conversely, if $C(p,q)$ is a copula and $F(x)$ and $G(y)$ are distribution functions, then the function $H(x,y)$ defined above is a joint distribution function with margins $F(x)$ and $G(y)$.

The best known example of a copula is the product $\Pi(p,q) = p \cdot q$, associated with stochastic independence. Two other important examples are $W(p,q) = \max(p+q-1,0)$ and $M(p,q) = \min(p,q)$. For any copula $C(p,q)$ and any (p,q) in $[0,1]^2$, it is the case that $W(p,q) \leq C(p,q) \leq M(p,q)$. Intuitively, M is the copula associated with the strongest possible positive dependence between X and Y, given the marginal distributions F and G ; similarly, W describes the strongest possible negative dependence. The copulas W and M are known as the Fréchet lower and upper bound, respectively.

5.3 Model and results

In its simplest version, our model for two-person bargaining is as abstract as the Nash model. We postulate that each feasible proposal x is mapped to a pair of probabilities (p_1,p_2). At this stage, it suffices to think of p_i as the individual acceptance probability that a third-party called the mediator attributes to Agent $i = 1,2$ when he is offered the proposal x. Section 5.4.2 discusses two compatible interpretations for this mapping.

For our purposes, a *bargaining problem* is represented by a compact set B in $[0,1]^2$ where each point $\mathbf{p}$ in B corresponds to a pair of (acceptance) probabilities. A *solution* is a map that for any problem B delivers (at least) one point in B.

We consider the preferences of the mediator over the set of lotteries on pairs of acceptance probabilities, and derive a behavioural characterisation under which she evaluates a proposal by the probability that both bargainers agree to it. More formally, we assume that the mediator has a preference relation $\succsim$ on the lotteries on $[0,1]^2$, and provide a representation theorem under which the solution for B corresponds to a $\succsim$–maximal point in B.

5.3.1 Assumptions on preferences

We denote an element (p_1, p_2) in $[0,1]^2$ by $\mathbf{p}$. We view $[0,1]^2$ as a mixture space for the $\oplus$ operation, under the standard interpretation where $\alpha\mathbf{p} \oplus (1-\alpha)\mathbf{q}$ is a lottery that delivers $\mathbf{p}$ in $[0,1]^2$ with probability α in $[0,1]$ and $\mathbf{q}$ in $[0,1]^2$ with the complementary probability $1-\alpha$; see Herstein and Milnor [1953]. Moreover, let $\mathbf{p} \vee \mathbf{q} = (\max(p_1, q_1), \max(p_2, q_2))$ and $\mathbf{p} \wedge \mathbf{q} = (\min(p_1, q_1), \min(p_2, q_2))$ denote the standard lattice-theoretical join and meet for the usual component-wise monotonic partial ordering $\geqq$ in $\mathbb{R}^2$.

We make the following assumptions about the mediator's preference $\succsim$ over $[0,1]^2$, where $\succ$ and $\sim$ have the usual meaning. For simplicity, we write "for sure" instead of the more accurate "with probability 1".

A.1 (Regularity) $\succsim$ is a complete preorder, continuous and mixture independent.

This implies that there exists a real-valued function $V : [0,1]^2 \to [0,1]$, unique up to positive affine transformations, that represents $\succsim$ and is linear with respect to $\oplus$; that is, $V(\alpha\mathbf{p} \oplus (1-\alpha)\mathbf{q}) = \alpha V(\mathbf{p}) + (1-\alpha)V(\mathbf{q})$, for any α in $[0,1]$ and any $\mathbf{p}, \mathbf{q}$ in $[0,1]^2$. See Theorem 8.4 in Fishburn [1970]. Quite interestingly, [Nash, 1950, p. 157] explicitly points out how an analog of A.1 is implied in his model by the assumption that both bargainers are expected utility maximizers.

A.2 (Non-triviality) $(1,1) \succ (0,0)$.

This rules out the trivial case where the mediator is indifferent between a proposal that is accepted for sure by both bargainers and another proposal that is refused for sure by both bargainers.

A.3 (Disagreement indifference) for any p, q in $[0,1]$, $(p,0) \sim (0,q)$.

This is named after Assumption DI in Border and Segal [1997], who also study a preference relation over solutions. Their paper is discussed in Section 5.5.2. Framed within the Nash model, Assumption DI states the following : a solution that assigns to either player the same utility he gets at the disagreement point is as good as the disagreement point itself. In simple words, a solution that gives one player the worst individually rational outcome is equivalent to a solution that gives both bargainers the same utility as the disagreement point. In our probability-based framework, it states that having one of the bargainers refusing for sure is equivalent to having both refusing for sure. A proposal is accepted if and only if both bargainers agree to it.

A.4 (Consistency over individual probabilities) for any p in $[0,1]$,

$$p(1,1) \oplus (1-p)(0,1) \sim (p,1) \quad \text{and} \quad p(1,1) \oplus (1-p)(1,0) \sim (1,p).$$

This states the following. Assume that one bargainer is known to accept for sure. Then the mediator is indifferent between a lottery that has the second bargainer accepting for sure with probability p and refusing for sure with probability $(1-p)$, or a proposal where the second bargainer accepts with probability p. Intuitively, the first lottery has an "objective" probability p of success, while the second proposal has a "subjective" probability with the same value p. We assume that the mediator is indifferent between the two modalities. Technically speaking, the assumption aligns the marginal acceptance probabilities with the corresponding joint acceptance probability if one of the two bargainers accepts for sure.

A.5 (Weak complementarity) for any $\mathbf{p}, \mathbf{q}$ in $[0,1]^2$,

$$(1/2)\,(\mathbf{p} \vee \mathbf{q}) \oplus (1/2)\,(\mathbf{p} \wedge \mathbf{q}) \succsim (1/2)\mathbf{p} \oplus (1/2)\mathbf{q}$$

This is named after Axiom S in Francetich [2013]. It states that a fifty-fifty lottery between two pairs of acceptance probabilities $\mathbf{p}$ and $\mathbf{q}$ is weakly inferior to a fifty-fifty lottery between their extremes (under the component–wise ordering). The interpretation is the following. Suppose $p_1 \geq q_1$ and $q_2 \geq p_2$. When the individual acceptance probabilities improve from (q_1, p_2) to (p_1, p_2), the increase in the probability of success for a proposal cannot be greater than when they change from (q_1, q_2) to (p_1, q_2). Whatever advantage is gained when the first bargainer's acceptance probability increases by $p_1 - q_1$, it adds more to the probability of success when the second bargainer is more likely to accept. In simple words, the individual acceptance probabilities are (weakly) complementary towards getting to an agreement. For a particularly sharp illustration, let $p_1 = q_2 = 1$ and $p_2 = q_1 = 0$: clearly, joint acceptance occurs only at $(1,1)$, and a fifty-fifty lottery between $(1,1)$ and $(0,0)$ is strictly better than a fifty-fifty lottery between $(1,0)$ and $(0,1)$.

We show by example in Section 5.3.6 that, under A.1, the four assumptions A.2–A.5 are logically independent.

5.3.2 A general characterisation

Our first result gives a behavioural characterisation for the preferences of the mediator. Under A1–A5, there exists a unique copula that represents $\succsim$. Given the marginal distributions, any copula identifies a joint probability distribution consistent with them ; see Section 5.2 and Nelsen [2006]. For any possible dependence structure linking the marginals, there is a copula that describes it.

Theorem 2. *The preference relation $\succsim$ satisfies A.1–A.5 if and only if there exists a unique copula $C : [0,1]^2 \to [0,1]$ that represents $\succsim$, in the sense that*

$$\mathbf{p} \succsim \mathbf{q} \quad \text{if and only if} \quad C(\mathbf{p}) \geq C(\mathbf{q}).$$

Démonstration. Necessity being obvious, we prove only sufficiency. By $A.1$, the Mixture Space Theorem (Herstein and Milnor [1953]) implies that there exists a unique (up to positive affine transformations) function $V : [0,1]^2 \to \mathbb{R}$ that represents $\succsim$ and is linear with respect to $\oplus$. By A.2, $V(1,1) - V(0,0) > 0$. Apply the appropriate positive affine transformation and consider the (unique) function $C : [0,1]^2 \to [0,1]$ defined by

$$C(p,q) = \frac{V(p,q) - V(0,0)}{V(1,1) - V(0,0)}.$$

We show that C satisfies the two defining properties C1–C2 of a copula given in Section 5.2.

By $A.3$, we have $C(p,0) = C(0,p) = C(0,0) = 0$, for any p in $[0,1]$. Moreover, clearly $C(1,1) = 1$. By A.4 and linearity, for any p in $[0,1]$, we get $C(p,1) = C(p(1,1) \oplus (1-p)(0,1)) = pC(1,1) + (1-p)C(0,1) = p$; a similar argument shows that $C(1,p) = p$. This proves C1.

By $A.5$ and the linearity of C, for all $\mathbf{p}, \mathbf{q}$ in $[0,1]^2$, it follows that

$$\frac{1}{2}C(\mathbf{p} \vee \mathbf{q}) + \frac{1}{2}C(\mathbf{p} \wedge \mathbf{q}) = C\left(\frac{1}{2}(\mathbf{p} \vee \mathbf{q}) \oplus \frac{1}{2}(\mathbf{p} \wedge \mathbf{q})\right) \geq C(\frac{1}{2}\mathbf{p} \oplus \frac{1}{2}\mathbf{q}) = \frac{1}{2}C(\mathbf{p}) + \frac{1}{2}C(\mathbf{q}),$$

so that C is supermodular, and C2 holds. □

In our setup, this result has a straightforward interpretation. A pair (p,q) of acceptance probabilities in B represents the individual probability that each bargainer accepts the underlying proposal. When the mediator's preferences satisfy axioms A.1–A.5, she behaves as if she aggregates these individual probabilities by consistently using a (unique) copula C and computes the joint probability $C(p,q)$ that the bargainers strike an agreement. Since the copula is arbitrary, the mediator may entertain any subjective opinion regarding the dependence structure (as embedded in the copula) between the individual acceptance probabilities. The proposals in B are ranked accordingly to the resulting joint acceptance probability.

A $\succsim$-maximal element in B is a choice that maximises the probability that the bargainers accept the underlying proposal and strike an agreement. Since any copula is necessarily Lipschitz continuous and B is compact, the set of $\succsim$-maximal elements in B is not empty and a solution exists. On the other hand, our assumptions do not imply its uniqueness. In general, the solution in B corresponds to an equivalence class of pairs of individual acceptance probabilities (including lotteries over those) for which the mediator assesses the same joint acceptance probability. It is worth noting that B is not required to be convex or even connected.

The copula representing the preferences of the mediator in Theorem 2 is linear with respect to $\oplus$, in the sense that $C(\alpha\mathbf{p}\oplus(1-\alpha)\mathbf{q}) = \alpha C(\mathbf{p})+(1-\alpha)C(\mathbf{q})$. When the mediator evaluates the probability of success for a lottery, she assesses first the probabilities of success for $\mathbf{p}$ and $\mathbf{q}$ through $C(\mathbf{p})$ and $C(\mathbf{q})$, and then she mixes them with the same weights defining the lottery. On the contrary, our assumptions do not imply linearity with respect to convex combinations of points in $[0,1]^2$ based on the standard $+$ operator.

Two simple variations on Theorem 2 are worth mentioning. Recall that a copula $C(p,q)$ is (weakly) increasing in each argument. Therefore, the preference relation $\succsim$ does not violate the (weak) Pareto ordering : if $p_1 \geq q_1$ and $p_2 \geq q_2$, then $(p_1,p_2) \succsim (q_1,q_2)$. But it may not satisfy the (strong) Pareto ordering, under which the condition $(p_1-q_1)(p_2-q_2) > 0$ implies $(p_1,p_2) \succ (q_1,q_2)$. Consider a mild strengthening of A.5, where we write $\mathbf{p} \bowtie \mathbf{q}$ to indicate that $\mathbf{p} \neq \mathbf{q}$ and that $\mathbf{p}$ and $\mathbf{q}$ are not comparable with respect to the usual partial ordering $\geqq$.

A.5* (Complementarity) for any $\mathbf{p},\mathbf{q}$ in $[0,1]^2$,

$$(1/2)\,(\mathbf{p}\vee\mathbf{q}) \oplus (1/2)\,(\mathbf{p}\wedge\mathbf{q}) \succsim (1/2)\mathbf{p} \oplus (1/2)\mathbf{q}.$$

Moreover, if $\mathbf{p} \bowtie \mathbf{q}$,

$$(1/2)\,(\mathbf{p}\vee\mathbf{q}) \oplus (1/2)\,(\mathbf{p}\wedge\mathbf{q}) \succ (1/2)\mathbf{p} \oplus (1/2)\mathbf{q}.$$

In combination with the other assumptions, this rules out "thick" indifference curves and makes $\succsim$ consistent with the (strong) Pareto ordering : if $p_1 \geq q_1$ and $p_2 \geq q_2$, then $(p_1,p_2) \succsim (q_1,q_2)$ and, moreover, $(p_1,p_2) \succ (q_1,q_2)$ if $(p_1-q_1)(p_2-q_2) > 0$. This follows from the next result, because any strictly supermodular copula is strictly increasing in each argument.

Theorem 3. *The preference relation $\succsim$ satisfies A.1–A.5* if and only if there exists a unique strictly supermodular copula $C:[0,1]^2 \to [0,1]$ that represents $\succsim$.*

A second variation embodies an elementary notion of fairness.

A.6 (Anonymity) for any p,q in $[0,1]$, $(p,q) \sim (q,p)$.

This states that the evaluation for any pair (p,q) of individual acceptance probabilities is unaffected by permutations, and hence is anonymous with respect to the bargainers' identity. The following result is immediate.

Theorem 4. *The preference relation $\succsim$ satisfies A.1–A.5 and A.6 if and only if there exists a unique symmetric copula $C:[0,1]^2 \to [0,1]$ that represents $\succsim$.*

Symmetric copulas are the most frequently studied class, and Anonymity seems to be a natural requirement. However, it remains a special case. The analog of some solutions (including the Nash solution) require Complementarity and imply Anonymity, and hence are associated with strictly supermodular and symmetric copulas.

Theorem 2 shows that we can characterise different solution concepts for cooperative bargaining as the aggregation (via copula) of two individual acceptance probabilities into a joint probability of success. As a special case, the following Theorem 5 proves that stochastic independence is the key assumption for deriving the product operator underlying the Nash solution.

Since each copula models a different dependence structure, other solutions for cooperative bargaining may be recovered under alternative assumptions. In particular, the two extreme assumptions of maximal positive dependence and maximal negative dependence between the individual acceptance probabilities bring about the egalitarian solution and a variant of the (truncated) utilitarian solution, discussed respectively in Sections 5.3.4 and 5.3.5.

5.3.3 The Nash solution

The Nash solution is a special case of Theorem 2 when the copula chosen by the mediator presumes stochastic independence among the individual acceptance probabilities. That is, the Nash solution emerges whenever we assume that these individual probabilities are independent. This seems by far a very natural requirement, and in our view it gives the Nash solution a central position among the special cases.

Under A.5*, we need only one additional assumption to characterise the Nash solution.

A.7 (Rescaling indifference) for any α, p, q in $[0, 1]$, $(\alpha p, q) \sim (p, \alpha q)$.

This states that the mediator is indifferent whether the same proportional reduction in the acceptance probability is applied to one bargainer or to the other one. The probability of joint acceptance is equally affected when downsizing (by the same factor) the individual propensity to accept of either bargainer. Clearly, A.7 implies A.3 (Disagreement indifference) and A.6 (Anonymity).

Theorem 5. *The preference relation $\succsim$ satisfies A.1–A2, A4–A.5*, and A.7 if and only if it is represented by the copula $\Pi(p, q) = p \cdot q$.*

Démonstration. Necessity is obvious. Sufficiency follows if we show that all the indifference curves for the representing copula are hyperbolas. First, suppose that $\mathbf{p} = (p_1, p_2)$ and $\mathbf{q} = (q_1, q_2)$ satisfy $p_1 p_2 = q_1 q_2$; without loss of generality, assume $p_1 > q_1$ and $p_2 < q_2$. Consider $\mathbf{p} \vee \mathbf{q} = (p_1, q_2)$ and let $\alpha = q_1/p_1 = q_2/p_2 < 1$. By *A.7*, we obtain $(\alpha p_1, q_2) \sim (p_1, \alpha q_2)$ and, substituting for α, we find $\mathbf{q} = (q_1, q_2) \sim (p_1, p_2) = \mathbf{p}$. Therefore, two points on the same hyperbola are indifferent.

Conversely, suppose without loss of generality that $p_1p_2 > q_1q_2$. We show that $(p_1, p_2) \succ (q_1, q_2)$. Let $\alpha = (q_1q_2)/(p_1p_2) < 1$. By strong Pareto dominance, $(p_1, p_2) \succ (\alpha p_1, p_2)$. Since the point $(\alpha p_1, p_2)$ lies on the same hyperbola as (q_1, q_2), by the first part of this proof $(\alpha p_1, p_2) \sim (q_1, q_2)$. Hence, $(p_1, p_2) \succ (\alpha p_1, p_2) \sim (q_1, q_2)$. $\square$

This result uncovers an appealing interpretation for the product operator underlying the Nash solution for cooperative bargaining. When the bargaining problem is framed with respect to pairs of acceptance probabilities (instead of utilities), the product operator is the natural consequence of the assumption that the contribution of these individual probabilities to the joint acceptance probability satisfies stochastic independence.

The key behavioural implication of Theorem 5 is the following. Suppose that the mediator evaluates that the probability of joint acceptance for a proposal (p, q) is π. For any α in $[0, 1]$, by the linearity of C, the mediator attributes a probability of success $\alpha\pi$ to the lottery $\alpha(p, q) \oplus (1-\alpha)\mathbf{0}$. Intuitively, the introduction of the α-randomisation reduces the probability π of success by a factor α. The reduction is multiplicative because the randomising device is (tacitly assumed as) stochastically independent. Under the Nash copula, any α-reduction to either of the individual acceptance probabilities has the same effect on the mediator's evaluation as an α-randomisation : $\alpha(p, q) \oplus (1-\alpha)\mathbf{0} \sim (\alpha p, q) \sim (p, \alpha q)$. It is immaterial whether the reduction comes from an (objective) lottery or from a (subjective) assessment.

5.3.4 The egalitarian solution

In the utility–based Nash model, the egalitarian solution (Kalai [1977b]) recommends the maximal point at which utility gains from the disagreement point d are equal. More simply, for a Nash problem (S, d), the egalitarian solution selects the maximiser of the function $\min\{(u_1 - d_1), (u_2 - d_2)\}$ for (u_1, u_2) in S and $u_i \geq d_i$ for $i = 1, 2$.

In our probability-based formulation, this translates to the requirement that the solution for a bargaining problem B is the (set of) maximiser(s) of the function $\min(p_1, p_2)$ for $\mathbf{p}$ in B. Consider the following assumption.

A.8 (Meet indifference) for any p, q in $[0, 1]$, $(p, p \wedge q) \sim (p \wedge q, q)$.

This states that the mediator is indifferent between two pairs of acceptance probabilities as far as they have the same meet. Intuitively, preferences over (p, q) depend only on the smallest value between p and q. Clearly, A.8 implies A.3 (Disagreement indifference) and A.6 (Anonymity). The manifest analogies between A.7 and A.8 reappear in the formulation of the following result, whose proof is similar and thus can be omitted.

Theorem 6. *The preference relation $\succsim$ satisfies A.1–A2, A4–A.5*, and A.8 if and only if it is represented by the copula $M(p, q) = \min(p, q)$.*

This characterises the representing copula under Meet indifference as the Fréchet upper bound $M(p, q) = \min(p, q)$, that provides the strongest possible positive dependence between two marginal distributions. Therefore, we can reinterpret the egalitarian solution as the recommendation that maximises the probability of joint acceptance when the mediator assumes that the individual acceptance probabilities are maximally positively dependent.

5.3.5 The utilitarian solution

There exist alternative formulations for the utilitarian solutions in the Nash model. They share the general principle that the solution should recommend an alternative that maximises the sum of utilities, or of utility increments over the disagreement point. The main obstacle impeding a unified definition is that utilities are defined only up to positive affine transformations. We consider relative utilitarianism, that normalises the individual utilities to have infimum zero and supremum one before considering their sum. This reasonable normalisation choice was first considered in Arrow [1963] ; see Dhillon and Mertens [1999].

In our probability-based framework, the normalisation issue is immaterial and we can simply map the utilitarian precept into the (still) generic recommendation of maximising the sum of individual acceptance probabilities. Clearly, if we are to reinterpret this sum as a probability of joint success, this generic recommendation needs to be suitably qualified. Consider the following assumption.

A.9 (Average indifference) for any p, q in $[0, 1]$, $(p, q) \sim (\frac{p+q}{2}, \frac{p+q}{2})$.

Consider all pairs of acceptance probabilities on the segment between (p, q) and $(\frac{p+q}{2}, \frac{p+q}{2})$. As we move inward towards the bisector, the components are "less spread out" and one individual acceptance probability decreases at the expense of the other. Assumption A.9 states that the mediator is indifferent among all these pairs of acceptance probabilities, because the increase of one exactly compensates the diminution of the other. Intuitively, the two individual probabilities behave as substitutes towards the joint probability of acceptance. Clearly, this is at odds with A.5*, but it is compatible with A.5. Moreover, A.9 implies A.6 (Anonymity) but not A.3 (Disagreement indifference).

Theorem 7. *The preference relation $\succsim$ satisfies A.1–A.5 and A.9 if and only if it is represented by the copula $W(p, q) = \max(p + q - 1, 0)$.*

Démonstration. Necessity is obvious. As for sufficiency, by Theorem 2 there exists a copula C representing $\succsim$. It is known that W is the only quasi-convex copula ; see Example 3.27 in Nelsen [2006]. Hence, it suffices to show that A.9 implies that C is quasi-convex ; that is, for any α in $(0, 1)$ and $\mathbf{p}, \mathbf{q}$ in $[0, 1]^2$, we have $C(\alpha\mathbf{p} + (1 - \alpha)\mathbf{q}) \leq \max\{C(\mathbf{p}), C(\mathbf{q})\}$.

For ease of notation, given $\mathbf{p} = (p_1, p_2)$, let $\overline{\mathbf{p}} = \left(\frac{p_1+p_2}{2}, \frac{p_1+p_2}{2}\right)$ denote its symmetrised counterpart lying on the main diagonal. Then A.9 states that $\mathbf{p} \sim \overline{\mathbf{p}}$ and thus $C(\mathbf{p}) = C(\overline{\mathbf{p}})$. Assume without loss of generality that $q_1 + q_2 \leq p_1 + p_2$. Since C is increasing over the main diagonal, we obtain

$$C\left(\alpha\mathbf{p} + (1-\alpha)\mathbf{q}\right) = C\left(\overline{\alpha\mathbf{p} + (1-\alpha)\mathbf{q}}\right) \leq C\left(\overline{\mathbf{p}}\right) = C(\mathbf{p}) \leq \max\left\{C(\mathbf{p}), C(\mathbf{q})\right\},$$

and thus C is quasi-convex. □

This characterises the representing copula under Average indifference as the Fréchet lower bound $W(p, q) = \max(p + q - 1, 0)$, that provides the strongest possibile negative dependence between two marginal distributions. Therefore, we can reinterpret this form of (truncated) utilitarian solution as the recommendation that maximises the probability of joint acceptance when the mediator assumes that the individual acceptance probabilities are maximally negatively correlated.

This interpretation requires a comment. The copula $W(p, q)$ is strongly Pareto increasing on the triangle above the diagonal from $(0, 1)$ to $(1, 0)$, and is zero on the rest of its domain. If the mediator's preferences are represented by this copula, she behaves as an utilitarian for all pairs above the diagonal, and is indifferent for the pairs below the diagonal, suggesting that she may violate (strong) Pareto dominance. The reason for the mediator's indifference is that, under the assumptions of Theorem 7, she believes that any feasible proposal mapping to a pair (p, q) below the diagonal will be refused for sure. Hence, from her viewpoint, it is neither better nor worse than $\mathbf{0}$.

5.3.6 Logical independence of the assumptions

This section provides simple examples to show that, under A.1, the four assumptions A.2–A.5 used in Theorem 2 are logically independent. Recall that A.1 implies the existence of a real-valued function $V : [0, 1]^2 \to [0, 1]$, unique up to positive affine transformations, that represents $\succsim$ and is linear with respect to $\oplus$. We recall each assumption and list the associated counterexample immediately after. We omit quantifiers when they are obvious.

A.2 (Non-triviality) $(1, 1) \succ (0, 0)$.

Consider $V(p, q) = k$, for some constant k in $[0, 1]$. Then A.3 holds because $V(p, 0) = k = V(0, q)$. A.4 holds because $pV(1, 1) + (1 - p)V(0, 1) = k = V(p, 1)$, and similarly for the second relation. And A.5 holds because $(1/2)V(\mathbf{p} \vee \mathbf{q}) + (1/2)V(\mathbf{p} \wedge \mathbf{q}) = k = (1/2)V(\mathbf{p}) + (1/2)V(\mathbf{q})$. However, A.2 does not hold because $V(1, 1) = k = V(0, 0)$.

A.3 (Disagreement indifference) for any p, q in $[0, 1]$, $(p, 0) \sim (0, q)$.

Consider $V(p, q) = p$. Then A.2 holds because $V(1, 1) = 1 > 0 = V(0, 0)$. A.4 holds because $pV(1, 1) + (1-p)V(0, 1) = p = V(p, 1)$, and $pV(1, 1) + (1-p)V(1, 0) = 1 = V(1, p)$.

And A.5 holds because

$$\frac{1}{2}V(\mathbf{p}\vee\mathbf{q})+\frac{1}{2}V(\mathbf{p}\wedge\mathbf{q})=\frac{1}{2}\left(p_1\vee q_1\right)+\frac{1}{2}\left(p_1\wedge q_1\right)=\frac{1}{2}p_1+\frac{1}{2}q_1=\frac{1}{2}V(\mathbf{p})+\frac{1}{2}V(\mathbf{q})$$

However, $A.3$ does not hold : for $p>0$ and any q, we have $V(p,0)=p>0=V(0,q)$.

A.4 (Consistency over individual probabilities) for any p in $[0,1]$,

$$p(1,1)\oplus(1-p)(0,1)\sim(p,1) \quad \text{and} \quad p(1,1)\oplus(1-p)(1,0)\sim(1,p).$$

Consider $V(p,q)=p^2q$. Then A.2 holds because $V(1,1)=1>0=V(0,0)$. A.3 holds because $V(p,0)=0=V(0,q)$. A.5 holds because the first mixed derivative of V is positive, and hence V is supermodular. However, $A.4$ does not hold because $pV(1,1)+(1-p)V(0,1)=p>p^2=V(p,1)$.

A.5 (Weak complementarity) for any $\mathbf{p},\mathbf{q}$ in $[0,1]^2$,

$$(1/2)\left(\mathbf{p}\vee\mathbf{q}\right)\oplus(1/2)\left(\mathbf{p}\wedge\mathbf{q}\right)\succsim(1/2)\mathbf{p}\oplus(1/2)\mathbf{q}$$

Consider the function

$$V(p,q)=\begin{cases}\min\left(p,q,\frac{1}{3},p+q-\frac{2}{3}\right) & \text{if } \frac{2}{3}\leq p+q\leq\frac{4}{3}\\ \max(p+q-1,0) & \text{otherwise,}\end{cases}$$

borrowed from Exercise 2.11 in Nelsen [2006]. Then A.2 holds because $V(1,1)=1>0=V(0,0)$. A.3 holds because $V(p,0)=0=V(0,q)$. A.4 holds because $pV(1,1)+(1-p)V(0,1)=p=V(p,1)$, and similarly for the second relation. But $A.5$ does not hold : let $\mathbf{p}=(1/3,2/3)$ and $\mathbf{q}=(2/3,1/3)$, so that $\mathbf{p}\vee\mathbf{q}=(2/3,2/3)$ and $\mathbf{p}\wedge\mathbf{q}=(1/3,1/3)$. Then $V(\mathbf{p}\vee\mathbf{q})+V(\mathbf{p}\wedge\mathbf{q})-V(\mathbf{p})-V(\mathbf{q})=-1/3<0$, contradicting supermodularity.

5.4 Commentary

This section discusses the model and two interpretations of the results presented in Section 5.3, contrasting them with the related literature.

5.4.1 Fundamentals

The Nash model is an abstraction of real bargaining situations. As mentioned, Rubinstein et al. [1992] — from now on, RST — have re-examined the Nash model, moving away from the utility-based language of the Nash model towards the fundamentals of a two-person bargaining problem. This work indirectly provides a foundation for the Nash model that is independent of the assumption that bargainers maximise expected utility. We provide a similar description of the fundamentals for our model, and argue that they nest RST's formulation.

There are two agents and a set A of feasible alternatives, described in physical terms and viewed as deterministic outcomes. The set A is a nonempty compact subset of a connected topological space X. Each agent $i = 1, 2$ has an ordinal preference order $\succsim_i$ over X, and hence over A. Preferences are continuous : therefore, there exist ordinal (i.e., unique up to increasing transformations) and continuous value functions $v_i(x)$ such that $x \succsim_i y$ if and only if $v_i(x) \geq v_i(y)$. Moreover, agents' preferences are (jointly) non-trivial : there are alternatives x, y in A such that $x \succ_i y$ for both i. We add a final simplifying assumption : there are no alternatives x, y in A such that $x \sim_i y$ for both i. This avoids the need to rephrase definitions and results in terms of equivalence classes.

A *native bargaining problem* is a triple $(A, \succsim_1, \succsim_2)$ that satisfies the assumptions above. Compared to RST, we assume neither a commonly known disagreement point δ (in physical terms) nor that agents' preferences are defined over lotteries where the prizes are elements of X. In particular, we do not require that agents are expected utility maximisers or, more generally, that their preferences over A are representable by (cardinal) utility functions that are invariant only to positive affine transformation. Compared to the six restrictions made in RST (Section 2), we maintain (i)-(ii)-(v), weaken (iii) to non-triviality, and drop (iv)-(vi).

5.4.2 Interpretations

We provide two compatible interpretations for the bare-bones model of Section 5.3. The first one is behavioural and casts the mediator's problem as a decision problem with incomplete information. The second interpretation hijacks this setup, and shows that the classical Nash model (based on utilities) is mathematically equivalent to our probability-based formulation.

The target-based interpretation

The first interpretation takes the viewpoint of a mediator, hired by two bargainers to recommend them a feasible proposal over which they could strike an agreement. Given a native bargaining problem $(A, \succsim_1, \succsim_2)$, the mediator may suggest any feasible alternative in A, but cannot impose it. Her goal is to put on the table a proposal that each bargainer will individually evaluate and decide whether to accept. She may take into account issues of fairness or other considerations, but eventually her task is to select a proposal from A and her success is defined by its joint acceptance on the part of the bargainers. The mediator wants to maximise her probability of success.

We cast this situation as a decision problem under incomplete information. A bargainer $i = 1, 2$ accepts a proposal x when $x \succsim_i t_i$. We say that t_i is the minimum acceptance *target* for i. We may think of t_i as the minimal "fair" outcome that the bargainer has in mind, or as a proxy for his toughness (akin to his *type*), or as the outcome of a deliberative process over the risk of disagreement. The crucial assumption is that the mediator has incomplete information about t_i. She knows that $\succsim_i$ is continuous, and she believes that the type T_i of i has a random distribution with an (order-)convex support on $(X, \succsim_i)$.

Given that $\succsim_i$ is continuous on X, it admits a real-valued representation by a continuous function $v_i : X \to \mathbb{R}$. Therefore, $x \succsim_i t_i$ if and only if $v_i(x) \geq v_i(t_i)$ and we can equivalently reformulate the incomplete information about T_i as a random variable $v(T_i)$ on $(\mathbb{R}, \geq)$ with c.d.f. F_i and convex support. Consequently, $P(x \succsim_i T_i) = F_i \circ v_i(x)$, where F_i is a strictly increasing and continuous c.d.f. on $\mathbb{R}$. Note that v_i is unique only up to increasing transformations, so F_i is not uniquely defined ; however, the function $F_i \circ v_i$ is unique. Therefore, we assume that the incomplete information of the mediator about each bargainer's type is summarised by $F_i \circ v_i$; in particular, given a proposal x, this is mapped into an individual acceptance probability $p_i = F_i \circ v_i(x)$.

The model in Section 5.3 takes this as its point of departure and provides a behavioural characterisation for the mediator's preferences over proposals in A. She ranks the feasible alternatives by their probability of being accepted by both bargainers, after combining their individual acceptance probabilities into a joint probability of success based on their dependence structure. For any native bargaining problem on A, the solution proposed by the mediator is a feasible alternative that maximises her induced preference order $\succsim$ over A.

We note two advantages for this behavioural interpretation. It is compatible with (but does not require) the assumption that agents are expected utility maximisers. In fact, it is not even necessary to include lotteries over feasible alternatives among the objects of choice for the bargainers. A similar comment applies for the disagreement point : in the literature, it is customary to mention it and immediately dispatch it by normalising its value to zero. Our model does not presume that a disagreement point δ (in physical terms) is known ; however, if one is given, then the agent's individual rationality implies that his target has zero probability to lie below δ, and thus $P(\delta \succ_i T_i) = 0$.

Nash bargaining redux

The original Nash model selects a solution by maximising the product of two von Neumann–Morgenstern utilities (from now on, NM). Our approach picks a solution by maximising a copula that aggregates two individual probabilities. The goal of this section is to show that these two approaches are consistent and strictly linked.

The key decision-theoretic observation is that the utility-based NM model may be recast in an exclusively probability-based language ; see Castagnoli and LiCalzi [1996] and Bordley and LiCalzi [2000]. For simplicity, consider preferences over a compact nonempty interval $B = [x_*, x^*]$ in $\mathbb{R}$. Suppose that the Bernoulli index U is strictly increasing, bounded, and continuous ; see Grandmont [1972] for an axiomatization. Applying if necessary a positive affine transformation, let $U(x_*) = 0$ and $U(x^*) = 1$. Then $U(x)$ has the formal properties of a cumulative distribution function (c.d.f.), and on some appropriate probability space there exists a random variable T with c.d.f. $U(x) = P(T \leq x)$. We call T a (random) target.

Let $\mathcal{L}$ the space of lotteries over B. If a lottery X in $\mathcal{L}$ has c.d.f. F and is stochastically independent of T, the chain of equalities

$$EU(X) = \int U(x)\, \mathrm{d}F(x) = \int P(T \leq x)\, \mathrm{d}F(x) = P(X \geq T) \tag{5.1}$$

shows that the expected utility $EU(X)$ is formally equivalent to the probability that the lottery X scores better than the target T. Hence, a claim made in a utility-based language for $EU(X)$ maps to an equally valid statement in a target-based language for $P(X \geq T)$. In particular, we can replace the notion of a cardinal Bernoulli index $U(x)$ that is unique only up to positive affine transformations by the simpler concept of a c.d.f $P(T \leq x)$ for the target T. The NM model for preferences under risk postulates that preferences are linear in probabilities. It can be equivalently interpreted as a procedure that ranks lotteries by the expected value of their Bernoulli index or by the probability that they score better than a target T; see LiCalzi [1999].

We are ready to consider the Nash bargaining problem (S, d). Recall that S is a compact and convex subset of feasible NM utility pairs, while d represents bargainers' utilities in case of breakdown. The crucial, but often implicit, assumption of the Nash model is that the NM-utility functions U_1 and U_2 are commonly known. Using (5.1), this reads as the assumption that the distributions of the bargainers' targets T_1 and T_2 are *ex ante* commonly known, whereas the targets are private information.

In our former interpretation, given a proposal x, the mapping $p_i = F_i \circ v_i(x)$ is based on the mediator's beliefs. Under common knowledge, the alternative interpretation is that the bargainers themselves agree on their own individual acceptance probabilities and may directly use these as input in constructing a bargaining solution. The missing step for the two bargainers is how to aggregate the commonly known individual probabilities and evaluate the proposals. This aggregation problem may be attacked in different ways. Ours is a behavioural characterisation : if the agents have common knowledge of the joint distribution of their targets *ex ante*, they maximise the probability of success by settling on the commonly known copula. In particular, if it is common knowledge that their two targets are *ex ante* stochastically independent, they should settle for the Nash solution. A related normative approach is pursued by Border and Segal [1997], discussed in Section 5.5.2.

Related literature

A relevant byproduct of our approach is the interpretation of the product operator in the Nash solution as the consequence of an assumption of stochastic independence between individual acceptance probabilities. To the best of our knowledge, the utility-based literature offers two competing interpretations for the product operator. We recall them briefly, for comparison.

[Roth, 1979, Section I.C] shows that we can frame the bargaining model as a single-person decision problem, where each agent chooses how much to claim by maximising his expected utility under the assumption that the claim of the other bargainer i is uniformly distributed between his disagreement point d_i and his ideal point m_i (defined as i's highest feasible utility). The Nash solution emerges from the independent choices of the two bargainers. A very similar interpretation is also in Glycopantis and Muir [1994], who make no reference to Roth [1979]. As Roth himself acknowledges, this approach is outside the game-theoretic tradition, because the processes by which the two agents form their expectations are not mutually consistent. On the other hand, similarly to ours, this interpretation is grounded on an assumption of independence between the evaluations made by the two agents.

A second interpretation for the Nash product is proposed in Trockel [2008]. He views the Nash product as a special case of a social welfare function that aggregates the admissible payoff pairs into a social ranking. It evaluates a recommendation $\mathbf{u}$ by the Lebesgue measure of the set of utility pairs that are Pareto-dominated by $\mathbf{u}$. Without advocating more than a formal analogy, we note that any copula in Theorem 2 may be interpreted as a social welfare function adopted by the mediator to select her recommendation.

5.4.3 Domain

The Nash model is framed in the space of utilities : it implicitly assumes that all native bargaining problems with the same utility representation are indistinguishable, and thus must have the same solution. This hidden assumption is probably extreme. RST discuss at length its implications, and lay bare the tradeoff between the power of Nash's axioms and the granularity of the domain. Switching to a preference-based language, they remould the assumption by keeping fixed the set A of alternatives and varying the bargainers' preferences. In their approach, a bargaining solution is a function that assigns a unique element of a given set A to every pair of bargainers' preferences over lotteries on $A \cup \{d\}$.

Consistent with Nash [1950], however, the standard way to specify the domain of the solution is to hold bargainers' preferences fixed and let the set A of alternatives vary. The typical formulation considers all the Nash problems based on the same disagreement point (in utilities). For generality, we cast our presentation assuming that the domain of our model contains all compact (but not necessarily convex) subsets of $[0, 1]^2$. However, this domain may be considerably shrunk and, in practical applications, it is reasonable to do so. The domain must be rich enough to include enough problems and elicit preferences over $[0,1]^2$. A smaller domain may be more appropriate to ensure that the dependence embedded in the copula refers to comparable situations : for instance, the two bargainers should be the same, and the problems submitted to the mediator should justify similar answers.

Here is an exemplification. Fix two bargainers and their preferences on X. For any feasible (compact) set A, let d_i be the acceptance probability for a $\succsim_i$-minimal proposal in A ; similarly, let m_i be the acceptance probability for a $\succsim_i$-maximal proposal. Borrowing language from the Nash model, let $\mathbf{d}$ and $\mathbf{m}$ be called the disagreement point and the ideal point (in probabilities). A rich domain for our model may consider only the compact subsets in $[0,1]^2$ associated with the same $\mathbf{d}$, and for those we may characterise a preference relation $\succsim_{\mathbf{d}}$ represented by a copula C such that $C(\mathbf{d}) = 0$. (We assume that an agent rejects a minimal proposal for sure.) A smaller but still rich domain is formed by the bargaining problems with the same $\mathbf{d}$ and $\mathbf{m}$: if we assume that an agent accepts his ideal point for sure, the representing copula would have both $C(\mathbf{d}) = 0$ and $C(\mathbf{m}) = 1$; see Cao [1982] for a similar normalisation over utility functions in the Nash model.

5.5 Applications and extensions

This section illustrates the richness and robustness of the target-based approach. We present a few applications and extensions, including comparative statics, testable restrictions, and prescriptive advice.

5.5.1 Comparative statics

A small but elegant literature deals with the comparative statics of the Nash solution. For a typical result, consider Theorem 1 in Kihlstrom et al. [1980] : "The utility which Nash's solution assigns to a player increases as his opponent becomes more risk averse." Let us consider the implications of this result for the target-based approach, when the Bernoulli index function $U_i(x) = F_i \circ v_i(x)$ is interpreted as the cumulative distribution function $P(x \succsim_i T_i)$.

An agent with a Bernoulli index function $V_1(x)$ is more risk averse than an agent characterised by $U_1(x)$ if and only if there exists an increasing concave transformation K such that $V_1(x) = K \circ U_1(x)$. Under the target-based interpretation, both V_1 and U_1 are distribution functions over the same domain and with the same range in $[0,1]$. Therefore, the function K is bounded in $[0,1]$ with $\lim_{x \downarrow 0} K(x) \geq 0$ and $\lim_{x \uparrow 1} K(x) = 1$; by concavity, it follows that $K(x) \geq x$ for all x. Hence, $V_1(x) \geq U_1(x)$ for all x ; that is, viewed as c.d.f.'s, V_1 is stochastically dominated by U_1. In simple words, the target associated with V_1 can be regarded as "less demanding" than the target associated with U_1. Theorem 1 may be reformulated as follows. At the Nash solution, $i = 1, 2$ is offered a better proposal and his individual acceptance probability increases when the target of the other agent $j = 3 - i$ becomes less demanding. If an agent becomes more accommodating, the Nash solution ends up rewarding the other one.

A related comment concerns the observation recently made in Alon and Lehrer [2014] that the Nash solution is not ordinally equivalent. Suppose that two agents are to divide a unit amount of money. Let x_i in $[0, 1]$ be the quantity attributed to Agent i, with $x_1 + x_2 = 1$. From an ordinal viewpoint, assume that each agent has increasing (and continuous) preferences in the amount x_i he secures. A model with ordinal preferences $U_1(x) = U_2(x) = x$ is equivalent to a model with ordinal preferences $U_1(x) = \sqrt{x}$ and $U_2(x) = 1 - \sqrt{1 - x}$. However, when applied literally, the Nash solution predicts $x_1 = x_2 = 1/2$ for the first model, and $x_1 = 1/4$ and $x_2 = 3/4$ for the second one. In our approach, the function $U_i(x) = F_i \circ v_i(x)$ embodies both the ordinal preference expressed by v_i and the individual acceptance probability represented by F_i. The two models are ordinally equivalent, but in the second model Agent 1 has a less demanding target while Agent 2 has a more demanding target. (In a utility-based language, what drives the difference is that in the second model Agent 1 is more risk averse while Agent 2 is less risk averse.)

These examples deal with comparative statics over the distribution of individual acceptance probabilities. The target-based approach, however, allows to study analogous results based on (partial) orderings for the copulas underlying the dependence that links individual probabilities to the joint probability of success. We consider only a simple example. Given two copulas C_1 and C_2, the *concordance* ordering states that C_1 is more concordant than C_2 if $C_1(\mathbf{p}) \geq C_2(\mathbf{p})$ for all $\mathbf{p}$ in $[0, 1]^2$. Suppose this is the case for C_1 and C_2, and let $\mathbf{p}_i^*$ be a solution under the copula C_i, for $i = 1, 2$ and the same bargaining problem B. Clearly,

$$C_1(\mathbf{p}_1^*) \geq C_1(\mathbf{p}_2^*) \geq C_2(\mathbf{p}_2^*)$$

so that the joint probability of success at the solution is increasing in the concordance ordering. Agents with concordant targets are more likely to strike a deal.

5.5.2 Social preferences and implementation

The model in Section 5.3 characterises a preference relation $\succsim$ and derives the bargaining solution through the maximisation of a copula. A similar approach is followed by Blackorby et al. [1994], who define a bargaining solution as the (set of) maximisers for a generalised Gini ordering, represented by a quasi-concave, increasing function that is linear on the rank-ordered subsets of $[0, 1]^2$. The class of generalised Gini orderings spans a continuum of solutions, exhibiting different levels of inequality aversion and including the egalitarian and the utilitarian solutions as extreme cases.

We use their model as an example to illustrate the flexibility and limitations of our copula-based approach. For $0 \leq \alpha \leq 1/2$, the family of symmetric copulas

$$C_{\alpha,\alpha}(u,v) = \begin{cases} u + \frac{\alpha}{1-\alpha}v - \frac{\alpha}{1-\alpha} & \text{if } \alpha \leq u \leq v \leq 1 \\ v + \frac{\alpha}{1-\alpha}u - \frac{\alpha}{1-\alpha} & \text{if } \alpha \leq v \leq u \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

represents a continuum of affine (truncated) generalised Gini orderings. This family includes the egalitarian solution for $\alpha = 0$ and the utilitarian solution for $\alpha = 1/2$. Accordingly, the parameter α may be interpreted as an index of (increasing) inequality aversion. (This family is nested in a larger class of asymmetric copulas $C_{\alpha,\beta}$ with $\alpha, \beta > 0$ and $\alpha + \beta < 1$; see Exercise 3.8 in Nelsen [2006].) We say that this family is truncated because $C(u,v) = 0$ if $\min\{u,v\} < \alpha$. The same comment made for the utilitarian solution in Section 5.3.5 applies here, and clearly these copulas do not satisfy A5* when $\alpha > 0$.

In our model, the copulas aggregate individual acceptance probabilities. One should be careful not to carry formal analogies too far, but the copulas $C_{\alpha,\alpha}$ may be used to model the mediator's aversion to inequalities among the individual acceptance probabilities. The alternative interpretation is that the copulas represent the stochastic dependence between agents' targets and that this dependence drives the choices. If the mediator's opinion is captured by $C_{\alpha,\alpha}$, then her recommended solution exhibits (some degree of) inequality aversion between the individual acceptance probabilities. If $C_{\alpha,\alpha}$ rationalises the mediator's recommendations, these two interpretations are behaviourally indistinguishable.

More generally, when $\succsim$ is viewed as a social preference relation, one may recast the representing copula as a social welfare function. This approach was pioneered by Kaneko and Nakamura [1979], who define and characterise a Nash social welfare function that evaluates the relative increases in individuals' welfare from a state δ (in physical terms) unanimously considered as the worst possible. In a subsequent paper, Kaneko [1980] takes care to point out the conceptual differences between a social welfare function and a bargaining solution. Put simply, however, the underlying idea is to pick a solution by maximising a function : this works also over non-convex (compact) sets, but it may generate set-valued solutions. The applicability of the Nash model is extended at the cost of forfeiting uniqueness.

Border and Segal [1997] provide an axiomatisation of the Nash (bargaining) solution that is very close in spirit both to Kaneko's insight and to our model. The natural interpretation for their setup is that "the two bargainers hire an arbitrator to make choices for them" (p. 1) and that the arbitrator has a preference order $\succsim$ over solutions. Border and Segal [1997] motivate these preferences by analogy to social choice, suggesting that the arbitrator relies on her notions of fairness to come up with a decision rule for all the bargaining problems. An ancillary interpretation views the axioms as guidelines that both bargainers should find acceptable before they agree to hire her. In their approach, the arbitrator's selection is justified by (normatively) binding axioms. Instead, we share with RST the search for a behavioural characterisation that downplays the normative undertones : the mediator in our model issues recommendations but cannot impose a solution.

A related but independent issue concerns the implementation problem. Our model assumes that the mediator knows the individual acceptance probabilities or, more precisely, that she holds subjective beliefs about the agents' targets. However, it may be in the agents' interest to misrepresent their objectives and manipulate the mediator's beliefs. Therefore, one should worry about devising mechanisms that help elicit correct information from the agents. An important step forward in this direction is made by Miyagawa [2002], who studies this problem in the context of two-person bargaining. He provides a simple four-stage sequential game that fully implements a reasonably large class of two-person bargaining solutions in subgame-perfect equilibrium.

Miyagawa [2002] relies on four restrictive assumptions to derive his results. First, the bargaining problem B must contain an alternative d that both agents consider least preferred. Second, any most preferred alternative for i is judged by $j = 3-i$ as indifferent to d. Third, the Pareto frontier of B is strictly convex. Finally, the bargaining solution must be generated by the maximisation of a (component-wise) increasing and quasi-concave function. This generating function is an analog of the social welfare function just discussed. Coincidentally, Miyagawa (2002) also assumes that this function is normalized to $[0, 1]$ as in the copula-based model. Each assumption has technical implications ; for instance, the combination of strict convexity for the Pareto frontier and quasi-concavity for the generating function entails the uniqueness of the solution.

Compared to our setup, the crucial restriction is that the generating function must be quasi-concave. A copula need not be quasi-concave ; for instance, the function $C(u, v) = [M(u, v) + W(u, v)]/2$ is a copula, but it is not quasi-concave. Hence, the applicability of Miyagawa (2002) cannot generally be taken for granted. However, many common examples, and all the functional forms presented so far, are quasi-concave. Therefore, the class of solutions that are implementable includes the Nash, the egalitarian, and the (truncated) utilitarian solutions.

5.5.3 Testable restrictions

An important requirement for a theory is the ability to generate falsifiable predictions. A direct test exists for the target-based approach. We first illustrate the idea, and then formalise it. As introduced in Section 5.3, a *bargaining problem* is represented by a compact set B in $[0,1]^2$ where each point $\mathbf{p}$ in B corresponds to a pair of (acceptance) probabilities. A *solution* is a map that for any problem B selects (at least) one point in B. The target-based approach recommends a solution by maximising a suitable copula C over B.

Suppose that B contains a point $\mathbf{p} = (p_1, p_2)$ with $p_1 + p_2 > 1$. The Fréchet lower bound implies $C(\mathbf{p}) \geq p_1 + p_2 - 1 > 0$ for any C. Consider another feasible point $\mathbf{q} = (q_1, q_2)$ in B. The Fréchet upper bound implies $C(\mathbf{q}) \leq \min(q_1, q_2)$. Therefore, if $\min(q_1, q_2) < p_1 + p_2 - 1$, the point $\mathbf{p}$ must be strictly preferred to $\mathbf{q}$, and $\mathbf{q}$ cannot be a solution for any copula C. More generally, by picking a point $\mathbf{p}^*$ that maximises $p_1 + p_2 - 1$ in B, we can formulate the following more stringent test.

Proposition 5.5.1. *Let* $\mathbf{p}^* \in \arg\max_B (p_1 + p_2 - 1)$. *Define the quadrant*

$$Q := \left\{ \mathbf{q} \in [0,1]^2 : \min(q_1, q_2) \geq p_1^* + p_2^* - 1 \right\}.$$

Then the solution must belong to $B \cap Q$.

Clearly, under A.5*, a second immediate test is that a solution for B cannot be dominated (in the strong Pareto ordering) by another point available in B.

A different class of restrictions is the following. As discussed in Section 5.5.2, our model characterises bargaining solutions that can be rationalised by a copula. Roughly speaking, this implies that we can generate only solutions that satisfy independence of irrelevant alternatives. For instance, the well-known Kalai-Smorodinsky (1975) solution (for short, KS solution) cannot be derived in our model. Formally speaking, this solution violates A1 as can be shown by a simple example. For the bargaining problem B_1 represented by the convex hull of the three points $(0,0)$, $(0,1)$, and $(1,0)$, the KS solution uniquely prescribes the point $\mathbf{p} = (1/2, 1/2)$. On the other hand, for the bargaining problem B_2 represented by the convex hull of the four points $(0,0)$, $(0,1)$, $(1/2, 1/2)$, and $(1/2, 0)$, the KS solution uniquely prescribes the point $\mathbf{q} = (1/3, 2/3)$. Since both $\mathbf{p}$ and $\mathbf{q}$ belong to $B_2 \subset B_1$, the first KS solution reveals $\mathbf{p} \succ \mathbf{q}$ while the second one reveals $\mathbf{q} \succ \mathbf{p}$, in violation of A1. Although it is formally possible to rationalise the KS solution as the outcome of a lexicographic maximisation, we find this approach interpretively unsatisfactory and thus we do not pursue it here.

5.5.4 Bargaining power

Since Binmore et al. [1986], the economic literature has made extensive use of an asymmetric version of the Nash Solution as a reduced form for the differences in the bargaining power of the agents. Formally, given a Nash bargaining problem (S, d), the *asymmetric Nash solution* is defined as the maximiser of the product $(u_1 - d_1)^a (u_2 - d_2)^{1-a}$, for a in $[0,1]$. As a increases from 0 to 1, the asymmetric solution increasingly favours the first agent, with symmetry holding at $a = 1/2$. For instance, consider the illustrative example from Section 5.5.1 where two agents are to divide a unit amount of money, and $U_1(x) = U_2(x) = x$. The asymmetric Nash solution gives $x_1 = a$ and $x_2 = 1 - a$.

It is natural to suggest using the function $N_a(p, q) = p^a q^{1-a}$ but this is not a copula, because it fails the second half of C1 in Section 5.2. Therefore, one cannot import the asymmetric Nash solution under the copula-based approach. On the other hand, since asymmetric copulas exist, we might search for some suitable (possibly parametric) substitute copula with similar properties. For instance, by Theorem 2.1 in Liebscher [2008], for any copula C the expression

$$C^a(p, q) = p^a q^{1-a} C(p^{1-a}, q^a) = N_a(p, q) \cdot C(p^{1-a}, q^a)$$

defines an asymmetric copula, bearing an obvious relationship with N_a. We argue that this approach cannot work either.

Given a bargaining problem B, let $\Delta_B(C) = \sup_{(p,q)\in B} |C(p, q) - C(q, p)|$ be the degree of asymmetry for a function C over B. In our illustrative example, we obtain $\Delta_B(N_a) = 1$ at $a = 0$ and $a = 1$; that is, the family of asymmetric Nash solutions includes the case of maximal asymmetry. On the other hand, it is known that over $B = [0, 1]^2$ we have $\Delta_B(C) \leq 1/3$ for any copula C (Nelsen [2007]); moreover, $\Delta_B(C) \leq 1/5$ if C is also quasi-concave (see Alvoni and Papini [2007]). Therefore, there exist no copula that can mimic the extreme asymmetry allowed by N_a. In our illustrative example, for instance, $\Delta_B(N_a)$ is symmetric around $a = 1/2$ and increasing in $|a - 1/2|$, with $\Delta_B(N_a) = 1/3$ at $|a - 1/2| \approx 0.219$; therefore, the copula-based approach cannot replicate N_a for a outside of the interval $[.281, .719]$.

In fact, a sound interpretation for modelling bargaining power requires more care. Consider again the standard utility-based approach. The widespread use of the asymmetric Nash solution is most likely due to its technical convenience, without much concern about the lack of any intuitive meaning for the weighted geometric mean of utilities represented by N_a. However, utility-based axiomatic foundations exist in the literature. Kalai [1977a] provides a characterisation for the asymmetric solution based on a replication argument for symmetric solutions. His approach is technically clean and simple, but provides little intuition.

A richer and more convincing approach is Harsanyi and Selten [1972], who study the case of fixed threats under incomplete information. This latter assumption, absent in the Nash model, explicitly recognises that each player has private information about some of his characteristics that are relevant for the bargaining process. Harsanyi and Selten [1972] show that differential information provides a foundation for asymmetric solutions. We follow in their footsteps and discuss a reduced form for modelling bargaining power in our illustrative example, based on the copula-based approach.

Recall the distinction between the native bargaining problem $(A, \succsim_1, \succsim_2)$ and the derived bargaining problem B ; see Section 5.4.1. Bargaining power must be traced back to the native problem. The individual acceptance probabilities provide the link between this and the derived problem, while the choice of a copula applies only to B as a tool to reconcile the individual acceptance probabilities into a joint probability of success. Therefore, bargaining power should affect the shape of the derived problem B rather than the functional form C.

In the illustrative example, the feasible set $A = \{(x_1, x_2) \in [0,1]^2 : \, x_1 + x_2 \leq 1\}$ for the native problem is especially simple to describe. In particular, it can be reduced to a unidimensional problem $\{(x_1, 1 - x_1) \in [0,1]^2 : \, 0 \leq x_1 \leq 1\}$, where each agent only cares about the money he receives. Assume Theorem 5 ; that is, the mediator aggregates agents' acceptance probabilities assuming stochastic independence and $C(p, q) = p \cdot q$. When Agent 1 has more bargaining power than Agent 2, he can aspire to higher targets. (As in Harsanyi and Selten [1972], bargaining power is only one among several issues that may cause this relationship to hold.) Everything else being equal, this us captured by the assumption that T_1 stochastically dominates T_2.

For practical purposes, it is convenient to adopt a parametric formulation. For instance, let $P(x \succsim_i T_i) = x^{a_i}$ where $a_i \geq 0$ can be interpreted as the bargaining strength of Agent i, and assume $a_1 + a_2 > 0$. Clearly, T_1 stochastically dominates T_2 if and only if $a_1 \geq a_2$. The shape of the derived bargaining problem

$$B(a_1, a_2) = \left\{(p_1, p_2) \in [0,1]^2 : \, p_1 = x^{a_1} \text{ and } p_2 = (1 - x)^{a_2}, \text{ for } 0 \leq x \leq 1\right\}$$

depends on how the bargaining strength a_i of each agent affects his own acceptance probability in the eyes of the mediator. Given the product copula, the acceptance probabilities aggregate into the function $x^{a_1} \cdot (1 - x)^{a_2}$, and the corresponding recommendation is $x_1^* = a_1/(a_1 + a_2)$ and $x_2^* = a_2/(a_1 + a_2)$. We recover the same solution set associated with the asymmetric Nash solution.

From an interpretive viewpoint, we emphasise that the asymmetry associated with differences in bargaining power affects the individual acceptance probabilities. The approach described at the beginning of this section fails because it tries to impose the asymmetry on the copula, that instead aggregates those into the joint probability of success. Although our approach allows asymmetric copulas, they are not meant as a tool to capture modelling issues concerning the native problem or its relation with the derived problem.

5.5.5 Negotiation analysis

This last section touches upon the relationships of our approach with the practice of negotiations. When [Roth, 1979, Section I.B] writes that "a solution can be interpreted as a model of the bargaining process", he argues for considering its descriptive implications that can be tested empirically. However, he points out that there is an alternative prescriptive approach which views the solution as "a rule which tells an arbitrator what outcome to select" ; e.g., Border and Segal [1997] aim to this second goal.

Our model is concerned with a mediator rather than an arbitrator : the latter can impose a solution, while the former can only recommend it. This puts the mediator in a hybrid position. Maximising the joint probability of success appears very natural, but its prescriptive value for the mediator may not be shared by the agents who presumably worry about more than just striking an agreement. This is not the place for a full discussion, but we wish to add a few remarks to highlight the potential of our approach.

[Subramanian, 2010, pp. 109–110] finds "that the implications for negotiation strategy change dramatically when we move away from the assumption that dealmakers will accept deals that are just better than their BATNA [Best Alternative to a Negotiated Agreement, equivalent to the disagreement point] to the more realistic and nuanced assumption that the likelihood the other side will say yes increases with the incentives to do so." The target-based interpretation makes the role of individual acceptance probabilities explicit. The mediator should acquaint herself with the agents well enough to understand how each of them views the native bargaining problem and code her understanding in terms of individual acceptance probability.

A second step requires the mediator to make a conscious choice about the possible connections between the individual acceptance probabilities and the joint probability of success. While the assumption of stochastic independence seems by far the simple and most natural one, it is not the only possible one. Different copulas may rationalise alternative recommendations, and thus this element should be given attention by the mediator.

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